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HARMONIC MAPPINGS OF NEGATIVELY CURVED MANIFOLDS

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1. Introduction. Volume decreasing properties of harmonic mappings of space forms were investigated by S. S. Chern and S. I. Goldberg [**3**] and the author. In a previous paper [**6**], a step toward generalization of the results was made proving the following theorem:

THEOREM. *Let $f: M \to N$ be a harmonic mapping of n-dimensional Riemannian manifolds, with $C \leqq 0$. Suppose the scalar curvature of M is not less than $-S$, and the Ricci curvature of N is not greater than $-\bar{S}/n$, where $S \geqq 0$ and $\bar{S} > 0$ are constants. Then, if u has a maximum on M,*

$$u \leqq (S/\bar{S})^n$$

i.e., f is volume decreasing up to a constant.

In this theorem, u is the square of the ratio of volume elements of N and M, and C is a scalar invariant of the mapping (cf. [**3**]).

In the present paper, the case in which M is a complete manifold with non-positive sectional curvature is considered by exhausting M with concentric geodesic balls carrying certain convex functions. These functions are used to define metrics in which the squared ratio of volume elements has a maximum, as the last theorem requires. In this way it is shown that harmonic mappings of negatively curved manifolds are volume decreasing, thus generalizing Chern and Goldberg's result for the ball. We thank the referee for mentioning that a similar result follows using methods developed by S. T. Yau [**8**].

2. The geodesic radius.

Definition. Let M be a Riemannian manifold, and p_0 be a fixed point in M. The *geodesic radius* on M (relative to the origin p_0) is the function $\tau: M \to \mathbf{R}$ defined by

$$\tau(p) = d(p, p_0),$$

where d is the Riemannian distance on M.

It is well known that τ is differentiable in $M - \Omega - \{p_0\}$, where Ω is the cut locus of p_0. In the case that M is complete and simply connected and has non-

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positive sectional curvature, $\Omega = \emptyset$ and τ is differentiable in $M - \{p_0\}$. According to Bishop-O'Neill [**2**], τ is also *convex* there, i.e., its Hessian $\nabla^2\tau$ is positive semi-definite. As a result, the inequality $\Delta\tau \geqq 0$ holds in $M - \{p_0\}$. Better bounds on $\Delta\tau$, the Laplacian of τ, which are valid in the general case, are given in the following theorem.

THEOREM 1. *Let M be a complete Riemannian manifold, and let Ω be the cut locus of $p_0 \in M$. Then, on $M - \Omega - \{p_0\}$,*

(a) *if the Ricci curvature of M is $\geqq (n-1)K$,*

$$\Delta\tau \leqq (n-1)\sqrt{K}\cot\sqrt{K}\tau;$$

(b) *if the sectional curvatures of M are $\leqq K$,*

$$\Delta\tau \geqq (n-1)\sqrt{K}\cot\sqrt{K}\tau \quad (\sqrt{K}\tau < \pi).$$

In the above formulas, $\sqrt{K}\cot\sqrt{K}\tau$ should be read $\sqrt{-K}\coth\sqrt{-K}\tau$ for $K < 0$ and $1/\tau$ for $K = 0$. The restriction $\sqrt{K}\tau < \pi$ should be omitted for $K \leqq 0$.

For a proof, see Aubin [**1**], although a proof which doesn't involve the second variation of arclength is also available (cf. [**7**]).

3. Harmonic maps. In this section, we review some properties of harmonic maps as found in [**3**] and [**6**], which will be required later.

Let M and N be Riemannian manifolds, and f: $M \to N$ a smooth mapping. Let $\{X_i\}$ and $\{\bar{X}_\alpha\}$ be (local) orthonormal frames in $T(M)$, $T(N)$ respectively†. Let $\{s_\alpha\}$ be the frame in the vector bundle $E = f^{-1}T(N)$ induced from $\{\bar{X}_\alpha\}$. Let $\{\omega_i\}$, $\{\bar{\omega}_\alpha\}$, $\{f^*\bar{\omega}_\alpha\}$ be the corresponding coframes. A matrix $(a_{\alpha i})$ is defined by the equation

$$f^*\bar{\omega}_\alpha = a_{\alpha i}\omega_i.$$

Consider the differential of the mapping, f_*, as a section of $f^{-1}T(N) \otimes T(M)^*$, that is an E-valued differential form on M. Then,

$$f_* = a_{\alpha i}s_\alpha \otimes \omega_i.$$

Let $\tilde{\nabla}$ be the covariant differentiation operator of E-valued differential forms defined by

$$\tilde{\nabla}(s_\alpha \otimes \Phi^\alpha) = Ds_\alpha \otimes \Phi^\alpha + s_\alpha \otimes \nabla\Phi^\alpha$$

where the Φ^α are real forms, ∇ is the covariant differential in M, and D is the linear connection in E induced from the Riemannian connection in $T(N)$. Then,

$$\tilde{\nabla}f_* = a_{\alpha ij}s_\alpha \otimes \omega_j \otimes \omega_i$$

†Latin indices have the range $1 \leqq i, j, k, \ldots \leqq \dim M$, while Greek ones have the range $1 \leqq \alpha, \beta, \gamma, \ldots \leqq \dim N$. Quantities on N are distinguished by an upper bar. The Einstein summation convention is used.

where the components $a_{\alpha ij}$ satisfy

$$(1) \quad a_{\alpha ij}\omega_j = da_{\alpha i} + a_{\alpha j}\omega_{ji} + a_{\beta i}f^*\bar{\omega}_{\beta\alpha}.$$

Here ω_{ji} and $\bar{\omega}_{\beta\alpha}$ are the connection forms in M and N respectively.

In the same way, we have

$$\tilde{\nabla}\tilde{\nabla}f_* = a_{\alpha ijk}s_\alpha \otimes \omega_k \otimes \omega_j \otimes \omega_i$$

where

$$(2) \quad a_{\alpha ijk}\omega_k = da_{\alpha ij} + a_{\alpha ik}\omega_{kj} + a_{\alpha kj}\omega_{ki} + a_{\beta ij}f^*\bar{\omega}_{\beta\alpha},$$

and so forth.

A mapping f: $M \to N$ is called *harmonic*, if

$$\operatorname{tr}(\tilde{\nabla}f_*) \equiv a_{\alpha ii}s_\alpha = 0.$$

Suppose M and N are n-dimensional Riemannian manifolds. Let $v_M = \omega_1 \wedge \ldots \wedge \omega_n$ and $v_N = \bar{\omega}_1 \wedge \ldots \wedge \bar{\omega}_n$ be their respective volume elements. Let $A = f^*v_N/v_M$ be the ratio of volume elements. Then

$$A = \det(a_{\alpha i}).$$

Let $(B_{i\beta})$ be the adjoint matrix of $(a_{\alpha i})$, i.e., $B_{i\alpha}a_{\alpha j} = \delta_{ij}A$. Finally let $u = A^2$. Then we have the formula

$$(3) \quad \tfrac{1}{4}\Delta u = \langle dA, dA\rangle + \tfrac{1}{2}u(R - a_{\beta k}a_{\gamma k}\bar{R}_{\beta\gamma}) - \tfrac{1}{2}(C_{lij}C_{ilj} - AB_{i\alpha}a_{\alpha jji}),$$

where R is the scalar curvature of M, $\bar{R}_{\beta\gamma}$ is the Ricci tensor in N, and $C_{lij} = B_{l\alpha}a_{\alpha ij}$. If f is harmonic, $a_{\alpha jji} = 0$ and therefore

$$(4) \quad \tfrac{1}{4}\Delta u = \langle dA, dA\rangle + \tfrac{1}{2}u(R - a_{\beta k}a_{\gamma k}\bar{R}_{\beta\gamma}) - \tfrac{1}{2}C,$$

where $C = C_{lij}C_{ilj}$.

4. The exhaustion method. Let $f : M \to N$ be a harmonic mapping of n-dimensional Riemannian manifolds. We suppose that M is complete and simply connected and has nonnegative sectional curvature.

Let ρ be an arbitrary positive constant. Consider the open submanifold

$$M^\rho = \{p \in M | \tau(p) < \rho\}.$$

As $\exp_{p_0}$ is a diffeomorphism by Hadamand-Cartan theorem, $\bar{M}^\rho$ (the closure of M^ρ) is compact. Define on M^ρ the conformal metric

$$(5) \quad ds^\rho = e^{v_\rho} ds$$

where

$$v_\rho = \log\left(\frac{\rho^2}{\rho^2 - \tau^2}\right).$$

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It is easy to check that the function v_ρ is C^∞ and strictly convex on M^ρ, and has the properties

(a) $v_\rho \geqq 0$ on M^ρ,
(b) $v_\rho \to \infty$ on ∂M^ρ, the boundary of M^ρ,
(c) for a fixed $p \in M$, $v_\rho(p) \to 0$ for $\rho \to \infty$.

Let f^ρ be the restriction of $f: M \to N$ to M^ρ. f^ρ is not harmonic with respect to the metric (5). Nevertheless, we consider the square of the ratio of volume elements

$$u^\rho = e^{-2nv_\rho}u$$

where the superscript ρ denotes quantities which are related to f^ρ in the same way as the quantities without the superscript are related to f. The function u^ρ is nonnegative and continuous on $\bar{M}^\rho$ and vanishes on ∂M^ρ, thus attaining a maximum on M^ρ.

Consider the scalar curvature on M^ρ (cf. [**4**, p. 115]):

$$\begin{aligned} R^\rho &= e^{-2v_\rho}\{R - (n-1)(2\Delta v_\rho + (n-2)\langle dv_\rho, dv_\rho\rangle)\} \\ &= \left(\frac{\rho^2 - \tau^2}{\rho^2}\right)^2 R - \frac{4(n-1)}{\rho^4}\{(\rho^2 - \tau^2)\tau\Delta\tau + \rho^2 + (n-1)\tau^2\} \end{aligned}$$

where we have used the identities

$$\begin{aligned} \Delta v_\rho &= v_\rho'\Delta\tau + v_\rho''\langle d\tau, d\tau\rangle = v_\rho'\Delta\tau + v_\rho'' \\ \langle dv_\rho, dv_\rho\rangle &= (v_\rho')^2\langle d\tau, d\tau\rangle = (v_\rho')^2. \end{aligned}$$

Suppose that $R \geqq -S$ where S, is a nonnegative constant. If M has nonpositive curvature, its Ricci curvature is $\geqq -S$ also, and from Theorem 1

$$(\rho^2 - \tau^2)\tau\Delta\tau \leqq (n-1)\rho^2\kappa\tau \coth \kappa\tau \leqq (n-1)\rho^2\kappa\rho \coth \kappa\rho,$$

where $S = (n-1)\kappa^2$. Also

$$\rho^2 + (n-1)\tau^2 \leqq n\rho^2$$

and so

$$R^\rho = \left(\frac{\rho^2 - \tau^2}{\rho^2}\right) R - \epsilon_\rho \tag{6}$$

where ϵ_ρ is a function on M^ρ satisfying

$$0 < \epsilon_\rho \leqq 4(n-1)\{(n-1)\kappa\rho^{-1} \coth \kappa\rho + n\rho^{-2}\}. \tag{7}$$

Now u^ρ satisfies the following identity, which is identical to (3):

$$\tfrac{1}{4}\Delta u^\rho = \langle dA^\rho, dA^\rho\rangle + \frac{u^\rho}{2}(R^\rho - a_{\beta k}{}^\rho a_{\gamma k}{}^\rho \bar{R}_{\beta\gamma}) - \tfrac{1}{2}C^\rho, \tag{8}$$

where

$$C^\rho = C_{lij}{}^\rho C_{ilj}{}^\rho - A^\rho B_{i\alpha}{}^\rho a_{\alpha jji}{}^\rho, \tag{9}$$

and

$$C_{lij}{}^{\rho} = B_{l\alpha}{}^{\rho} a_{\alpha ij}{}^{\rho}.$$

The second term in (9) appears due to the fact that f^{ρ} is not harmonic.

The following lemma has been essentially furnished to me by S. I. Goldberg (cf. [**5**])

LEMMA. *At the maximum point of u^{ρ} on M^{ρ},*

$$C^{\rho} = e^{-2(n+1)v_{\rho}}\{C - (n-2)u(\Delta v_{\rho} + (n-1)\langle dv_{\rho}, dv_{\rho}\rangle)\}.$$

Proof. Let $(ds^{\rho})^2 = \sum_i (\omega_i{}^{\rho})^2$, where $\omega_i{}^{\rho} = e^{v_{\rho}}\omega_i$ are fundamental forms on (M^{ρ}, ds^{ρ}). The appropriate connection forms are

$$\omega_{ij}{}^{\rho} = \omega_{ij} + v_{\rho i}\omega_j - v_{\rho j}\omega_i.$$

Furthermore,

$$a_{\alpha i}{}^{\rho} = e^{-v_{\rho}}a_{\alpha i}, \quad B_{i\alpha}{}^{\rho} = e^{-(n-1)v_{\rho}}B_{i\alpha}, \quad \text{and} \quad A^{\rho} = e^{-nv_{\rho}}A.$$

We then have from (1) and (2),

$$e^{2v_{\rho}}a_{\alpha ij}{}^{\rho} = a_{\alpha ij} + \delta_{ij}a_{\alpha k}v_{\rho k} - a_{\alpha i}v_{\rho j} - a_{\alpha j}v_{\rho i},$$

$$e^{3v_{\rho}}a_{\alpha jji}{}^{\rho} = a_{\alpha jji} - 2a_{\alpha jj}v_{\rho i} + (n-2)a_{\alpha ij}v_{\rho j} + (n-2)a_{\alpha j}v_{\rho ji} - 2(n-2)a_{\alpha j}v_{\rho j}v_{\rho i}.$$

Therefore,

$$(10) \quad C_{lij}{}^{\rho}C_{ilj}{}^{\rho} = e^{-(2n+2)v_{\rho}}(C - (n-2)u\langle dv_{\rho}, dv_{\rho}\rangle),$$

$$(11) \quad A^{\rho}B_{i\alpha}{}^{\rho}a_{\alpha jji}{}^{\rho} = (n-2)e^{-(2n+2)v_{\rho}}(AB_{i\alpha}a_{\alpha ij}v_{\rho j} + u\Delta v_{\rho} - 2u\langle dv_{\rho}, dv_{\rho}\rangle).$$

On the other hand, the components $u_j{}^{\rho}$ of du^{ρ},

$$u_j{}^{\rho} = 2A^{\rho}B_{i\alpha}{}^{\rho}a_{\alpha ij}{}^{\rho} = 2e^{-(2n-1)v_{\rho}}(AB_{i\alpha}a_{\alpha ij} - nuv_{\rho j}),$$

vanish at the maximum point of u^{ρ}. Substituting in (11) gives

$$(12) \quad A^{\rho}B_{i\alpha}{}^{\rho}a_{\alpha jji}{}^{\rho}|_{\max u^{\rho}} = (n-2)e^{-(2n+2)v_{\rho}}u(\Delta v_{\rho} + (n-2)\langle dv_{\rho}, dv_{\rho}\rangle).$$

Substituting (10) and (12) in (9), the lemma follows.

All the preparations completed, we arrive at our goal.

THEOREM 2. *Let $f: M \to N$ be a harmonic mapping of n-dimensional Riemannian manifolds, with $C \leqq 0$. Suppose that M is complete with nonpositive sectional curvature. If the scalar curvature of M is not less than $-S$, and the Ricci curvature of N is not greater than $-\bar{S}/n$, where $S \geqq 0$ and $\bar{S} > 0$ are constants, then*

$$u \leqq (S/\bar{S})^n,$$

i.e., f is volume decreasing up to a constant.

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Proof. We assume that M is simply connected, for otherwise, we may take its simply connected covering, and use the fact that the covering map is an isometric immersion. Define M^ρ as above, and choose an arbitrary $\epsilon > 0$. From (6) and (7), there exists a constant ρ_0, $0 < \rho_0 < \infty$, such that on M^ρ, for any $\rho > \rho_0$

(13) $R^\rho \geqq -S - \epsilon$.

From the lemma, the convexity of v_ρ implies that $C^\rho \leqq 0$ at the maximum point of u^ρ. At the same point, we also have $\Delta u^\rho \leqq 0$ and $du^\rho = 0$, and (8) reduces to the inequality

(14) $R^\rho \leqq a_{\beta k}{}^\rho a_{\gamma k}{}^\rho \bar{R}_{\beta\gamma}$.

Furthermore,

(15) $a_{\beta k}{}^\rho a_{\gamma k}{}^\rho \bar{R}_{\beta\gamma} \leqq -(\bar{S}/n) \sum_{\beta,k} (a_{\beta k}{}^\rho)^2 \leqq -\bar{S}(u^\rho)^{1/n}$.

Hence, from (13), (14) and (15),

$$u^\rho \leqq ((S + \epsilon)/\bar{S})^n$$

at the maximum point of u^ρ, so obviously everywhere in M^ρ.

But, $u = e^{2nv_\rho}u^\rho$. Let p be a fixed point of M, then for ρ large enough (greater than ρ_0 and $\tau(p)$, where the latter condition ensures $p \in M^\rho$)

$$u \leqq e^{2nv_\rho}((S + \epsilon)/\bar{S})^n.$$

Letting $\rho \to \infty$, $v_\rho \to 0$, so

$$u \leqq ((S + \epsilon)/\bar{S})^n.$$

As the last inequality is valid for an arbitrary ϵ, the theorem follows.

Note. Since almost complex maps of almost Kaehler manifolds are harmonic, it might be interesting to interpret the above result in this case. Furthermore, it can be shown that Theorem 2 holds for holomorphic maps of hermitian manifolds (not necessarily Kaehlerian) with the same curvature conditions. (In this case, the scalar C does not appear.) Details will appear elsewhere.

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