SUPERDIAGONAL FORMS FOR RELATED LINEAR OPERATORS

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The concept of superdiagonal forms for $n \times n$ matrices T with complex entries has been extended by J. R. Ringrose [**4**] to the setting of compact linear operators $T: X \to X$ acting on a complex Banach space X. In a recent paper D. Koros [**2**] generalized Ringrose's approach to the case of compact linear operators $T: X \to X$ on a complex locally convex space X. The reason why both authors confine their attention to the class of compact linear operators is that the existence of proper closed invariant subspaces is, aside from Riesz–Schauder theory, the main tool in their construction. In the present paper it is shown that the existence of superdiagonal forms possesses a certain permanence property in the following sense.

Let X and Y denote two locally convex spaces, let $P: X \to Y$ and $Q: Y \to X$ denote two continuous linear operators. Then according to A. Pietsch [**3**] the operators $T := QP: X \to X$ and $S := PQ: Y \to Y$ are said to be related. Roughly speaking we shall prove that T has a superdiagonal form in the sense of Ringrose if and only if S has a superdiagonal form. Since every compact linear operator $T: X \to X$ on a complex locally convex space X is related to a compact linear operator $S: Y \to Y$ on a complex Banach space Y, we especially obtain an independent approach to Koros's result without using any locally convex arguments concerning Riesz–Schauder theory and invariant subspaces.

1. Notation. Throughout this paper X and Y denote locally convex spaces over the complex numbers, $L(X, Y)$ denotes the space of all continuous linear operators from X to Y, writing $L(X)$ for $L(X, X)$. The term subspace will always mean closed linear subspace, and a proper subspace will be a subspace different from $\{0\}$ and the whole space. A subspace M of X is said to be invariant under $T: X \to X$ if $Tx \in M$ for all $x \in M$. A nest $\mathcal{F}$ of subspaces of X will be a family of subspaces which is totally ordered by inclusion. If in addition every $M \in \mathcal{F}$ is invariant under T, $\mathcal{F}$ will be said to be an invariant nest. The symbol "$\subset$" will be reserved for proper inclusion. If $S \subset X$ is a subset, let cl S denote the closure of S with respect to the topology of X.

Given a nest $\mathcal{F}$ of subspaces of X containing $\{0\}$ and X, for $M \in \mathcal{F}$ define

$$M_- = \mathrm{cl} \bigcup \{L : L \in \mathcal{F}, L \subset M\},$$
$$M_+ = \bigcap \{L : L \in \mathcal{F}, L \supset M\}.$$

Obviously M_- and M_+ are subspaces of X, and it may happen that $M_- = \{0\}$ while $M_+ = X$. M_- and M_+ are invariant under T provided $\mathcal{F}$ is an invariant nest.

DEFINITION 1.1. A nest $\mathcal{F}$ of subspaces of X is said to be *simple* if
(i) $\{0\}, X \in \mathcal{F}$;

(ii) if $\mathcal{F}_0$ is any subfamily of $\mathcal{F}$, then the subspaces $\bigcap\{L: L\in\mathcal{F}_0\}$ and cl $\bigcup\{L: L\in\mathcal{F}_0\}$ are in $\mathcal{F}$;

(iii) if $M\in\mathcal{F}$, then $\dim M/M_- \leq 1$.

Condition (iii) can be replaced by the equivalent condition

(iii)′ if $M\in\mathcal{F}$, then $\dim M_+/M\leq 1$.

In order to see this, assume $M\subset M_+$. Then $M\subseteq M_{+-}$. On the other hand, if $L\in\mathcal{F}$ is such that $L\subset M_+$, then $L\subseteq M$, for otherwise $L\supseteq M_+$. Hence $M_{+-}=M$ and (iii) implies (iii)′. Conversely assume that (iii)′ holds. Thus let $M_-\subset M$. By definition we have $M_{-+}\subseteq M$. On the other hand, if $L\in\mathcal{F}$ and $L\supset M_-$, then $L\supseteq M$, for otherwise $L\subset M$ and hence $L\subset M_-$. This gives $M_{-+}=M$.

A linear operator $T: X\to X$ is said to be compact if there exists a neighbourhood U of zero in X such that $T(U)$ is relatively compact. Letting Y denote the linear span of cl $T(U)$ provided with the Minkowski-norm

$$m_Y(y):=\inf\{c>0: c^{-1}y\in \text{cl } T(U)\},$$

it is easily seen that T factors compactly through Y. Indeed $T=QP$, where $P: X\to Y$ is given by $Px=Tx$ for $x\in X$, and Q denotes the embedding of Y into X, this map also being compact. Therefore T is related to the compact operator S defined to be PQ acting on the Banach space Y.

2. Simple invariant nests for related linear operators. Given an arbitrary operator $T\in L(X)$ and an invariant nest $\mathcal{F}$ we can always find a maximal invariant nest $\mathcal{F}_{max}$ containing $\mathcal{F}$ by an argument based on Zorn's lemma, but in general a maximal invariant nest will not be simple. Indeed by Enflo's counterexample to the invariant subspace problem [**1**] there exists a Banach space X and $T\in L(X)$ such that $\{0\}$, X is a maximal invariant nest for T. Since superdiagonal forms are given for simple nest only, we confine our attention to those operators for which a maximal invariant nest is automatically simple and related operators.

THEOREM 2.1. *Let $T\in L(X)$ and $S\in L(Y)$ denote two related linear operators. If every maximal invariant nest for T is simple, then the same is true for S.*

Proof. Let $\mathcal{F}$ be a maximal invariant nest for S, let $M\in\mathcal{F}$ and assume that $\dim M/M_->1$. Obviously $P^{-1}(M)$ and $P^{-1}(M_-)$ are invariant subspaces for T, and since $QM\subseteq P^{-1}(M)$ (M is invariant under S), the restriction of T to $P^{-1}(M)$ is related to the restriction of S to M. We shall prove that $\dim M/M_->1$ is impossible. For that purpose we may assume that $X=P^{-1}(M)$ and $Y=M$. On the other hand $\hat{T}: X/P^{-1}(Y_-)\to X/P^{-1}(Y_-)$ given by $\hat{T}([x])=[Tx]$ and $\hat{S}: Y/Y_-\to Y/Y_-$ given by $\hat{S}(\langle y\rangle)=\langle Sy\rangle$ are related, $[.]$ and $\langle.\rangle$ denoting the cosets in $X/P^{-1}(Y_-)$ and Y/Y_- respectively. Indeed $\hat{T}=\hat{Q}\hat{P}$, $\hat{S}=\hat{P}\hat{Q}$ with $\hat{P}([x])=\langle Px\rangle$ and $\hat{Q}(\langle y\rangle)=[Qy]$. Thus, if $\dim X/P^{-1}(Y_-)\leq 1$, then $\hat{S}$ has at most rank one. Consequently, we find a proper invariant subspace $\hat{N}$ of $\hat{S}$ and $N:=\{y:\langle y\rangle\in\hat{N}\}$ is an invariant subspace of S such that $Y_-\subset N\subset Y$. It is easily checked that $\{N\}\cup\mathcal{F}$ is an invariant nest containing $\mathcal{F}$ properly, contradicting the maximality of $\mathcal{F}$. Therefore let us assume that $\dim X/P^{-1}(Y_-)>1$. Since X and $P^{-1}(Y_-)$ are members of a suitable simple invariant nest $\mathcal{G}$, there exists $N\in\mathcal{G}$ such that $P^{-1}(Y_-)\subset N\subset X$. Then $Q^{-1}(N)$ is an invariant subspace for S. We shall show that N can be chosen in such a way that $Y_-\subset Q^{-1}(N)\subset Y$. This again will contradict the maximality of $\mathcal{F}$ and hence finish the proof. First of all we remark that for $L\in\mathcal{G}$, $Q^{-1}(L)=Y_-$ implies $L\subseteq T^{-1}(L)=P^{-1}(Y_-)$. Consider

$$P^{-1}(Y_-)_+=\bigcap\{K:K\in\mathcal{G}, K\supset P^{-1}(Y_-)\}.$$

If we had $Q^{-1}(K)=Y$ for all such K, then $Q^{-1}(P^{-1}(Y_-)_+)=Y$ (otherwise we are done!). We distinguish two cases.

(i) If $P^{-1}(Y_-)_+=P^{-1}(Y_-)$, then $S^{-1}(Y_-)=Q^{-1}(P^{-1}(Y_-)_+)=Y$. Thus $\hat{S}:Y/Y_-\to Y/Y_-$ is identically zero.

(ii) If $P^{-1}(Y_-)_+\neq P^{-1}(Y_-)$, then $P^{-1}(Y_-)_+=P^{-1}(Y_-)\oplus\mathbb{C}x_0$ with a suitable $x_0\in X$, because $\mathcal{G}$ was a simple nest. Therefore

$$SY=PQ(Q^{-1}(P^{-1}(Y_-)\oplus Cx_0))\subseteq Y_-+\mathbb{C}Px_0,$$

and hence $\hat{S}:Y/Y_-\to Y/Y_-$ is at most of rank one. In both cases $\hat{S}$ has a proper invariant subspace. By the argument used at the beginning of the proof we obtain a contradiction to $\mathcal{F}$ being maximal. Thus $\dim M/M_-$ is at most one and $\mathcal{G}$ is simple.

Throughout the remainder of this section, let $T=QP\in L(X)$ and $S=PQ\in L(Y)$ denote two related linear operators such that every maximal invariant nest of subspaces is simple. Let $\mathcal{F}(T)$ denote a simple invariant nest for T; then $\mathcal{F}(S)$ and $\mathcal{F}_{-1}(T)$ denote simple invariant nests for S and T containing $\{Q^{-1}(M):M\in\mathcal{F}(T)\}$ and $\{P^{-1}(K):K\in\mathcal{F}(S)\}$, respectively.

If $M\in\mathcal{F}(T)$, we have $M=M_-$ or $\dim M/M_-=1$. Let us assume $M\neq M_-$, $z_M\in M\setminus M_-$. Then $Tz_M\in M$ can be expressed uniquely in the form

$$Tz_M=\alpha_M(T)z_M+y_M,$$

where $\alpha_M(T)\in\mathbb{C}$ and $y_M\in M_-$. The scalar $\alpha_M(T)$ does not depend on the choice of z_M. In this way we can associate to each $M\in\mathcal{F}(T)$ a complex number $\alpha_M(T)$ called the *diagonal coefficient* of T at M. Let α be a scalar. We define the *diagonal multiplicity* of α to be the (possibly infinite) number of distinct subspaces $M\in\mathcal{F}(T)$ for which $\alpha_M(T)=\alpha$.

THEOREM 2.2. *Let $T\in L(X)$ and $S\in L(Y)$ denote two related linear operators. Then there is a one-to-one correspondence between the diagonal coefficient of T with respect to $\mathcal{F}(T)$ and those of S with respect to $\mathcal{F}(S)$. More precisely: given $\alpha\in\mathbb{C}\setminus\{0\}$, the diagonal multiplicity of α is the same with respect to both $\mathcal{F}(T)$ and $\mathcal{F}(S)$.*

Proof. Let $M\in\mathcal{F}(T)$ and assume that $\alpha_M(T)\neq 0$. Then $M=M_-\oplus\mathbb{C}z_M$ with a suitable $z_M\in M\setminus M_-$. Then $Tz_M\notin M_-$, and hence $Pz_M\notin Q^{-1}(M_-)$. On the other hand $M=M_-\oplus\mathbb{C}Tz_M$, and therefore

$$Q^{-1}(M)=Q^{-1}(M\oplus\mathbb{C}QPz_M)=Q^{-1}(M_-)\oplus\mathbb{C}Pz_M.$$

This implies $Q^{-1}(M)_-=Q^{-1}(M_-)$. Since $SPz_M=PTz_M=P(\alpha_M(T)z_M+y_M)=\alpha_M(T)Pz_M+Py_M(Py_M\in Q^{-1}(M_-))$, we have $\alpha_M(T)=\alpha_{Q^{-1}(M)}(S)$. If $M_1\subset M_2(M_i\in\mathcal{F}(T))$, then $\alpha_{M_1}(T)\neq 0$ implies $Q^{-1}(M_1)\subset Q^{-1}(M_2)$, for otherwise $TM_2\subseteq M_1$, which gives $\alpha_{M_2}(T)=0$, a contradiction. Therefore the diagonal multiplicity of $\alpha\neq 0$ with respect to $\mathcal{F}(S)$ exceeds that of α with respect to $\mathcal{F}(T)$.

Conversely let $K\in\mathcal{F}(S)$ and $\alpha_K(S)\neq 0$. Define

$$M_\sim=\bigcap\{M:M\in\mathcal{F}(T),\,Q^{-1}(M)\supset K\}.$$

Of course $M_\sim\in\mathcal{F}(T)$. On the other hand we prove that $L=M_\sim\cap P^{-1}(K)\in\mathcal{F}(T)$. So let $N\in\mathcal{F}(T)$. If $Q^{-1}(N)\subset K$, then $Q^{-1}(N)\subseteq Q^{-1}(M_\sim)$, and $N\subseteq M_\sim$, $T^{-1}(N)\subseteq P^{-1}(K)$. This gives $N\subseteq M_\sim\cap P^{-1}(K)$. If $Q^{-1}(N)\supseteq K$, then $Q^{-1}(N)\supseteq Q^{-1}(M_\sim)$ and $N\supseteq M_\sim$, which implies $N\supseteq M_\sim\cap P^{-1}(K)$. By the maximality of $\mathcal{F}(T)$, $L\in\mathcal{F}(T)$. Moreover $M_\sim\cap P^{-1}(K)_-$ has at most codimension one in L. Note that $P^{-1}(K)_-$ (with respect to $\mathcal{F}_{-1}(T)$ of course!) equals $P^{-1}(K_-)$ by the same argument as in the first step of this proof. If we had $L=M_\sim\cap P^{-1}(K_-)$, then $Q^{-1}(L)=S^{-1}(K_-)\cap Q^{-1}(M_\sim)$, and hence $K\cap S^{-1}(K)\cap Q^{-1}(M_\sim)=K\cap S^{-1}(K_-)\cap Q^{-1}(M_\sim)$; i.e. $K=K_-$, contradicting $\alpha_K(S)\neq 0$. Thus $L_-=P^{-1}(K_-)\cap M_\sim$. If we had $TL\subseteq L_-$, then $S^2(K)=S^2(K\cap Q^{-1}(M_\sim))\subseteq S(SK\cap S(Q^{-1}(M_\sim))\subseteq S(K\cap PQ(Q^{-1}(M_\sim))\subseteq S(K\cap PM_\sim)=S(P(P^{-1}(K)\cap PM_\sim)=PT(P^{-1}(K)\cap M_\sim)=PT(L)\subseteq PL_-=P(P^{-1}(K_-)\cap M_\sim)\subseteq K_-$; i.e. S^2 maps K into K_- contradicting $\alpha_K(S)\neq 0$. Thus $\alpha_L(T)\neq 0$. Let $y_0\in K\diagdown K_-$. Then $Qy_0\in P^{-1}(K)\diagdown P^{-1}(K_-)$ as in the first step and $L=L_-\oplus\mathbb{C}Qy_0$. Hence $TQy_0=QSy_0=Q(\alpha_K(S)y_0+y_M)=\alpha_K(S)Qy_0+Qy_k(y_k\in K_-,\,Qy_K\in P^{-1}(K_-)\cap M_\sim)$. On the other hand $TQy_0=\alpha_L(T)Qy_0+z_L(z_L\in L_-)$, and thus $\alpha_L(T)=\alpha_K(S)$. This proves that the diagonal multiplicity of $\alpha\neq 0$ with respect to $\mathcal{F}(T)$ exceeds that of α with respect to $\mathcal{F}(S)$. This proves the theorem.

3. Superdiagonal forms for compact linear operators. If $T\in L(X)$ is a compact linear operator acting on a complex locally convex space X, then (cf. section 1) T is related to a compact linear operator $S\in L(Y)$ acting on a complex Banach space Y. J. R. Ringrose [**4**, cf. proof of Theorem 1] implicitly proved that every maximal invariant nest $\mathcal{F}(S)$ is simple. Hence Theorem 2.1 implies that there exists a simple invariant nest $\mathcal{F}(T)$.

For Banach spaces X the following result is due to Ringrose.

THEOREM 3.1 (D. Koros [**2**, Theorem 2]). *Let $T\in L(X)$ be a compact linear operator acting in a complex locally convex space X, and let $\mathcal{F}(T)$ be a simple nest of subspaces of X, each of which is invariant under T. Then*

(i) *a non-zero scalar α is an eigenvalue of T if and only if α is a diagonal coefficient of T;*

(ii) *the diagonal multiplicity of α is equal to its algebraic multiplicity as an eigenvalue of T;*

(iii) *the operator T is quasi-nilpotent if and only if $\alpha_M(T)=0$ $(M\in\mathcal{F}(T))$; or equivalently if and only if $T(M)\subseteq M_-$ $(M\in\mathcal{F}(T))$.*

Proof. Since T and S are related, $\alpha\neq 0$ is an eigenvalue of T if and only if α is an eigenvalue of S with the same algebraic multiplicity $d(\alpha)$. This is an easy consequence of the definition of related operators (cf. Wrobel [**5**]). By Theorem 2 of Ringrose [**4**], $d(\alpha)$ is equal to the diagonal multiplicity of α with respect to $\mathcal{F}(S)$, and hence $d(\alpha)$ is equal to the diagonal multiplicity of α with respect to $\mathscr{F}(T)$ by Theorem 2.2. Since T is quasi-nilpotent if and only if S is quasi-nilpotent, (iii) follows from Ringrose's result and Theorem 2.2 as well.

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