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COMPACTNESS IN TOPOLOGICAL HJELMSLEV PLANES

BY

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ABSTRACT. In the theory of ordinary topological affine and projective planes it is known that (1) An affine plane is never compact (2) a locally compact ordered projective plane is compact and archimedean (3) a locally compact connected projective plane is compact and (4) a locally compact projective plane over a coordinate ring with bi-associative multiplication is compact. In this paper we re-examine these results within the theory of topological Hjelmslev Planes and observe that while (1) remains valid (2), (3) and (4) are false. At first glance these negative results seem to suggest we are working in too general a setting. However a closer examination reveals that the absence of compactness in our setting is a natural and expected feature which in no way precludes the possibility of obtaining significant results.

A *topological projective* (*affine*) *plane* is a projective (affine) plane whose point and line sets are topological spaces so that the joining of points and the intersection of lines (and parallelism) are continuous operations. Historically, this continuity was introduced into the classical geometries by means of topological coordinate fields ([5]; see also [12, 7]) or order ([14], [10], [13]).

The euclidean plane is not compact, but the classical projective planes over the reals, complexes, quaternions and Cayley numbers have compact topologies. More generally Salzmann has proved that if a topological plane is neither discrete nor indiscrete then,

I. An affine plane is never compact. ([12, page 48])
II. A locally compact ordered projective plane is compact and archimedean. ([11, page 450])
III. A locally compact connected projective plane is compact. ([11, page 448])
IV. A locally compact projective plane, over a coordinate ring whose multiplication is bi-associative; is compact. ([11 page 452])

In this paper we consider these results within the theory of topological Hjelmslev planes and show that, while I remains valid, II, III and IV are false.

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At first glance these negative results seem to suggest we are working in too general a setting. However, a closer examination reveals that the absence of compactness in our setting is a natural, and expected feature, which in no way precludes the possibility of obtaining significant results. In case II, it is known historically ([4]) that ordered projective Hjelmslev planes are non-archimedean; and hence it is not surprising that such planes are not compact. Finally, we remark that all the classical examples of projective Hjelmslev planes (the planes over the topological rings $\mathbf{R}[x]/(x^n)$ where $\mathbf{R}$ is the reals) contain topological closed copies of the euclidean plane and hence, though locally compact, are never compact.

1. **Topological Hjelmslev planes.** Incidence structures and their homomorphisms are defined as in Dembowski's Finite Geometries [2]. For any incidence structure $\langle P, L, I\rangle$: points are denoted by $P, Q, R, \ldots$. and lines by $l, m, n, \ldots$. L_p denotes the set of lines incident with P, $l\wedge m$ set of points incident with l and m, and $P\vee Q$ the set of lines incident with P and Q. A *parallelism* of an incidence structure $\langle \mathbf{P}, \mathbf{L}, \mathbf{I}\rangle$ is an equivalence relation $\|\subseteq \mathbf{L}\times\mathbf{L}$. A homomorphism $\alpha=(\alpha_1, \alpha_2)$ of incidence structures with parallelism also preserves parallelism i.e. $l\,\|\, m \Rightarrow \alpha_2(l)\,\|\,\alpha_2(m)$. $\langle \mathbf{P}, \mathbf{L}, \mathbf{I}\rangle$ is a *topological incidence structure* if $\mathbf{P}$ and $\mathbf{L}$ are topological spaces; and we say it has a topological property (*) if its point set has the property (*). Finally, a homomorphism, (α_1, α_2) between topological incidence structures is continuous or open if both α_1 and α_2 are continuous or open.

For the definition and basic properties of a projective or affine Hjelmslev plane (PH or AH-plane for short) we refer the reader to [8] or [9]. We also adhere to the notational conventions of these two papers. In particular, $\sim$ is the neighbour relation and $\bar{P}$ is the set of points neighbouring to P.

(1.2) Definition. (a) A topological incidence structure, $\mathbf{H}=\langle \mathbf{P}, \mathbf{L}, \mathbf{I}\rangle$ is a *topological* PH-*plane* (TPH-*plane*) if $\mathbf{H}$ is a PH-plane with the following additional axioms:

(TPH1). The maps $\vee: \mathbf{P}\times\mathbf{P}\backslash\sim_{\mathbf{P}}\rightarrow\mathbf{P}$ and, $\wedge: \mathbf{L}\times\mathbf{L}\backslash\sim_{\mathbf{L}}\rightarrow\mathbf{P}$ are continuous.

(TPH2). $\sim_{\mathbf{P}}$ and $\sim_{\mathbf{L}}$ are closed in $\mathbf{P}\times\mathbf{P}$ and $\mathbf{L}\times\mathbf{L}$ respectively.

(b) A topological incidence structure with parallelism, $\mathbf{H}=\langle \mathbf{P}, \mathbf{L}, \mathbf{I}, \|\rangle$ is a *topological* AH-*plane* (TAH-*plane*) if H is an AH-plane with the following additional axioms:

(TAH1). The maps $\vee: \mathbf{P}\times\mathbf{P}\backslash\sim_{\mathbf{P}}\rightarrow\mathbf{L}$, $\wedge: \mathbf{L}\times\mathbf{L}\backslash\{(l, m): |l\wedge m|\neq 1\}\rightarrow\mathbf{P}$ and $L: \mathbf{P}\times\mathbf{L}\rightarrow\mathbf{L}$ are continuous.

(TAH2) = (TPH2).

From now on, unless otherwise stated, if H is a TPH-*plane or a* TAH-*plane, then we consider the canonical image* $\mathbf{H}/\sim$ *as a topological incidence structure with respect to the quotient topologies of the neighbour relations.*

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2. **Compactness in topological affine Hjelmslev planes.** We easily see that I is also true in TAH-planes.

(2.1) THEOREM. *A topological* AH-*plane*, **H**, *where* $\mathbf{H}/\sim$ *is not discrete, is never compact.*

Proof. Since $\mathbf{H}/\sim$ is a hausdorff affine plane and $\pi:\mathbf{H}\to\mathbf{H}/\sim$ is open-continuous [8, 1.9], then I clearly implies that **H** is never compact.

3. **Compactness in topological PH-planes.** In this section we prove some results on compactness which help to explain why many interesting classes of topological projective Hjelmslev planes are not compact.

(3.1) PROPOSITION. *Let* **H** *be a* TPH-*plane with* $\mathbf{H}/\sim$ *not discrete. Then,* **H** *is locally compact and hausdorff if and only if it possesses a locally compact hausdorff line.*

Proof. The necessity is clear from [6, 7.9] since a closed set of a locally compact space is locally compact. For the sufficiency we assume **H** has one (and hence all) locally compact hausdorff line.

Now to show **H** is locally compact hausdorff it suffices to show any 2 points lie in an open locally compact hausdorff subspace. But any 2 points lie in an affine subplane ([7, 6.7]) whose point set is homeomorphic to the cartesian product of any of its (affine) lines with itself [6, 1.5]. Also, by [8, 1.6(a)], an affine line, $l\backslash\bar{P}$, is an open subset of l and since an open subset of a locally compact space is locally compact, we are done.

We next generalize a result of Salzmann's for ordinary planes ([11]) using the ideas in [13].

(3.2) THEOREM *Let* **H** *be a hausdorff* TPH-*plane.*

(a) *If* $\mathbf{H}/\sim$ *is discrete and infinite, then* **H** is not compact.

(b) If $\mathbf{H}/\sim$ is not discrete, then **H** is compact if and only if it possesses a compact line.

Proof. (a) Since $\mathbf{H}/\sim$ is discrete, each $\bar{P}$ is an open set. The open cover $\{\bar{P}\}$, then has no finite subcover, because $\mathbf{H}/\sim$ is infinite. Thus, **H** is not compact.

(b) The necessity is immediate from [8, 1.6(c)]. For the sufficiency, assume the lines are compact. By (3.1.) and [8, 2.7.1] we conclude that **H** is a locally compact hausdorff separable metric space. Hence, to show that **P** is compact it suffices to establish that every sequence of points has a convergent subsequence.

Before we do this, we make the following two observations:

(i) For each point P, $\mathbf{L}_p$ is a compact set.

This follows since $\mathbf{L}_p$ is homeomorphic to a line with no points neighbouring to P i.e. $l\to L_p(X\to X\vee P)$.

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(ii) For any sequence of points, $\{A_n\}$ in $\mathbf{P}$,

$$\bigcup_{i=1}^{\infty} \bar{A}_n \neq \mathbf{P}.$$

This follows, because by [8, 1.10.1] each $\bar{A}_n$ is a nowhere dense set. The statement is then just a consequence of the Baire category theorem for locally compact spaces ([3, pages 142–145]).

Now, let $\{A_n\}$ be a sequence in $\mathbf{P}$. By (ii) we may choose $P \not\sim A_n$ for each n. Then, the sequence $\{p_n\}$ with $p_n = P \vee A_n \in L_p$, has, by (i) a convergent subsequence, say $p_m \to p$, with $p \in L_p$. For any point $P: \sum(p) = \{X : X$ is a neighbour of some point of $p\}$. Now choose $Q \in \mathbf{P} \backslash \sum(p)$ so that $Q \not\sim A_m$ for each m. This is possible because $\sum(p)$ is also a nowhere dense set [7, 6.7] and so $\bigcup_{i=1}^{\infty} \bar{A}_r \cup \sum(p) \neq \mathbf{P}$. Thus, $\{q_m\}$ with $q_m = Q \vee A_m \in L_Q$ has a convergent subsequence $Q \vee A_i = q_i \to q$ with $q \in L_Q$ and so $q \not\sim p$. Then, $p_i \to p$ and so $(p_i, q_i) \to (p, q)$. Since $\sim$ is closed and $p \not\sim q$, we can assume, without loss of generality, that $P_i \not\sim q_i$ for each i. Thus, $p_i \vee q_i = A_i \to p \vee q$.

We next determine several conditions which are equivalent to compactness in a PH-plane with connected lines. (Note that we do not know if a connected point set implies that the lines are connected. A weaker form of connectedness appears to be required for such a result. See [7] and [8]). We then use this result in section five to show that the topological desarguesian PH-plane over the dual numbers violates both III and IV.

(3.3) THEOREM. *Let* $\mathbf{H}$ *be a topological hausdorff* PH-*plane, with* $\mathbf{H}/\sim$ *not discrete and whose lines are connected.*

The following statements are equivalent.

(A) $\mathbf{P}$ *is compact.*

(B) *For each point* P, *there exists a compact neighbourhood* $\mathbf{V}$ *of* P *so that* $\bar{P} \subseteq \mathbf{V}$.

(C) $\mathbf{P}$ *is locally compact and* $\bar{P}$ *is compact for each point* P.

Proof. (A) $\Rightarrow$ (B) is obvious and (B) $\Rightarrow$ (C) since $\bar{P}$ is closed. (C) $\Rightarrow$ (A). To verify (A) we need only show, because of (3.2.), that $\mathbf{H}$ has compact lines. Let g be any line and choose P so that $P \sim X$ for each $X I g$ Since $\mathbf{H}$ is hausdorff, g is closed by [8, 1.6(c)]. Since $\mathbf{H}$ is locally compact, $\bar{P}$ is compact and $P \subseteq \mathbf{P} \backslash g$. Then by [3, Theorem 2, page 135] there exists a compact neighbour $\mathbf{U}$ of P so that $\bar{P} \subseteq \operatorname{int} \mathbf{U} \subseteq \mathbf{U} \subseteq \mathbf{P} \backslash g$. Hence, $\mathbf{P} \cap \partial(\mathbf{U}) = 0 = \mathbf{U} \cap g$. Moreover, $\partial(\mathbf{U}) \subseteq \mathbf{U}$ and so $\partial(\mathbf{U})$ is compact.

Claim. $V_p : \partial(\mathbf{U}) \to g (X \to (X \vee P) \wedge g)$ is an epimorphism.

The claim immediately implies that g is compact. To verify our claim it suffices to show that for each $Y \in g$, $\partial(\mathbf{U}) \cap (P \vee Y) \neq 0$: for if $X \in \partial(\mathbf{U}) \cap (P \vee Y)$ then $X \sim P$ and $(P \vee X) \cap g = V_p(X) = Y$. Now suppose $\partial(\mathbf{U}) \cap (P \vee Y) = 0$. We then claim that $\mathbf{U} \cap (P \vee Y)$ is clopen in the relative topology of $P \vee Y = h$. First

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observe that $\mathbf{U}\cap(P\vee Y)$ is clopen in h if and only if $\partial_h(\mathbf{U}\cap h)=0$, where ∂_h is the relative boundary operator in h. Let $\mathbf{C}_h$, Γ_h and int_h be the relative compliment, closure and interior operators in h. Then,

$$\partial_h(\mathbf{U}\cap h)=\Gamma_h(\mathbf{U}\cap h)\cap\mathbf{C}_h\ \text{int}_h(\mathbf{U}\cap h)=(\mathbf{U}\cap h)\cap\Gamma_h)\,\mathbf{C}_h(\mathbf{U}\cap h).$$

But

$$\mathbf{C}_h(\mathbf{U}\cap h)=h\cap\mathbf{C}(\mathbf{U}\cap h)=h\cap(\mathbf{CU}\cup\mathbf{C}h)=h\cap\mathbf{CU}\subseteq h\cap\Gamma\mathbf{CU}.$$

Thus,

$$\begin{aligned}\partial_h(\mathbf{U}\cap h)&\subseteq(\mathbf{U}\cap h)\cap h\cap\Gamma(h\cap\mathbf{CU})\subseteq\mathbf{U}\cap(h\cap\Gamma\mathbf{CU})\\&=h\cap(\mathbf{U}\cap\Gamma\mathbf{CU})\subseteq h\cap(\mathbf{U}\cap\mathbf{C}\,\text{int}\,\mathbf{U})=h\cap\partial(\mathbf{U})=0\end{aligned}$$

Since $\mathbf{U}\cap g=0$, then $Y\notin U$, and so $P\in U\cap h\subseteq h$, which contradicts the connectedness of h.

4. **Compactness in ordered projective Hjelmslev planes.** The details of the result in this section already appear in [1], but for the sake of completeness we briefly discuss the ideas here.

Let $\mathbf{H}$ be a PH-plane. If l, m are two lines and P is a point so that $P\mathbf{I}l$ and P is not a neighbour of any point on m, then the set $\hat{P}=\{(X,Y):X\in l,\ Y\in m$ and, (X,Y,P) are collinear$\}$ is a projective relation. In ordinary planes $\hat{P}$ is a (bijective) projection. H is an *ordered* PH-*plane* if each line l possesses a cyclic ordering $|\subseteq l^2\times l^2$, and the orderings are invariant under projective relations.

For each line, l the segments (intervals) $(P,Q)_S=\{X\mid PQ\mid XS\}$ form a base for the interval topologies on l. The dual plane, $\mathbf{H}^d$, is also ordered and the dual of a segment is an angle $(P,q)r$. This angle determines the sector $\mathbf{S}=\{X\mid pq\mid(V\vee X)r\}$ where $V\mathbf{I}p\vee q$. The intersection of two sectors defines what Wyler calls a quadrangle. The quandrangles form a base for the *order topology* on the point set $\mathbf{P}$. Dually, we have an *order topology* on the line set $\mathbf{L}$. An open set in $\mathbf{P}$ intersects any line in an open set of the interval topology.

If we endow $\mathbf{H}$ with the order topologies, then it is a TPH-plane. Moreover, in the order topologies, each $\bar{P}$ is an open set and so the quotient topology on $\mathbf{H}/\sim$ is discrete. But $\mathbf{H}/\sim$ is also ordered and so infinite. Thus, from (3.2.) (a) and [1] we have

(4.1) THEOREM. *An ordered* PH-*plane* $\mathbf{H}$ (*endowed with the order topologies*) *is never compact. Moreover,* $\mathbf{H}$ *is never archimedean nor connected*; *and* $\mathbf{H}/\sim$ *is discrete.*

5. **The counter example.** In this section we prove that III and IV are false by showing that Hjelmslev's original geometry ([4]) over the dual numbers violates both assertions. Details concerning the following discussion can be found in [9].

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$\mathbf{D}=\mathbf{R}+t\mathbf{R}$, where $\mathbf{R}$ is the reals, $t^2=0$ and $\text{tr}=\text{rt}$ for all $r\in\mathbf{R}$ is the *real algebra of dual numbers.* Another model is

$$\mathbf{D}=\left\{\begin{pmatrix} ab \\ oa \end{pmatrix}: a, b\in\mathbf{R}\right\}=\left\{[ab]: a, b\in\mathbf{R}\right\} \text{ with } t=[o1].$$

Then, $\mathbf{D}$ is a local ring with jacobson radical, $\mathbf{J}=\{[ob]: b\in\mathbf{R}\}$, equal to the set of non-units. Also, $\mathbf{D}/\mathbf{J}\cong\mathbf{R}$. The geometry over $\mathbf{D}$ is defined as follows. Let $\approx_l$ and $\approx_r$ be the equivalence relations on $\mathbf{D}\times\mathbf{D}\times\mathbf{D}\backslash\mathbf{J}\times\mathbf{J}\times\mathbf{J}$ whose equivalence classes are

$$\langle abc\rangle=\{\lambda(a, b, c): \lambda\in\mathbf{D}\backslash\mathbf{J}\}$$

and

$$[uvw]=\{(u, v, w)\lambda: \lambda\in\mathbf{D}\backslash\mathbf{J}\}.$$

Then, the incidence structure $H(D)=\langle\mathbf{P}, \mathbf{L}, \mathrm{I}\rangle$ defined by

$$\mathbf{P}=(\mathbf{D}\times\mathbf{D}\times\mathbf{D}\backslash\mathbf{J}\times\mathbf{J}\times\mathbf{J})/\approx_l$$
$$\mathbf{L}=(\mathbf{D}\times\mathbf{D}\times\mathbf{D}\backslash\mathbf{J}\times\mathbf{J}\times\mathbf{J})/\approx_r$$

and $\langle a, b, c\rangle I\ [uvw]\Leftrightarrow au+bv+cw=0$, is a PH-plane.

Neighbours in $\mathbf{H}(\mathbf{D})$ are described algebraically by $\langle abc\rangle\sim\langle a'b'c'\rangle\Leftrightarrow$ there exists $\lambda\in\mathbf{D}\backslash\mathbf{J}$ so that $(abc)-\lambda(a'b'c')\in\mathbf{J}\times\mathbf{J}\times\mathbf{J}$. Then $\mathbf{H}(\mathbf{D})/\sim\cong\mathbf{H}(\mathbf{D}/\mathbf{J})\cong\mathbf{H}(\mathbf{R})$, the real projective plane.

If we endow $\mathbf{D}$ with the subspace topology from $\mathbf{R}^4$, then $\mathbf{D}$ is a topological ring whose radical $\mathbf{J}$ is closed with a void interior. Then, $\mathbf{H}(\mathbf{D})$ is a topological PH-plane when endowed with the quotient topologies from $\approx_l$ and $\approx_r$. Moreover, $\mathbf{H}(\mathbf{D})/\sim$ is not discrete and H is locally compact hausdorff with connected lines. Now, the closed set $\overline{\langle[00][00][10]\rangle}=\{\langle[0x][0y][10]\rangle: x, y\in\mathbf{R}\}$ is easily seen to be homeomorphic to $\mathbf{R}^2$ and so is not compact. Thus, by 3.3, $\mathbf{H}$ is not compact, and so is a counter example to both III and IV.

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