# TWO INTEGRAL TRANSFORM PAIRS INVOLVING HYPERGEOMETRIC FUNCTIONS†

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**1. Introduction.** In this note, we first establish an integral transform pair where the kernel of each integral involves the Gaussian hypergeometric function. Special cases of Theorem 1 have been studied by several authors [**1, 2, 5, 6**]. In Theorem 2 a similar integral transform pair involving a confluent hypergeometric function is given.

We conclude with several examples.

**2. Results.**

THEOREM 1. *Let $n$ be an integer, $n > \mathrm{Re}(c) > 0$; let $0 < y \leqq 1$; let $F(x) \in C^n$ and $G(x)$ be absolutely continuous, $0 \leqq x \leqq 1$, and let $F(1) = F'(1) = \ldots = F^{(n-1)}(1) = 0$. Then either of the statements*

$$F(y) = \int_y^1 (x-y)^{c-1}{}_2F_1(a, b;\ c;\ 1-y/x)G(x)\,dx, \tag{1}$$

$$G(y) = \frac{(-)^n}{\Gamma(c)\Gamma(n-c)}\int_y^1 (x-y)^{n-c-1}{}_2F_1(-a, -b;\ n-c;\ 1-x/y)F^{(n)}(x)\,dx, \tag{2}$$

*implies the other.*

*Proof.* In (1) and (2) let $y = e^{-t}$, $x = e^{-u}$, $F(e^{-t}) = f(t)$, $G(e^{-u})e^{-uc} = g(u)$. Equations (1) and (2) become

$$f(t) = \int_0^t [1-e^{-(t-u)}]^{c-1}{}_2F_1(a, b;\ c;\ 1-e^{-(t-u)})g(u)\,du, \tag{3}$$

$$e^{ct}g(t) = \frac{1}{\Gamma(c)\Gamma(n-c)}\int_0^t [1-e^{-(t-u)}]^{n-c-1}e^{-u(n-c)}{}_2F_1(-a, -b;\ n-c;\ 1-e^{-(u-t)}) \times\left\{e^u\frac{d}{du}\right\}^n f(u)\,du. \tag{4}$$

Because of Euler's relationship

$${}_2F_1(a, b;\ c;\ z) = (1-z)^{-b}{}_2F_1(c-a, b;\ c;\ z/(z-1)), \tag{5}$$

(4) may be written

$$e^{t(c-b)}g(t) = \frac{1}{\Gamma(c)\Gamma(n-c)}\int_0^t [1-e^{-(t-u)}]^{n-c-1}{}_2F_1(n-c+a, -b;\ n-c;\ 1-e^{-(t-u)}) \times e^{-u(n-c+b)}\left\{e^u\frac{d}{du}\right\}^n f(u)\,du. \tag{6}$$

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Equations (3) and (6) are integral equations of convolution type and each may be solved by the use of Laplace transforms under the given hypotheses.

Let

$$\mathscr{L}\{f(t)\} = \bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt. \tag{7}$$

We need the following known formulae [3]:

$$\mathscr{L}\left\{\left(e^t \frac{d}{dt}\right)^n f(t)\right\} = (p-1)(p-2)\ldots(p-n)\bar{f}(p-n),$$

$$f^{(k)}(0) = 0 \quad (k = 0, 1, \ldots, n-1); \tag{8}$$

$$\mathscr{L}\{e^{-at} f(t)\} = \bar{f}(p+a); \tag{9}$$

$$\mathscr{L}\{(1-e^{-t})^{\gamma-1} {}_2F_1(\alpha, \beta;\ \gamma;\ 1-e^{-t})\} = \frac{\Gamma(p)\Gamma(\gamma-\alpha-\beta+p)\Gamma(\gamma)}{\Gamma(\gamma-\alpha+p)\Gamma(\gamma-\beta+p)} \quad (\mathrm{Re}\,(\gamma) > 0). \tag{10}$$

The Laplace transform of (3) is

$$\bar{f}(p) = \frac{\Gamma(p)\Gamma(c-a-b+p)\Gamma(c)\bar{g}(p)}{\Gamma(c-a+p)\Gamma(c-b+p)} \quad (\mathrm{Re}\,(p) > 0), \tag{11}$$

while (6) yields

$$\bar{g}(p+b-c) = \frac{\Gamma(p)\Gamma(b-a+p)(n+p+b-c-1)(n+p+b-c-2)\ldots(p+b-c)\bar{f}(p+b-c)}{\Gamma(c)\Gamma(p-a)\Gamma(n+p+b-c)}$$

$$= \frac{\Gamma(p)\Gamma(b-a+p)\bar{f}(p+b-c)}{\Gamma(c)\Gamma(p-a)\Gamma(p+b-c)} \quad (\mathrm{Re}\,(p+b-c) > 0), \tag{12}$$

which holds since $f^{(k)}(0) = F^{(k)}(1) = 0$ $(k = 0, 1, \ldots, n-1)$. But (11) and (12) are equivalent statements, and thus the theorem is proved.

To obtain the result in [1], let $a = -\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}\mu$, $b = -\frac{1}{2}\nu+\frac{1}{2}\mu$, $c = \mu$ and make the obvious changes of variable.† Likewise the transform pairs given in [2] and [5] follow by the proper identification of parameters.

Let us write (3) in the form

$$f(t) = \int_0^t k(t-u)g(u)\, du. \tag{13}$$

A feature of the present study is that the inverse Laplace transform of $\bar{k}(p)^{-1}$ has a simple form. Whenever this is true, the solution of (13) often yields a simple integral transform pair. Our second theorem, which involves a confluent hypergeometric function, demonstrates this.

† The powers of $(t^2 - x^2)$ in the first two equations of this reference should read $(\mu - 1)/2$ instead of $(1 - \mu)/2$. There the conditions of validity on $F$ were omitted.

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THEOREM 2. *Suppose that* $f(t) \in C^n$ *with* $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$. *Let* $g(t)$ *be absolutely continuous for* $t \geqq 0$. *Then, for* $n > \operatorname{Re}(c) > 0$, *either of the statements below implies the other.*

$$f(t) = \int_0^t (t-u)^{c-1} \Phi(a, c;\ \lambda(t-u)) g(u)\, du, \tag{14}$$

$$g(t) = \frac{1}{\Gamma(c)\Gamma(n-c)} \int_0^t (t-u)^{n-c-1} \Phi(-a, n-c;\ \lambda(t-u)) f^{(n)}(u)\, du. \tag{15}$$

(Here $\Phi(a, b;\ z)$ is Kummer's confluent hypergeometric function ${}_1F_1(a, b;\ z)$.)

*Proof.* The proof follows the manner of that for Theorem 1 (see (3) and (4)). The transform pairs needed are given in [**3**, 4. 2. 3 (1), 4. 1 (8)].

**3. Applications.** We give two examples of (1) and (2) where the kernels involve elementary transcendents. The conditions for validity may be inferred from Theorem 1.

If $a = \frac{1}{2}$, $b = 1$, $c = \frac{3}{2}$ and $n = 2$, then

$$\left.\begin{aligned} F(y) &= \int_y^1 \ln\left\{\frac{x+\sqrt{(x^2-y^2)}}{x-\sqrt{(x^2-y^2)}}\right\} G(x)\, dx, \\ G(y) &= \frac{1}{\pi}\int_y^1 (x^2-y^2)^{-\frac{1}{2}} \left(\frac{2y^2}{x^2}-1\right)[xF''(x)-F'(x)]\, dx. \end{aligned}\right\} \tag{16}$$

If $b = a + \frac{1}{2}$, $c = \frac{1}{2}$ and $n = 1$, then

$$\left.\begin{aligned} F(y) &= \int_y^1 (x^2-y^2)^{-\frac{1}{2}} \{[x+\sqrt{(x^2-y^2)}]^{-2a} + [x-\sqrt{(x^2-y^2)}]^{-2a}\} G(x)\, dx, \\ G(y) &= -\frac{1}{\pi}\int_y^1 (x^2-y^2)^{-\frac{1}{2}} \operatorname{Re}\{[y+i\sqrt{(x^2-y^2)}]^{2a+1}\} F'(x)\, dx. \end{aligned}\right\} \tag{17}$$

Other examples of (1) and (2) may be obtained by applying the formulae in [**4**, Ch. 2], and examples of (14), (15) follow by using the results in [**4**, Ch. 6].

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