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DISCONTINUOUS HOMOMORPHISMS AND THE SEPARATING SPACE

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0. Introduction

Let $T\colon A \to B$ be a linear operator between two Banach algebras A and B. The basic problem in the theory of automatic continuity is to find algebraic conditions on T, A, and B which ensure that T is continuous. As a means to study continuity properties of T the separating space of T has played a crucial role. It is defined as

$$\mathfrak{S}(T) = \{b \in B \mid \text{there exists a sequence } x_n \to 0 \text{ in } A \text{ such that } Tx_n \to b\}.$$

By the closed graph theorem T is continuous if and only if $\mathfrak{S}(T) = (0)$. If T is an algebra homomorphism whose image is dense in the range algebra then $\mathfrak{S}(T)$ is a two-sided ideal, which satisfies the following stability property, known as the stability lemma. Given any sequence (b_n) in $T(A)$ there exists $N \in \mathbb{N}$ such that

$$(b_1 \ldots b_n \mathfrak{S}(T))^- = (b_1 \ldots b_N \mathfrak{S}(T))^- \quad \text{for all} \quad n \geqq N, \tag{*}$$

where $-$ denotes norm closure. If T is an algebra derivation then $\mathfrak{S}(T)$ is also a closed two-sided ideal and the sequence in (*) can be any sequence from the range algebra. For a thorough discussion of the separating space see [**6**].

In the conference on automatic continuity in Leeds, June 1976 the question was raised if (*) would imply that the closures of the powers of the separating space had finite descent, i.e. does there exist $N \in \mathbb{N}$ such that

$$(\mathfrak{S}(T)^n)^- = (\mathfrak{S}(T)^N)^- \quad \text{for all} \quad n \geqq N? \tag{**}$$

(If E is a subset of an algebra, E^n denotes the linear span of products of n elements from E.)

Although it may be well-known among people who work in the field that (**) holds for epimorphisms and derivations it has not, as far as we know, appeared in the literature before. We obtain it here as a corollary of a slightly more general result, and we give an example of strictly infinite descent where T is a homomorphism which is not surjective. Hence the property (**) is somewhat special and it is rather an elementwise property that is important for investigations of the possible continuity of an algebra homomorphism, namely the existence of divisible subspaces. Let V be a subspace of an algebra A and let $x \in A$. Then V is called x-divisible if

$$(x - \lambda)V \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

The presence of such subspaces in a Banach algebra is closely tied up with the stability lemma by means of an application of the Mittag–Leffler theorem on inverse limits, see Lemma 1.8 of [**6**]. In the case of a commutative Banach algebra this is further explored in a paper by W. G. Badé, P. C. Curtis, Jr., and K. B. Laursen, [**3**]. They considered the following two statements about a commutative Banach algebra B:

(i) Every homomorphism from any Banach algebra A into B is continuous.
(ii) B is semiprime and no $b \in B$ has a non-trivial divisible subspace.

Assuming that B has scattered maximal ideal space they proved that (i) is equivalent to (ii). The assumption that B has scattered maximal ideal space means that all elements in B have countable spectrum and is needed to bring the Mittag-Leffler theorem on inverse limits into use. Under various assumptions on the domain algebra their result remains true without assuming that B has scattered maximal ideal space and they conjecture that the theorem is valid in general. In this paper we prove that (i) always implies (ii). As to the implication (ii)$\Rightarrow$(i) let us note that Badé, Curtis, and Laursen actually prove that it holds under the weaker assumption that every element $b \in B$ has countable spectrum when regarded as the operator $L_b: r \to br$ on the (Jacobson-) radical of B.

In order to prove (i)$\Rightarrow$(ii) and to give the example of a homomorphism whose separating space has infinite descent of the closures of the powers we shall be concerned with the construction of discontinuous homomorphisms. To this end we shall follow techniques developed in a paper by M. Thomas, [**7**], which deals with the construction of discontinuous functional calculi. M. Thomas proves that if b is an element of a commutative Banach algebra B with identity and the spectrum of b has at most countably many connected components, then b possesses a discontinuous functional calculus if and only if there is a non-zero b-divisible subspace in B. Using the main technical result of this paper we will show that if B is a semiprime commutative Banach algebra which contains a non-zero b-divisible subspace for some $b \in B$ then there is a discontinuous homomorphism from the disc algebra into B and that we can construct this homomorphism such that its separating space is included in any predetermined closed ideal which contains a non-zero b-divisible subspace.

1. Construction of discontinuous homomorphisms

In order to construct discontinuous homomorphisms we will use a version of Corollary 2.6 of [**7**]. Before we state it we briefly review the terminology of that reference. In this section all algebras are assumed to be over the complex field, commutative, and with unit.

Let A be an algebra and let B be a Banach algebra. Suppose that $\theta: A \to B$ is a unital homomorphism. Then B is a unit linked A-module when the module action is given by

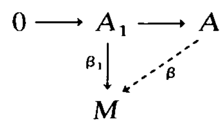
$$(a, b) \to \theta(a)b \equiv a.b$$

for all $a \in A$ and all $b \in B$. Every ideal in B is a submodule of B.

If E is a subset of A and S is a subspace of B we denote by $D(E, S)$ the largest subspace of S such that $eD(E, S) = D(E, S)$ for all $e \in E$. If $b \in B$ we denote by $D(b, S)$

the largest subspace of S such that for all $\lambda \in \mathbb{C}$ we have $(b-\lambda)D(b, S) = D(b, S)$, i.e. $D(b, S)$ is the largest b-divisible subspace of S according to the terminology in the introduction.

A linear map $\beta: A \to B$ satisfying $\beta(fg) = f \,.\, \beta(g) + g \,.\, \beta(f) + \beta(f)\beta(g)$ for all $f, g \in A$ is called a σ-derivation. Note that if β is a σ-derivation from A to B then $\theta + \beta$ is a homomorphism from A to B. Let M be a submodule of B. Then M is called σ-injective if for all subalgebras A_1 of A containing 1 and all σ-derivations $\beta_1: A_1 \to M$ we can extend β_1 to a σ-derivation $\beta: A \to M$. Schematically



The following lemma is a special case of Corollary 2.6 of [**7**].

Lemma 1.1. *Let b be an element of a Banach algebra B, let A_0 denote the complex polynomials, and let A be some algebra of germs of analytic functions on* Sp(b), *the spectrum of b, such that $A_0 \subseteq A$. We make B an A-module by means of the usual functional calculus of b. Suppose that*

$$A \subseteq \{pu \mid p \in A_0,\ u \text{ is invertible as a germ of an analytic function on } \mathrm{Sp}\,(b)\}.$$

If I is a closed ideal in B and $D(A_0, I)$ is torsionfree over A_0 then $D(A_0, I)$ is σ-injective as a module over A.

Lemma 1.2. *If B is semiprime then $D(A_0, I)$ is torsionfree over A_0.*

Proof. Let d be a non-zero torsion element in $D(A_0, I)$ and let $a \in A_0 \backslash (0)$ be such that $a \,.\, d = 0$. By definition of $D(A_0, I)$ there exists $d' \in D(A_0, I)$ such that $a \,.\, d' = d$. Then $d^2 = (a \,.\, d')d = d'(a \,.\, d) = 0$. Hence B is not semiprime.

We are now in the position to construct discontinuous homomorphisms into semiprime Banach algebras.

Theorem 1.3. *Let B be a semiprime Banach algebra and suppose there is $b \in B$ and a closed ideal $I \subseteq B$ such that $D(b, I) \neq (0)$. Then there is a discontinuous homomorphism, ν say, from the disc algebra into B such that the separating space $\mathfrak{S}(\nu)$ is contained in I.*

Proof. We may assume that B has an identity. Let Δ be a closed disc in $\mathbb{C}$ whose interior contains Sp(b) and let A be the uniform algebra of continuous functions on Δ which are analytic on the interior of Δ. As above, the algebra B becomes an A-module with the module action given by the functional calculus θ of b. Since polynomials factor into linear factors $D(b, I) = D(A_0, I)$ and since each $f \in A$ has at most finitely many zeros in Sp(b) we have, invoking Lemma 1.2, the conditions of Lemma 1.1 satisfied. Since A contains elements which are transcendental over the polynomials we get a non-trivial σ-derivation $\beta: A \to D(A_0, I)$ which vanishes on the polynomials. Define $\nu = \theta + \beta$. Then ν is discontinuous and $\mathfrak{S}(\nu) \subseteq (\beta A)^{-} \subseteq I$.

This theorem gives us the generalisation of Theorem 2.4 of [**3**].

Corollary 1.4. *Let B be a Banach algebra and assume that all homomorphisms from any Banach algebra into B are continuous. Then B is semiprime and $D(b, B) = (0)$ for all $b \in B$.*

Proof. This is verified by contraposition. If B contains nilpotent elements a discontinuous homomorphism into B is easily constructed so the corollary follows immediately from Theorem 1.3.

2. A result on powers of the separating space

An element x of a Banach algebra B is said to have finite closed descent (see [**2**]) if there exists $N \in \mathbb{N}$ such that for all $n \geqq N$ we have

$$(x^n B)^- = (x^N B)^-.$$

If x belongs to the radical of B and x has finite closed descent then $D(x, B) \neq (0)$ unless x is nilpotent. In [**1**] G. R. Allan uses this to construct a discontinuous homomorphism from the disc algebra into B. The following theorem should be seen in the light of the question raised at the conference in Leeds mentioned in the introduction.

Theorem 2.1. *Let B be a Banach algebra in which every element has finite closed descent. Then there exists $M \in \mathbb{N}$ such that*

$$\left(\bigcap_{n=1}^{\infty} B^n\right)^- = (B^M)^-.$$

Proof. Let $x \in B$ and let $N \in \mathbb{N}$ be such that $(x^n B)^- = (x^N B)^-$ for all $n \geqq N$. Put $J = (x^N B)^-$. Then $(x^n J)^- = J$ for all $n \in \mathbb{N}$, so by the Mittag–Leffler theorem on inverse limits, [**6**, Chapitre II, §3, Théorème 1], $\bigcap_{n=1}^{\infty} x^n J$ is dense in J. In particular $x^{N+1} \in \left(\bigcap_{n=1}^{\infty} B^n\right)^-$ and consequently the quotient algebra $B \big/ \left(\bigcap_{n=1}^{\infty} B^n\right)^-$ is a nil Banach algebra. By a result of S. Grabiner [**4**] it is a nilpotent Banach algebra. But this is just the statement that $B^M \subseteq \left(\bigcap_{n=1}^{\infty} B^n\right)^-$ for some $M \in \mathbb{N}$.

Corollary 2.2. *Let $\mathfrak{S}$ be the separating space of an epimorphism between Banach algebras or of an algebra derivation on a Banach algebra. Then there exists $M \in \mathbb{N}$ such that $\bigcap_{n=1}^{\infty} \mathfrak{S}^n$ is dense in $\mathfrak{S}^M$.*

Proof. We only have to remark that as a direct consequence of the stability lemma every element of $\mathfrak{S}$ has finite closed descent in $\mathfrak{S}$.

We now give an example which shows that if the assumption of surjectivity of the homomorphism is dropped we may have a strictly infinite descent of the closed powers of the separating space.

Example 2.3. Let $\omega(t)=e^{-t^2}$, $t\geqq 0$, and let $L^1(\omega)$ denote the Banach space of Lebesgue measurable functions on the positive halfline such that $\|f\|\equiv\int_0^\infty |f(t)|\,\omega(t)\,dt<\infty$. With convolution as algebra product, $L^1(\omega)$ becomes a commutative radical Banach algebra with bounded approximate identity. Let $f\in L^1(\omega)$ and define $\alpha(f)=\inf\operatorname{supp} f$. Let $M=\{f\in L^1(\omega)\mid \alpha(f)\geqq 1\}$. Then M is a closed ideal in $L^1(\omega)$ and by the Johnson-Varopoulos extension of Cohen's factorisation theorem ([**5**] and [**9**]) there is $f\in L^1(\omega)$ such that $(f^nM)^-=M$ for all $n\in\mathbb{N}$. As in the proof of Theorem 2.1, it follows that $\bigcap_{n=1}^{\infty} f^nM\equiv D$ is dense in M. We show that D is a divisible subspace for f. Let $d\in D$. Then there exists a sequence (m_n) in M such that $d=f^nm_n$ for all $n\in\mathbb{N}$. Since, by the Titchmarsh convolution theorem [**8**], $L^1(\omega)$ is an integral domain it follows that $m_1=f^{n-1}m_n$ for all $n\in\mathbb{N}$ so that $m_1\in D$. This shows that $fD=D$. From $\operatorname{Sp}(f)=\{0\}$ it follows that $(f-\lambda)D=D$ for all $\lambda\in\mathbb{C}$. Being an integral domain $L^1(\omega)$ is in particular semiprime so the hypotheses of Theorem 1.2 are fulfilled and we get a discontinuous homomorphism ν from the disc algebra into $L^1(\omega)$ such that $\mathfrak{S}(\nu)\subseteq M$. Define $C_n=\inf\{\alpha(g)\mid g\in(\mathfrak{S}(\nu)^n)^-\}$. Then $C_1\geqq 1$ and by the Titchmarsh convolution theorem $C_n=nC_1$ for all $n\in\mathbb{N}$ so $(\mathfrak{S}(\nu)^n)^-\neq(\mathfrak{S}(\nu)^{n+1})^-$ for all $n\in\mathbb{N}$.

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