EXPERIENCE OF NUMERICAL INTEGRATION AND APPROXIMATION WITH APPLYING CHEBYSHEV POLYNOMIALS FOR CONSTRUCTING EPHEMERIDES OF THE SOLAR SYSTEM NATURAL AND ARTIFICIAL BODIES

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Abstract.

Successful experience of applying the Chebyshev polynomials as a power "mathematical tool" for numerical integration and approximation techniques in celestial mechanics is presented. Detailed analysis of approximation function behavior inside an integration step allows to elaborate a special technique for high accuracy and rapid integration of piece-wise continuous functions, modeling the Earth's shadow effect for artificial satellite orbits. Original software is elaborated for creating the ephemeris file simultaneously with the process of numerical integration. This technique is applied for the construction of ephemerides of natural and artificial celestial bodies as well as for the compact polynomial representation of different geodynamic parameters.

1. Approximation method

Let the motion of a celestial body be described by a system of the differential equations

$$Y'' = F(Y', Y, t), \tag{1}$$

under initial conditions

S. Ferraz-Mello et al. (eds.), Dynamics, Ephemerides and Astrometry of the Solar System, 347–352.

$$Y\,|_{t=0}=Y_0,\quad Y'\,|_{t=0}=Y_0'. \tag{2}$$

Here Y, Y' and Y'' denote three-dimensional vectors of the position, velocity and acceleration of a body. In INCH method (Belikov, 1993) of numerical integration each right-hand side of the system (1) is presented as a truncated Chebyshev expansion

$$F(Y',Y,t)=\sum_{m=0}^{M} a_m T_m^*\left(\frac{t}{h}\right),\quad t^{(1)}\leq\ t\leq t^{(2)}. \tag{3}$$

Here T_m^* is shifted Chebyshev polynomials, $h=t^{(2)}-t^{(1)}$ is an integration step, and a_m are coefficients which are evaluated by an iteration procedure. The acceleration $Y''(t)$ is presented over an interval $T=t_2-t_1,(T>h)$ in form of another truncated series

$$F^{(N)}=Y''(t)=\sum_{n=0}^{N} A_n T_n^*\left(\frac{t}{T}\right),\quad t_1\leq t\leq\ t_2. \tag{4}$$

The initial conditions at the left-hand boundary of an approximation interval are

$$Y\,|_{t=t_1}=y_0,\quad Y'\,|_{t=t_1}=y_0'. \tag{5}$$

Thus the problem of the polynomial approximation is reduced to the determination of the coefficients A_n. The latter are presented in form of integrals

$$A_n=\frac{2\delta_n}{\pi}\int_0^1\frac{Y''(x)T_n^*(x)}{\sqrt{x(1-x)}}\,dx,\quad \delta_0=\frac{1}{2},\quad \delta_i=1\;(i>1). \tag{6}$$

Here $Y''(x)$ means a function which is defined by formula (3) over the whole interval T. In INCH method the value of function $Y''(x)$ can be determined with a sufficient accuracy at an arbitrary point of the time interval T. Thus for calculating the integral (6) one can use the quadrature formula of the highest accuracy class. The resulting formula is

$$\int_0^1\frac{Y''(x)T_n^*(x)}{\sqrt{x(1-x)}}dx\approx\alpha Y''(0)T_n^*(0)+\beta Y''(1)T_n^*(1)+\alpha^*\sum_{k=1}^{N}Y''(x_k)T_n^*(x_k), \tag{7}$$

where

$$x_k=\frac{1+cos(\pi k/(N+1))}{2},\quad k=1,2,...N, \tag{8}$$

$$\alpha = \beta = \frac{\pi}{2(N+1)}, \quad \alpha^* = \frac{\pi k}{N+1}. \tag{9}$$

By applying the quadrature formulae (7)-(9), one can determine the whole set of the desired coefficients A_n in (4). The value of function $Y''(x_k)$ in (7) is found as follows:

$$Y''(x_k) = \sum_{m=0}^{M} a_m T_m^*(z_k), \quad z_k = x_k T/h, \quad t_1 \le t_k \le t_2, \tag{10}$$

where a_m are the above mentioned coefficients computed by integrator INCH. Then, after twofold analytical integrating (4) with consideration of (5), one has:

$$Y'^{(N)} = \sum_{n=0}^{N+1} B_n T_n^*(t/T), \quad Y^{(N)} = \sum_{n=0}^{N+2} C_n T_n^*(t/T). \tag{11}$$

The high accuracy of the representation is proved by Powel's estimates establishing the closeness between the approximations by Chebyshev interpolating polynomial, Chebyshev truncated series and by polynomials of the best uniform approximation (Luke, 1975).

2. Applications of the method

The calculations were performed on PC Dell 486/66 computer with double precision (16 decimal digits of mantissa).

2.1. CONSTRUCTION OF EPHEMERIDES OF THE MOON, SUN AND PLANETS

The gravitational interaction of the Solar System bodies is modeled by Einstein- Infeld-Hofmann's equations, defining the orbital barycentric motion of the Sun, major planets and the Moon as non-rotating masses in the barycentric isotropic coordinate system (Eroshkin, Trubitsina, 1992). Additional perturbations in the motions are caused by the attraction of five most massive asteroids. The total system, consisting of 40 ordinary differential equations of the second order, forms the basis of the given model. 39 of these equations describe the barycentric motion of the Sun, 9 major planets and geocentric motion of the Moon, and remaining four equations describe the Moon's rotation around its own center of the masses. In Table 1 the results of the numerical testes of the problem is shown. The accuracy criterion for AE ephemerides is that an approximation error should be less than 1 millimeter at each point of approximation interval.

Similar tests have been conducted for constructing the specialized geocentric ephemerides of the Moon and Sun (AEMS) (Eroshkin, Taybatorov, Trubitsina, 1994). The ephemerides are optimized by accuracy and compactness for practical requirements of numerical integration for some types of artificial satellites.

TABLE 1. Maximum residuals between the ephemeride AE94 and DE200/LE200 over 50 year span

Object	Residuals (mm)	Approx.interval (days) AE(LE/DE)	Polyn.degree AE(LE/DE)
Mercury	52	8(8)	12(11)
Venus	7	32(32)	11(11)
E-M barycenter	10	16(16)	12(14)
Mars	18	32(32)	8(9)
Jupiter	16	32(32)	8(8)
Saturn	59	32(32)	7(7)
Uranus	31	32(32)	7(7)
Neptune	33	32(32)	5(5)
Pluto	14	32(32)	5(5)
Geocentr.Moon	5	8(4)	14(11)
Sun	0.034	32(32)	12(14)
Earth	9	16	12

2.2. ARTIFICIAL SATELLITE NUMERICAL EPHEMERIDES

The numerical tests were conducted for GPS satellite orbit (Taybatorov, Trubitsina, 1992). The results of both numerical integration over the time interval of 7 days and polynomial approximation for GPS satellites are presented in Table 2. As the reference ("exact") solution in "Integration" part of Table 2 the results of numerical integration are taken, obtained on computer ELBRUS-1CB with a longer mantissa (24 decimal digits). In the part of Table 2, titled as "Polynomial representation", the maximum deviations of the ephemeris position components from the numerical integration results are given, where Q denotes the ephemerides file size, δt is additional computer time for ephemeride file construction procedure, M, N, T, h are defined in (3)-(4).

TABLE 2. Comparison of integration and compact polynomial representation procedures for GPS satellites numerical ephemerides for (7 days arc)

Integration	M	h(days)	ΔR (m)	$\Delta\lambda$(sec)	$\Delta\beta$ (sec)		
	7	0.05	0.35e-3	0.11e-3	0.49e-4		
Polynomial	N	T(day)	ΔR (m)	$\Delta\lambda$ (sec)	$\Delta\beta$ (sec)	δt	Q(Kb)
Approximation	14	0.05	0.11e-7	0.23e-8	0.25e-8	4%	49.3
	14	0.15	0.51e-4	0.42e-6	0.31e-6	2%	16.2

3. Numerical integration of piecewise-continuous functions in satellite dynamics

If we need to take into account the solar pressure in problems of satellite dynamics we deal with piecewise-continuous functions. The characteristics of the numerical integration for such a problem were investigated on a model, simulating the disturbing motion of Lageos with taking into account the direct solar pressure and passing a satellite the Earth's shadow.

Three methods are compared with varying basis parameters: RA(15) (Everhart, 1985), INCH(9), INCHE(9) (Belikov, 1993). All the conclusions are made by comparison of the differences between the forward and backward solutions for one day time interval with four gap points.

The earlier elaborated method INCHE(9) (Belikov, 1993) is tested for the case with discontinuities. In this procedure the bisection principle is used to find the appropriate length of a variable step in the neighborhood of discontinuity. The high accuracy of solution is reached in this procedure at the expense of significant increasing CPU time due to a lot of step subdivisions. It is not convenient for practical application.

In this connection special procedure for an optimal subdivision of an integration step is elaborated at present work, in which the property of Chebyshev approximation to reach the maximum approximation error in the neighborhood of a gap (Gibbs phenomenon) is used for the optimal choice of an integration step. This modified INCHE method shows its essential time reducing, when solving the numerical problems for piecewise-continuous functions with sufficient accuracy. The results are summarized in Table 3, where N_{step} is number of intergation steps, N_{force} is number of calls of force subroutine (computer time measure).

Acknowledgements

The author is grateful to the Russian Foundation of Fundamental Research for financial support of this investigation (grant N95-02-04304-a).

TABLE 3. Comparison of the efficiency of the different numerical integration software in solving the system of differential equations with piecewise continuous functions

Method	$\frac{Y''_{sp}}{Y''_e}$	ΔR (m)	$\Delta\lambda$ (sec)	$\Delta\beta$	N_{step}	N_{force}
	0	0.34e-7	0.65e-8	0.34e-8	194	2974
RA(15)	1.e-9	0.30e-3	0.11e-4	0.28e-4	183	2809
	1.e-5	0.32e+1	0.12e+0	0.34e+0	190	2914
	0	0.23e-7	0.93e-8	0.59e-8	149	2886
INCHE(9)	1.e-9	0.56e-6	0.33e-9	0.58e-7	1060	22769
	1.e-5	0.46e-6	0.18e-7	0.36e-7	1060	22769
INCHE(9)	1.e-9	0.33e-7	0.11e-7	0.58e-8	231	5664
modified	1.e-5	0.33e-7	0.12e-7	0.25e-7	238	5570

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