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GOLDIE DIMENSIONS OF QUOTIENT MODULES

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Abstract

For an infinite cardinal $\aleph$, an associative ring R is quotient $\aleph^<$-dimensional if the generalized Goldie dimension of all right quotient modules of R_R are strictly less than $\aleph$. This latter quotient property of R_R is characterized in terms of certain essential submodules of cyclic modules being generated by less than $\aleph$ elements, and also in terms of weak injectivity and tightness properties of certain subdirect products of injective modules. The above is the higher cardinal analogue of the known theory in the finite $\aleph = \aleph_0$ case.

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Introduction

A ring R is quotient finite dimensional (q.f.d.) if for all right ideals $0 \leq K < R$, R/K has finite Goldie dimension. In [1, page 38, Theorem], Al-Huzali, Jain, and Lopéz-Permouth showed that the ring R as a right R-module is q.f.d. if and only if any direct sum of injective modules is weakly injective (w.i.). Let $\aleph \geq \aleph_0$ be an infinite regular cardinal. A ring R viewed as a right R-module is *quotient* $\aleph^<$*-dimensional* if for any $0 \leq K < R$, and any $\oplus\{V_i \mid i \in I\} \leq R/K$, $|\{i \in I \mid V_i \neq 0\}| < \aleph$ (abbreviation: q.$\aleph^<$-d.). The absolute value of a set denotes its cardinality. Thus q.$\aleph_0^<$-d. is the same as q.f.d. A ring R_R viewed as a right R-module satisfies the $\aleph^<$-A.C.C., or is $\aleph^<$*-Noetherian* if any properly ascending well ordered chain of right ideals has strictly less than $\aleph$ terms.

In Dauns [4, page 187, Theorem 4.1] it was shown that R_R is $\aleph^<$-Noetherian if and only if every $\aleph^<$-product of any injective modules remains injective. The concept 'q.f.d.' has the same kind of characterization in terms of direct sums of injectives as 'Noetherian'. However, no one has been able to prove the natural obvious generalization of this that R_R is q.$\aleph^<$-d. if and only if every $\aleph^<$-product of injectives is weakly injective. This note conjectures that the latter is not true. It gives an alternate characterization for a ring R to be q.$\aleph^<$-d. in Theorem 2.5 where 'weakly injective' is replaced by the slightly more stringent property of weakly very injective (Definition 2.1). This note does not generalize any previous work, it is new even in the finite $\aleph = \aleph_0$ q.f.d. case. Furthermore, every ring R is q.$\aleph^<$-d. for a unique smallest cardinal $\aleph$. Also, a new characterization of the property that R is q.f.d. (and q.$\aleph^<$-d.) is given in Theorem 1.5, which relates Goldie dimensions to the number of generators of modules.

1. Finitely generated modules

In this section, some characterizations of quotient $\aleph^<$-dimensional rings are given.

NOTATION 1.1. For a right unital R-module M, for a submodule $K < M$ and $m \in M$, let $m^{-1}K = (m+K)^{\perp} = \{r \in R \mid mr \in K\} < R$. Large submodules are denoted by '$\ll$'. Thus $M \ll \widehat{M}$, where the latter denotes the injective hull of M.

For any cardinals $\aleph$ and $|X|$, $\aleph^+$ or $|X|^+$ denote successor cardinals, and $\operatorname{cof}\aleph$ is the cofinality of $\aleph$.

DEFINITION 1.2. For any module M, its *Goldie dimension* $\operatorname{Gd} M$ is the cardinal $\operatorname{Gd} M = \sup\{|I| \mid \bigoplus_{i\in I} V_i \le M, \text{all } V_i \neq 0\}$. A *nontrivial direct sum* $\bigoplus_I V_i$ will be one with all $V_i \neq 0$ (see [6]). Its *Goldie plus dimension* $\operatorname{Gd}^+ M$ is the unique smallest infinite cardinal $\aleph \ge \aleph_0$ such that for any nontrivial $\oplus\{V_i \mid i \in I\} \le M$, it follows that $|I| < \aleph$. If $\operatorname{Gd} M$ is not inaccessible, then $\operatorname{Gd}^+ M = (\operatorname{Gd} M)^+$. Always $\operatorname{Gd}^+ M \le (\operatorname{Gd} M)^+$ (see [7] and [5, page 2880, 1.2]). We say R is q.$\aleph^<$-d. if R_R is.

For any module M, and any $X \subseteq M$, $\langle X\rangle \le M$ denotes the submodule generated by X; $\operatorname{gen} M$ is the minimum cardinality of a generating set of M; $\operatorname{gen} M$ is called the *generating dimension* of M.

For later use, the next lemma combines [5, page 2882, Lemma 2.2] with [5, page 2884, Lemma 2.7 and Theorem 2.8].

LEMMA 1.3. *For any cardinal* $\aleph \ge \aleph_0$, *the following are all equivalent*:

(i) *R is q.$\aleph^<$-d. (that is, for any cyclic M,* $\operatorname{Gd}^+ M \le \aleph$).
(ii) *For any finitely generated M,* $\operatorname{Gd}^+ M \le \aleph$.
(iii) *For any M, if* $\operatorname{gen} M < \operatorname{cof}\aleph$, *then* $\operatorname{Gd}^+ M \le \aleph$.
(iv) *There exists* $\aleph_0 \le \lambda \le \operatorname{cof}\aleph$ *such that for any M, if* $\operatorname{gen} M < \lambda$, *then* $\operatorname{Gd}^+ M \le \aleph$.

The following quotient module operation on pairs of modules will be used later twice.

CONSTRUCTION 1.4. *Suppose that we are given* $0 \leq J < R$ *and any direct sum of cyclics* $\bigoplus\{(y_\gamma + J)R \mid \gamma \in \Gamma\} \leq R/J$ *with* $y_\gamma \in R$, $\gamma \in \Gamma$. *Then there exists* $J \subseteq I < R$ *such that*

(i) $\Sigma\{(y_\gamma + I)R \mid \gamma \in \Gamma\} = \bigoplus\{(y_\gamma + I)R \mid \gamma \in \Gamma\} \leq R/I$;

(ii) $S_\gamma = (y_\gamma R + I)/I$ *is simple for all* $\gamma \in \Gamma$; *and*

(iii) $\bigoplus\{S_\gamma \mid \gamma \in \Gamma\} \ll R/I$.

PROOF. We will first produce an I satisfying only (i) and (ii), but call it K. (i) Take any maximal submodule $K_\gamma < y_\gamma R + J$ with $J \subseteq K_\gamma$. Then $(y_\gamma R + J)/K_\gamma = (y_\gamma R + K_\gamma)/K_\gamma$ is simple. Hence the right ideals $I_\gamma = (y_\gamma + K_\gamma)^\perp < R$ are maximal. Observe that $I_\gamma < R$ is the unique maximal right ideal such that $K_\gamma = y_\gamma I_\gamma + J = y_\gamma I_\gamma + K_\gamma$. Define $K = \sum\{y_\gamma I_\gamma + J \mid \gamma \in \Gamma\}$. Tedious elementwise arguments (using direct sums in R/J) prove that in R/K, the sum $\sum\{(y_\gamma R + K)/K \mid \gamma \in \Gamma\} = \bigoplus\{(y_\gamma R + K)/K \mid \gamma \in \Gamma\} \leq R/K$ remains direct, and also that $y_\gamma \notin K$ for all $\gamma \in \Gamma$.

Note that $\sum_{\gamma\in\Gamma}(y_\gamma R + J) = \sum_{\gamma\in\Gamma}(y_\gamma R + K)$. But then the natural isomorphism $(R/J)/(K/J) \longrightarrow R/K$ maps the quotient of the above two direct sums as follows

$$\frac{\bigoplus_{\gamma\in\Gamma}(y_\gamma R + J)/J}{\bigoplus_{\gamma\in\Gamma}(y_\gamma I_\gamma + J)/J} \cong \frac{\sum_{\gamma\in\Gamma}(y_\gamma R + K)}{K} = \bigoplus_{\gamma\in\Gamma}[(y_\gamma R + K)/K]$$

Since $(y_\gamma + K)I_\gamma \subseteq K_\gamma + K \subset K$ with $y_\gamma \notin K$, it follows that $(y_\gamma + K)^\perp = I_\gamma$. (ii) Hence $(y_\gamma R + K)/K$ is simple.

(iii) Let $I/K \leq R/K$ be a complement submodule such that $[\bigoplus_{\gamma\in\Gamma}(y_\gamma + K)R] \oplus I/K \ll R/K$. Set $S_\gamma = (y_\gamma R + I)/I$. Then $S_\gamma \cong (y_\gamma R + K)/K$ and $\bigoplus\{S_\gamma \mid \gamma \in \Gamma\} \ll R/I$. $\square$

The next theorem characterizes 'q.$\aleph^<$-d.' in terms of 'gen' alone. It is new even in the finite $\aleph = \aleph_0$ ordinary q.f.d. case. In specific concrete examples, the most natural easiest choice of λ below to use is $\lambda = 2$, in which case we only need to consider cyclics N in (ii).

THEOREM 1.5. *For any* $\aleph \geq \aleph_0$ *and ring* R, *let* λ *be any fixed cardinal either* $\lambda = 2$, *or* $\aleph_0 \leq \lambda \leq \operatorname{cof} \aleph$. *Then the following are equivalent*:

(i) R *is* q.$\aleph^<$-d.

(ii) *For any* N *(or any* $N \leq R$*) with* gen $N < \lambda$, *for every* $L \ll N$, *it follows that there exists* $D \ll L$ *with* gen $D < \aleph$.

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PROOF. (i) implies (ii). Use Lemma 1.3 (i) if $\lambda = 2$, or Lemma 1.3 (iv) otherwise to conclude that for any essential direct sum of cyclics $D = \bigoplus\{x_iR \mid i < \tau\} \ll L$ with τ a cardinal, necessarily $\tau < \aleph$. But then gen $D \leq \tau < \aleph$.

(ii) implies (i). For any $0 \leq J < R$, and any direct sum $\bigoplus\{(y_\gamma + J)R \mid \gamma \in \Gamma\} \leq R/J$, $y_\gamma \in R$, it suffices to show that $|\Gamma| < \aleph$. As in Construction 1.4, there exists an $J \subseteq I < R$ with $L = \bigoplus\{S_\gamma \mid \gamma \in \Gamma\} \ll R/I$. By (ii), with $\lambda = 2$, or $\aleph_0 \leq \lambda \leq \operatorname{cof} \aleph$, there is a $D \ll L \ll R/I$ with gen $D < \aleph$. Since all $D \cap S_\gamma \neq 0$, $D = \bigoplus\{S_\gamma \mid \gamma \in \Gamma\} \ll L$. Since $\aleph \geq \aleph_0$, if $|\Gamma| < \aleph_0$, then gen $D \leq |\Gamma| < \aleph$. If $\aleph_0 \leq |\Gamma|$, then D cannot be finitely generated, and hence [5, page 2881, Lemma 2.1] shows that $|\Gamma| = \text{gen } D < \aleph$. □

2. Weak injectivity

Special kinds of weakly injective and tight modules will be used here.

DEFINITION 2.1. For a module M, select and fix a copy of M in $M \ll \widehat{M}$, and below M always refers to this copy. Then M is *weakly very injective* (w.v.i.) if for any finitely generated submodule $N \leq \widehat{M}$, there exists a triple X, D, ψ, where $D \ll M \cap N \ll N \subseteq X \leq \widehat{M}$, and where $X \cong M$ under an isomorphism $\psi : X \longrightarrow M \ll \widehat{M}$ with $\psi | D = 1_D$:

$$\begin{array}{ccccccc} D & \ll & M \cap N & \ll & N \subseteq X & \leq & \widehat{M} \\ \| 1_D & & & & \cong \downarrow \psi & & \\ D & \leq & & & M & \ll & \widehat{M}. \end{array}$$

If in the above definition all reference to D is omitted, the result is the usual definition of M is *weakly injective* (w.i.).

Imprecisely and incompletely speaking, if in the above definition X is omitted, the result is the next definition.

DEFINITION 2.2. Again fix $M \ll \widehat{M}$. The module M is *very tight* (v.t.) if for any finitely generated $N \leq \widehat{M}$, there exist D, f with $D \ll M \cap N \ll N$ and a monic map $f : N \to M \ll \widehat{M}$ with $f | D = 1_D$:

$$\begin{array}{ccccccc} D & \ll & M \cap N & \ll & N & & \\ \| 1_D & & & & \downarrow f & & \\ D & \leq & & & M & \ll & \widehat{M}. \end{array}$$

And again, deletion of all reference to D defines the usual concept of M is *tight*. Thus we have the following implications:

$$\text{injective} \Longrightarrow \text{w.v.i.} \Longrightarrow \text{v.t.} \quad \text{and} \quad \text{w.v.i.} \Longrightarrow \text{w.i.} \Longrightarrow \text{tight}.$$

EXAMPLE 2.3. For $R = \mathbb{Z}$, $M = \mathbb{Z}$ is w.i. and tight, but not w.v.i., and hence not v.t.

In [1, page 38, Theorem] it was proved that (1) is equivalent to (2) in the next theorem with $\aleph = \aleph_0$ and with 'w.v.i.' replaced by 'v.i.'. Later, for $\aleph = \aleph_0$ and with 'v.t.' replaced by 'tight', Jain and Lopéz showed that (1), (2), (3), and (4) are all equivalent ([9, page 9, Theorem 2.6]).

DEFINITION 2.4. For a cardinal $\aleph \geq \aleph_0$, and a family $F_\gamma, \gamma \in \Gamma$ of modules, their $\aleph^<$-product is $\prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma = \{x = (x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} F_\gamma : |\operatorname{supp} x| < \aleph\}$, where $\operatorname{supp} x = \{\gamma \in \Gamma \mid x_\gamma \neq 0\}$. For any subset X of the $\aleph$-product, $\operatorname{Supp} X = \bigcup\{\operatorname{supp} x \mid x \in X\}$.

THEOREM 2.5 (Main Theorem). *Let $\aleph \geq \aleph_0$ be regular. Then for any ring R the following are all equivalent*:

(1) *R_R is q.$\aleph^<$-d.*
(2) *For any Γ and injective modules $F_\gamma, \gamma \in \Gamma$, $\prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma$ is weakly very injective.*
(3) *For any Γ and injective modules $F_\gamma, \gamma \in \Gamma$, $\prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma$ is very tight.*
(4) *For any Γ and indecomposable injectives $F_\gamma, \gamma \in \Gamma$, $\prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma$ is very tight.*
(5) *For any Γ and simple $S_\gamma, \gamma \in \Gamma$, $\prod_{\gamma\in\Gamma}^{<\aleph} \widehat{S}_\gamma$ is very tight.*

PROOF. (1) implies (2). Set $M = \prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma$, and let $N \leq \widehat{M}$ be finitely generated. First, take an essential direct sum of cyclics in $D = \bigoplus\{x_i R \mid i < \tau\} \ll M \cap N \ll N$, where τ is a cardinal. In view of (1), Lemma 1.3 (ii) now guarantees that $D = \langle\{x_i \mid i < \tau\}\rangle$ with $\operatorname{gen} D \leq \tau < \aleph$. Alternatively, Theorem 1.5 (ii) could be used here to guarantee the existence of such D. Set $\Omega = \bigcup\{\operatorname{supp} x_i \mid i < \tau\}$. Since $\aleph$ is regular, $|\Omega| < \aleph$. Then $D \leq \prod_{\gamma\in\Omega} F_\gamma \leq M$, where the product is a direct summand of M. Now let $\widehat{D}$ be any (not necessarily unique) injective hull of D inside the product, that is, $D \leq \widehat{D} \leq \prod_{\gamma\in\Omega} F_\gamma \leq M$. But since $D \ll N$, $\widehat{D} \cong \widehat{N}$ is isomorphic to any injective hull $\widehat{N}$ of N. Next, let $\widehat{N}$ be defined as that injective hull of N with $N \ll \widehat{N} \leq \widehat{M}$. In general, $\widehat{D}$ need not contain N, but note that $D \ll \widehat{N}$.

Since $\widehat{D} \leq M$, write $M = \widehat{D} \oplus K$ for some $K \leq M$. It is asserted that $K \cap \widehat{N} = 0$. If not, then $K \cap \widehat{N} \neq 0$, and since $D \ll \widehat{N}$, also $D \cap K \cap \widehat{N} \neq 0$. But then $D \cap K \subseteq \widehat{D} \cap K = 0$ is a contradiction. Note that $\widehat{D} \oplus K = M \leq \widehat{M}$ is the fixed copy of M inside $\widehat{M}$. Since $D \ll \widehat{N}$ as well as $D \ll \widehat{D}$, the identity map of D extends to an isomorphism $\varphi : \widehat{N} \longrightarrow \widehat{D}$, and hence to a further isomorphism $\psi = \varphi \oplus 1_K : X = \widehat{N} \oplus K \longrightarrow \widehat{D} \oplus K = M \ll \widehat{M}$. Furthermore, $\psi|D = \varphi|D = 1_D$ is the identity on D as required. (Note that since $D \ll \widehat{N}$, $D \oplus K \ll M$ as well as $D \oplus K \ll X$. Hence $X \ll \widehat{M}$.)

(2) implies (3) implies (4) implies (5). These are all trivial.

(5) implies (1). In the last three implications as well as the present one, the fact that $\aleph$ is regular is not used. It suffices to show that for any $0 \leq J < R$ and any

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nontrivial direct sum $\bigoplus\{(y_\gamma R + J)/J \mid \gamma \in \Gamma\} \leq R/J$, we have $|\Gamma| < \aleph$. Let $0 \leq J \subseteq I < R$ and $\bigoplus\{S_\gamma \mid \gamma \in \Gamma\} \leq R/I$ with $S_\gamma = (y_\gamma R + I)/I$ simple, $\gamma \in \Gamma$, satisfy Construction 1.4 (i) and (ii).

Let j be the natural inclusion map $j : \bigoplus_{\gamma\in\Gamma} S_\gamma \longrightarrow \prod^{<\aleph}_{\gamma\in\Gamma} \widehat{S}_\gamma \equiv M \ll \widehat{M}$. Let $i : \bigoplus_{\gamma\in\Gamma} S_\gamma \longrightarrow R/I$ be the natural inclusion. There exists a map $\varphi : R/I \longrightarrow \widehat{M}$ with $\varphi i = j$. In particular, $\varphi i \bigoplus_{\gamma\in\Gamma} S_\gamma = j \bigoplus_{\gamma\in\Gamma} S_\gamma = \bigoplus_{\gamma\in\Gamma} S_\gamma < \prod^{<\aleph}_{\gamma\in\Gamma} \widehat{S}_\gamma$ is the natural inclusion of the sum into the $\aleph$-product. The module $N = \varphi(R/I) \leq \widehat{M}$ is cyclic, and hence by (5), there exists a $D \ll M \cap N \ll N = \varphi(R/I) \leq \widehat{M}$, and a monomorphism $\psi : N \longrightarrow M \ll \widehat{M}$ such that $\psi|D = 1_D$ is the identity on D.

Since $D \ll N$, and since $0 \neq \varphi i S_\gamma = j S_\gamma \subseteq N$, necessarily also $0 \neq D \cap j S_\gamma$. Since $0 \neq D \cap j S_\gamma \subseteq j S_\gamma$ and $j S_\gamma$ is simple, $j S_\gamma = D \cap j S_\gamma \subseteq D$. But then $j(\bigoplus_{\gamma\in\Gamma} S_\gamma) \subseteq D \subseteq M$.

At this point, the elements $y_\gamma \in R$ with $S_\gamma = (y_\gamma R + I)/I = [(1+I)y_\gamma]R$ will be needed. Next, define $m = \psi\varphi(1+I) \in M$. Since $\psi|D = 1_D$, and $j S_\gamma \subseteq D$, $j S_\gamma = 1_D j S_\gamma = \psi j S_\gamma = \psi\varphi i S_\gamma = \psi\varphi i\{(1+I)y_\gamma]R\} = \{[\psi\varphi i(1+I)]y_\gamma\}R = (m y_\gamma)R$. Now $m, m y_\gamma \in M$, and $\operatorname{Supp} j S_\gamma = \{\gamma\}$. Since $j S_\gamma = m y_\gamma R$, also $\operatorname{supp} m y_\gamma = \{\gamma\}$. Thus for any $\gamma \in \Gamma$, $\{\gamma\} = \operatorname{supp} m y_\gamma \subseteq \operatorname{supp} m$. Hence $\Gamma \subseteq \operatorname{supp} m$. Finally, $m \in M$ with $|\Gamma| \leq |\operatorname{supp} m| < \aleph$ shows that R is q.$\aleph^<$-d. □

COROLLARY 2.6. *Theorem* 2.5 *remains valid if in the quantifications* (2)–(5), *only index sets* Γ *with* $|\Gamma| \leq R$ *are admitted.*

PROOF. In (5) implies (1), $\bigoplus\{S_\gamma \mid \gamma \in \Gamma\} \leq R/I$ implies that $|\Gamma| \leq |R/I| \leq |R|$. □

Not only the previous corollary, but the previous theorem as well is even new in the ordinary $\aleph = \aleph_0$ finite q.f.d. case. Note that in the Corollary 2.7 below, the quantifier 'for every' may not be removed.

COROLLARY 2.7. *Let* $F_\gamma, \gamma \in \Gamma$ *be any modules. Then*

(i) *For every* Γ, $F_\gamma, \gamma \in \Gamma$, $\bigoplus_{\gamma\in\Gamma} \widehat{F}_\gamma$ *is w.i. if and only if for every* Γ, $F_\gamma, \gamma \in \Gamma$, $\bigoplus_{\gamma\in\Gamma} \widehat{F}_\gamma$ *is w.v.i.*

(ii) *For every* Γ, F_γ, $\gamma \in \Gamma$, $\bigoplus_{\gamma\in\Gamma} \widehat{F}_\gamma$ *is tight if and only if for every* Γ, F_γ, $\gamma \in \Gamma$, $\bigoplus_{\gamma\in\Gamma} \widehat{F}_\gamma$ *is v.t.*

3. Conjectures and examples

The only way to validate the three conjectures in Conjecture 3.1 below would be to construct appropriate examples, or counterexamples.

CONJECTURE 3.1. For every (regular) $\aleph > \aleph_0$, the following hold:

(1) R is q.$\aleph^<$-d. does not imply that for any w.v.i. F_γ, $\gamma \in \Gamma$, $\prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma$ is w.v.i.

(2) If R has the property that for any injective modules F_γ, $\gamma \in \Gamma$, $\prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma$ is w.i., then that does not imply that R is q.$\aleph^<$-d.

(3) Definition 2.1 and Definition 2.2 cannot be simplified by taking $D \ll M \cap N$ always to be $D = M \cap N$. That is, the latter would define different concepts.

QUESTION 3.2. (4) For every (regular) $\aleph \geq \aleph_1$, does there exist a q.$\aleph^<$-d. ring R, and a family of weakly injective modules F_γ, $\gamma \in \Gamma$ such that $\prod_{\gamma\in\Gamma}^{<\aleph} F_\gamma$ is not weakly injective?

(5) Is there a very tight module that is not w.v.i.? There is an example in [11, page 352, Example 3.3] and [11, page 219, Example 2.11] of a tight not weakly injective, and not very tight module.

In the following examples $\mathbb{Z}$ and $\mathbb{Q}$ are the integers and rationals.

Any R is q.$|R|^{+<}$-d., and in the next example, R is not q.$|R|^<$-d. Let $\aleph(R)$ denote the unique smallest cardinal such that R is q.$\aleph(R)^<$-d.

EXAMPLE 3.3. Consider the following ring R, right ideal L, and quotient module R/L:

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} < R, \quad R/L \cong \begin{pmatrix} \mathbb{Z} & \oplus_p \mathbb{Z}(p^\infty) \\ 0 & 0 \end{pmatrix}.$$

Since $\operatorname{Gd} R/L = \aleph_0$ and $|R| = \aleph_0$, R is q.$\aleph_1^<$-d. Note that $\operatorname{Gd} R = 2$ but $\aleph(R) = \aleph_1$.

EXAMPLE 3.4. For V a $\mathbb{Q}$-vector space, let

$$R = \begin{pmatrix} \mathbb{Z} & V \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Any right ideal of R is of one of the following forms:

$$\begin{pmatrix} \mathbb{Z}n & V \\ 0 & \mathbb{Q} \end{pmatrix}, \begin{pmatrix} \mathbb{Z}n & V \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & W \\ 0 & \mathbb{Q} \end{pmatrix}, \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}, \quad 0 \neq n = 1, 2, \dots;$$

$W \subseteq V$, W a $\mathbb{Q}$-subspace. If $\dim_{\mathbb{Q}} V \geq 2$, $\aleph(R) = \dim_{\mathbb{Q}} V + 2$. If $V = \mathbb{Q}$ or (0), $\aleph(R) = 3$.

An example from [8, page 71] is used to show how the criterion in Theorem 1.5 may be used, and that it cannot be simplified by combining $D = L \ll N$ in Theorem 1.5.

EXAMPLE 3.5. For a fixed prime p, let R be the following commutative ring

$$R = \left\{ \begin{pmatrix} a & \xi \\ 0 & a \end{pmatrix} \,\middle|\, a \in \mathbb{Z},\ \xi \in \mathbb{Z}(p^\infty) \right\}, \quad L = \begin{pmatrix} 0 & \mathbb{Z}(p^\infty) \\ 0 & 0 \end{pmatrix} < R.$$

Take $\lambda = 2$ in Theorem 1.5, $N = R$, so that $L \ll N = R$. Then $genL = \aleph_0$. For $0 \neq c_1 \in \mathbb{Z}(p^\infty)$ with $pc_1 = 0$, the right ideal $D < R$ below

$$D = \begin{pmatrix} 0 & \mathbb{Z}c_1 \\ 0 & 0 \end{pmatrix} \ll L$$

is essential in L and cyclic.

Here R is uniform, and every ideal of R is of one of the following two forms I_1 or I_2, where $0 \leq V \subset_{\neq} \mathbb{Z}(p^\infty)$ is any subgroup

$$\begin{pmatrix} 0 & \mathbb{Z}(p^\infty) \\ 0 & 0 \end{pmatrix} \subseteq I_1 = \left\{ \begin{pmatrix} na & \xi \\ 0 & na \end{pmatrix} \,\middle|\, a \in \mathbb{Z},\ \xi \in \mathbb{Z}(p^\infty) \right\},$$

$$I_2 = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}, \quad n = 0, 1, 2, \ldots .$$

Then $R/I_1 \cong \mathbb{Z}/\mathbb{Z}n$; and $R/I_2 \cong \mathbb{Z} \times \mathbb{Z}(p^\infty)/V \cong \mathbb{Z} \times \mathbb{Z}(p^\infty)$, where $(b, q + V) \cdot r = (ba, b\xi + qa + V)$, $b \in \mathbb{Z}$, $q \in \mathbb{Z}(p^\infty)$. In Theorem 1.5 (ii), N is of one of the above types R/I_1', or R/I_2'. Then L is of the form I_1/I_1', I_2/I_2', I_1/I_2', or R/I_2' where $I_2' = V'e_{12}$ for a subgroup $0 \leq V' \subset V \subset \mathbb{Z}(p^\infty)$, where e_{12} is a matrix unit. Now Theorem 1.5 (ii) can be verified; $L = I_1/I_1' = \mathbb{Z}$ or is finite. In the other cases, $L = I_2/I_2'$, I_1/I_2', or R/I_2'; for any $0 \neq v \in V \backslash V' \cong Ve_{12} \backslash I_2' \subset L$, $0 \neq D = (ve_{12} + I_2')R \ll L$ with gen $D = 1$. Thus R is q.f.d. It has Krull dimension one, is not Noetherian, but is $\aleph_1^<$-Noetherian.

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