A UNIFIED EXISTENCE AND UNIQUENESS THEOREM FOR STOCHASTIC EVOLUTION EQUATIONS

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Abstract

An existence and uniqueness theorem for mild solutions of stochastic evolution equations is presented and proved. The diffusion coefficient is handled in a unified way which allows a unified theorem to be formulated for different cases, in particular, of multiplicative space–time white noise and trace-class noise.

1. Introduction

The theory of stochastic evolution equations, by which we mean essentially stochastic partial differential equations (SPDEs) of the parabolic or hyperbolic types, shares all of the complications of the corresponding theory for their deterministic counterparts together with many more due to the nature of the noise, in particular its infinite dimensionality. The literature contains many existence and uniqueness results for such SPDEs; see, for example, Da Prato and Zabczyk [3, 4] and Prévot and Röckner [9] and the papers cited therein.

In this note we consider SPDEs of the form

$$dX_t = [AX_t + F(X_t)]\, dt + B(X_t)\, dW_t, \quad X_0 = x_0 \tag{1.1}$$

on a Hilbert space H and where A is in general an unbounded linear operator (for example $A = \Delta$ with Dirichlet boundary conditions), F and B are (possibly) nonlinear functions and W_t is a cylindrical Wiener process. In particular, we consider the

SPDE (1.1) in the mild integral equation form

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)} F(X_s)\,ds + \int_0^t e^{A(t-s)} B(X_s)\,dW_s, \quad \text{almost surely,}$$

(1.2)

for $t \in [0, T]$. A precise description of that equation and the assumptions that we use are given in Section 2.

In this article we concentrate in particular on the diffusion coefficient. We assume that $B : H \to L(U, D)$ is a strongly measurable mapping from H into the space of bounded linear operators from U to D, where D is a (possibly) larger space than H. In addition we assume $e^{At}(D) \subset H$ for every $t > 0$ and therefore $e^{At}B(v)$ is a well-defined linear operator from U to H for every $v \in H, t > 0$. Then, instead of Lipschitz assumptions on B directly, we assume that the mapping $v \mapsto e^{At}B(v)$ satisfies a linear growth bound and a global Lipschitz condition with respect to the Hilbert–Schmidt norm for each $t > 0$ with the constants depending of a fractional power of t (see also [4, Theorem 5.3.1]). This enables us to handle space–time white noise and trace-class noise as well as additive and multiplicative noise in one setting. This is useful when, for example, one investigates Taylor expansions and numerical approximations of the solutions of such SPDEs [5, 6, 8].

This article is organized as follows. In the next section, we describe more exactly the SPDE that we are considering and state the assumptions that we require on its terms and coefficients and on the initial value. We state our existence and uniqueness theorem and a second theorem on the regularity of solutions in the third section. The proofs are given in the Section 5 after we give two examples which satisfy our assumptions in Section 4, one with space–time white noise and one with trace-class noise.

In conclusion, we note that we assume that the drift coefficient F satisfies a global Lipschitz condition, but this is only a convenience which allows us to focus on the noise terms here. Our results can be extended to more general drift coefficients such as those satisfying a dissipativity condition as in [1, 2, 7].

2. Setting and assumptions

Fix $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $\mathcal{F}_t, t \in [0, T]$, see for example [3] for details. In addition, let $(H, \langle \cdot, \cdot \rangle_H)$ and $(U, \langle \cdot, \cdot \rangle_U)$ be two separable $\mathbb{R}$-Hilbert spaces with norms denoted by $\| \cdot \|_H$ and $\| \cdot \|_U$, respectively. Moreover, let $(D, \| \cdot \|_D)$ be a separable $\mathbb{R}$-Banach space with $H \subset D$ continuously and let $L(U, D)$ be the $\mathbb{R}$-Banach space of all bounded linear operators from U to D.

We consider the SPDE (1.1) in the mild integral equation form (1.2) on H, where $W_t, t \in [0, T]$, is a cylindrical Q-Wiener process on U with respect to $\mathcal{F}_t, t \in [0, T]$ for which the covariance operator $Q = I$ is the identity on U (see [9]) and where the objects A, F, B and x_0 are specified through the following assumptions.

ASSUMPTION 2.1 (Linear operator A). Let $\mathcal{I}$ be a finite or countable set. Moreover, let $(\lambda_i)_{i \in \mathcal{I}}$ be a family of positive real numbers with $\inf_{i \in \mathcal{I}} \lambda_i > 0$ and let $(e_i)_{i \in \mathcal{I}}$ be an orthonormal basis of H. Then, suppose that the linear operator $A : D(A) \subset H \to H$ is given by

$$Av = \sum_{i \in \mathcal{I}} -\lambda_i \langle e_i, v \rangle_H e_i$$

for all $v \in D(A)$ with $D(A) = \left\{ v \in H : \sum_{i \in \mathcal{I}} |\lambda_i|^2 |\langle e_i, v \rangle_H|^2 < \infty \right\}$.

ASSUMPTION 2.2 (Drift F). The mapping $F : H \to H$ is globally Lipschitz continuous with respect to $\| \cdot \|_H$.

Let $D((-A)^r)$, $r \in \mathbb{R}$, denote the interpolation spaces of powers of the operator $-A$ (see, for example, [10]) and let $\| \cdot \|_{HS}$ denote the Hilbert–Schmidt norm for Hilbert–Schmidt operators from U to H.

ASSUMPTION 2.3 (Diffusion B). Suppose that $D \subset D((-A)^{-r})$ continuously for some $r \geq 0$ and that $B : H \to L(U, D)$ is a strongly measurable mapping such that $e^{At} B(v)$ is a Hilbert–Schmidt operator from U to H and

$$\|e^{At} B(v)\|_{HS} \leq L(1 + \|v\|_H) t^{(\varepsilon - 1/2)}$$
$$\|e^{At}(B(v) - B(w))\|_{HS} \leq L \|v - w\|_H t^{(\varepsilon - 1/2)}$$

for all $v, w \in H$ and $t \in (0, T]$, where $L > 0$ and $\varepsilon > 0$ are given constants.

By the assumed strong measurability we mean that the mappings

$$H \to D, \quad v \mapsto B(v)u, \quad v \in H$$

are $\mathcal{B}(H)/\mathcal{B}(D)$-measurable for every $u \in U$. If $B : H \to L(U, D)$ is assumed to be $\mathcal{B}(H)/\mathcal{B}(L(U, D))$-measurable, then in particular it is strongly measurable.

ASSUMPTION 2.4 (Initial value x_0). Let $p \in [2, \infty)$ be given and suppose that $x_0 : \Omega \to H$ is a $\mathcal{F}_0/\mathcal{B}(H)$-measurable mapping with $\mathbb{E}\|x_0\|_H^p < \infty$.

3. Existence, uniqueness and regularity results

The main result of this article is that the SPDE (1.1) has a unique solution up to modifications under the assumptions given in Section 2.

THEOREM 3.1. *Let assumptions 2.1–2.4 be satisfied. Then, there is a unique (up to modifications) predictable stochastic process $X : \Omega \times [0, T] \to H$ with $\sup_{0 \leq t \leq T} \mathbb{E}\|X_t\|_H^p < \infty$, where $p \geq 2$ is given in Assumption 2.4, such that*

$$\mathbb{P}\left[X_t = e^{At} x_0 + \int_0^t e^{A(t-s)} F(X_s) \, ds + \int_0^t e^{A(t-s)} B(X_s) \, dW_s \right] = 1 \qquad (3.1)$$

for all $t \in [0, T]$. Furthermore, X is the unique mild solution of the SPDE (1.1) in this sense.

Note that the integrals in Equation (3.1) are well defined under the Assumptions 2.1–2.4 (see the proof of Theorem 3.1). Finally, we also obtain the following regularity property of the solution if further assumptions on $e^{At}B(\cdot)$ are satisfied.

THEOREM 3.2. *Let Assumptions* 2.1–2.4 *be satisfied and let* $\gamma \in (0, 1)$ *be such that* $\mathbb{E}\|(-A)^{\gamma} x_0\|_H^p < \infty$. *Furthermore, suppose that* $(-A)^{\gamma} e^{At} B(v)$ *is a Hilbert–Schmidt operator from* U *to* H *with*

$$\|(-A)^{\gamma} e^{At} B(v)\|_{\mathrm{HS}} \le L(1 + \|v\|_H) t^{(\varepsilon - 1/2)}$$

for all $v \in H$ *and all* $t \in (0, T]$ *with constants* $L > 0$ *and* $\varepsilon > 0$. *Then, the unique solution process* $X : \Omega \times [0, T] \to H$ *of the SPDE* (1.1) *given by Theorem* 3.1 *satisfies* $\sup_{0 \le t \le T} \mathbb{E}\|(-A)^{\gamma} X_t\|_H^p < \infty$.

4. Examples

As an example in which our assumptions hold, we consider the following situation. Let $O := (0, 1)^d \subset \mathbb{R}^d$ with $d \in \mathbb{N}$ and let $H = L^2(O, \mathbb{R})$ be the Hilbert space of all square integrable functions from O to $\mathbb{R}$, with scalar product and the norm

$$\langle v, w \rangle_H = \int_O v(x) w(x)\, dx, \quad \|v\|_H = \left(\int_O v(x)^2\, dx \right)^{1/2}$$

for every $v, w \in H$. (In fact, H is the Hilbert space of equivalence classes of square integrable functions, but for convenience we do not take care of this differentiation.) We also define $U := H$. Moreover, let $A = \vartheta \Delta$ with $\vartheta > 0$ be a constant times the Laplacian with Dirichlet boundary conditions, so $\mathcal{I} = \mathbb{N}^d$ and

$$e_i(x) = 2^{d/2} \sin(i_1 \pi x_1) \cdots \sin(i_d \pi x_d), \quad \lambda_i = \vartheta \pi^2 (i_1^2 + \cdots + i_d^2)$$

for all $x = (x_1, \ldots, x_d) \in O$ and all $i = (i_1, \ldots, i_d) \in \mathcal{I}$. The operator A is then given by

$$Af = \sum_{i \in \mathcal{I}} -\lambda_i \langle e_i, f \rangle_H e_i$$

for all $f \in D(A)$ with

$$D(A) = \left\{ f \in H \mid \sum_{i \in \mathcal{I}} \lambda_i^2 |\langle e_i, f \rangle_H|^2 < \infty \right\}.$$

Thus, Assumption 2.1 holds here.

Furthermore, let $f, g : \mathbb{R} \to \mathbb{R}$ be two globally Lipschitz continuous functions, that is,

$$|g(x) - g(y)| \le L|x - y|, \quad |f(x) - f(y)| \le L|x - y|$$

for all x, $y \in \mathbb{R}$ with a constant $L > 0$, and define

$$F : H \to H, \quad F(v)(x) = f(v(x)), \quad x \in (0, 1)^d \tag{4.1}$$

and

$$G : H \to H, \quad G(v)(x) = g(v(x)), \quad x \in (0, 1)^d \tag{4.2}$$

for all $v \in H$. Hence, F and G are also globally Lipschitz continuous functions on H, that is,

$$\|G(v) - G(w)\|_H \leq L\|v - w\|_H, \quad \|F(v) - F(w)\|_H \leq L\|v - w\|_H$$

for all v, $w \in H$, and Assumption 2.2 holds.

To discuss Assumption 2.3 we now consider separately the two cases of space–time white noise and trace-class noise.

4.1. Space–time white noise. Let $d = 1$ and $D = L^1(0, 1)$. Then, we define B by

$$B : H \to L(H, D), \quad (B(v)(w))(x) := (G(v))(x) \cdot w(x) \tag{4.3}$$

for every $x \in (0, 1)$ and v, $w \in H$. Indeed, B is well defined, since, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\|B(v)(w)\|_D &= \int_0^1 |G(v)(x) \cdot w(x)| \, dx \\
&\leq \left(\int_0^1 |G(v)(x)|^2 \, dx \right)^{1/2} \left(\int_0^1 |w(x)|^2 \, dx \right)^{1/2} = \|G(v)\|_H \cdot \|w\|_H
\end{aligned}$$

for all v, $w \in H$, so $B(v)$ is indeed a bounded linear operator from H to D with the property

$$\|B(v)\|_{L(H,D)} \leq \|G(v)\|_H$$

for all $v \in H$. In the same way, we obtain

$$\|B(v) - B(w)\|_{L(H,D)} \leq \|G(v) - G(w)\|_H \leq L\|v - w\|_H$$

for all v, $w \in H$, since G is globally Lipschitz continuous. Hence, B is also a globally Lipschitz continuous function from H to $L(H, D)$ and is, in particular, measurable. Combining the definitions in (4.1)–(4.3), we obtain

$$B : H \to L(H, D), \quad (B(v)(w))(x) := g(v(x)) \cdot w(x) \quad \forall x \in (0, 1)$$

and every $v, w \in H$. In the next step, let $\gamma \geq 0$. Then, of course, $(-A)^\gamma e^{At} B(v)$ is a bounded linear operator from H to H for every $v \in H$ and $t \in (0, T]$. Moreover,

$$
\begin{aligned}
\|(-A)^\gamma e^{At} B(v)\|_{\mathrm{HS}}^2 &= \sum_{i \in \mathcal{I}} \|(-A)^\gamma e^{At} B(v) e_i\|_H^2 \\
&= \sum_{i \in \mathcal{I}} \left\| (-A)^\gamma e^{At} \left(\sum_{j \in \mathcal{I}} \langle e_j, B(v) e_i \rangle_H e_j \right) \right\|_H^2 \\
&= \sum_{i \in \mathcal{I}} \left\| \sum_{j \in \mathcal{I}} (-A)^\gamma e^{At} e_j \langle e_j, B(v) e_i \rangle_H \right\|_H^2 \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} ((\lambda_j)^{2\gamma} e^{-2\lambda_j t} |\langle e_j, B(v) e_i \rangle_H|^2)
\end{aligned}
$$

for all $v \in H$, $t \in (0, T]$. Therefore, the definition of B yields

$$
\begin{aligned}
\|(-A)^\gamma e^{At} B(v)\|_{\mathrm{HS}}^2 &= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} ((\lambda_j)^{2\gamma} e^{-2\lambda_j t} |\langle e_i, B(v) e_j \rangle_H|^2) \\
&= \sum_{j \in \mathcal{I}} \left((\lambda_j)^{2\gamma} e^{-2\lambda_j t} \left(\sum_{i \in \mathcal{I}} |\langle e_i, B(v) e_j \rangle_H|^2 \right) \right) \\
&= \sum_{j \in \mathcal{I}} ((\lambda_j)^{2\gamma} e^{-2\lambda_j t} \|B(v) e_j\|_H^2)
\end{aligned}
$$

for all $v \in H$ and all $t \in (0, T]$. Since $|e_j(x)| \leq \sqrt{2}$ for every $j \in \mathcal{I} = \mathbb{N}$ and $x \in (0, 1)$, we obtain

$$
\begin{aligned}
\|(-A)^\gamma e^{At} B(v)\|_{\mathrm{HS}} &\leq \left(\sum_{j \in \mathcal{I}} ((\lambda_j)^{2\gamma} e^{-2\lambda_j t} \, 2 \|G(v)\|_H^2) \right)^{1/2} \\
&= \sqrt{2} \|G(v)\|_H \|(-A)^\gamma e^{At}\|_{\mathrm{HS}}
\end{aligned}
$$

for all $v \in H$, $t \in (0, T]$. Suppose now that $\gamma \in [0, \frac{1}{4})$. Then, we obtain

$$
\begin{aligned}
\|(-A)^\gamma e^{At}\|_{\mathrm{HS}}^2 &= \sum_{j=1}^\infty (\lambda_j)^{2\gamma} \exp(-2\lambda_j t) = \sum_{j=1}^\infty (\vartheta j^2 \pi^2)^{2\gamma} \exp(-2\vartheta j^2 \pi^2 t) \\
&\leq \vartheta^{2\gamma} \pi^{4\gamma} \left(\int_0^\infty (x^{4\gamma} + 1) \exp\left(-\frac{1}{2} (2\sqrt{\vartheta t} \pi x)^2 \right) dx \right) \\
&\leq \frac{\vartheta^{2\gamma} \pi^{4\gamma}}{2\sqrt{\vartheta t} \pi} \left(\int_{-\infty}^\infty \left(\left(\frac{|x|}{2\sqrt{\vartheta t} \pi} \right)^{4\gamma} + 1 \right) \exp(-x^2/2) \, dx \right) \\
&= \frac{\vartheta^{(2\gamma - \frac{1}{2})} \pi^{(4\gamma - 1)}}{2\sqrt{t}} \left(\int_{-\infty}^\infty \left(\frac{|x|}{2\sqrt{\vartheta t} \pi} \right)^{4\gamma} \exp(-x^2/2) \, dx + \sqrt{2\pi} \right)
\end{aligned}
$$

for every $t \in (0, T]$. Hence,

$$
\begin{aligned}
\|(-A)^{\gamma} e^{At}\|_{\mathrm{HS}}^2 &\leq \frac{\vartheta^{(2\gamma - \frac{1}{2})}\sqrt{\pi}}{\sqrt{2t}}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{|x|}{\sqrt{\vartheta t}}\right)^{4\gamma}e^{-x^2/2}\,dx + 1\right) \\
&= \frac{\vartheta^{(2\gamma - \frac{1}{2})}\sqrt{\pi}}{\sqrt{2t}}\left(\frac{(\vartheta t)^{-2\gamma}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|^{4\gamma}e^{-x^2/2}\,dx + 1\right) \\
&\leq \frac{\vartheta^{(2\gamma - \frac{1}{2})}\sqrt{\pi}}{\sqrt{2t}}\left(\frac{(\vartheta t)^{-2\gamma}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1 + x^2)e^{-x^2/2}\,dx + 1\right) \\
&= \frac{\vartheta^{(2\gamma - \frac{1}{2})}\sqrt{\pi}}{\sqrt{2t}}(2(\vartheta t)^{-2\gamma} + 1) \leq \frac{\vartheta^{(2\gamma - \frac{1}{2})}\sqrt{2\pi}}{\sqrt{t}}((\vartheta t)^{-2\gamma} + 1) \\
&= \sqrt{2\pi}(\vartheta^{-\frac{1}{2}}t^{(-\frac{1}{2}-2\gamma)} + \vartheta^{(-\frac{1}{2}+2\gamma)}t^{-\frac{1}{2}}) \\
&\leq \frac{\sqrt{2\pi}(1 + T^{2\gamma})}{\min(\vartheta, 1)}t^{(-\frac{1}{2}-2\gamma)}
\end{aligned}
$$

which yields

$$
\|(-A)^{\gamma} e^{At}\|_{\mathrm{HS}} \leq \left(\frac{\sqrt{\sqrt{2\pi}}\sqrt{T+2}}{\min(\vartheta, 1)}\right)t^{-(\frac{1}{4}+\gamma)}
$$

for every $t \in (0, T]$. Hence, we obtain

$$
\|(-A)^{\gamma} e^{At} B(v)\|_{\mathrm{HS}} \leq \left(\frac{4(T+1)}{\min(\vartheta, 1)}\right)\|G(v)\|_H\, t^{-(\frac{1}{4}+\gamma)}
$$

for every $v \in H$, $t \in (0, T]$ and $\gamma \in [0, \frac{1}{4})$. In the same way, we can show that

$$
\|(-A)^{\gamma} e^{At}(B(v) - B(w))\|_{\mathrm{HS}} \leq \left(\frac{4(T+1)}{\min(\vartheta, 1)}\right)\|G(v) - G(w)\|_H t^{-(\frac{1}{4}+\gamma)}
$$

for every $v, w \in H$, $t \in (0, T]$ and $\gamma \in [0, \frac{1}{4})$. Thus, Assumption 2.3 holds. Finally, if the initial value satisfies Assumption 2.4, then the SPDE

$$
dX_t(x) = [\Delta X_t(x) + f(X_t(x))]\,dt + g(X_t(x))\,dW_t, \quad X_0(x) = x_0(x),\ x \in (0, 1)
$$

has a unique solution by Theorem 3.1 and if the initial value also satisfies

$$
\mathbb{E}\|(-A)^{\gamma} x_0\|^p < \infty \quad \text{with } \gamma \in [0, \tfrac{1}{4}) \text{ and } p \in [2, \infty),
$$

then the solution has values in $D((-A)^{\gamma})$ almost surely in the sense of Theorem 3.2. In particular, the equation with additive noise

$$
dX_t(x) = [\Delta X_t(x) + f(X_t(x))]\,dt + dW_t, \quad x \in (0, 1)
$$

with $g(y) \equiv 1$ and the stochastic heat equation with linear multiplicative noise

$$
dX_t(x) = \Delta X_t(x)\,dt + X_t(x)\,dW_t, \quad x \in (0, 1)
$$

with $g(y) = y$ and $f(y) = 0$ for all $y \in \mathbb{R}$ both have unique solutions.

4.2. Trace-class noise. Let $d \in \mathbb{N}$, $D = H$ and let $(f_i)_{i \in \mathcal{I}}$ be another orthonormal basis in H with the property that $f_i : \bar{O} \to \mathbb{R}$ are continuous functions, which satisfy

$$\sup_{i \in \mathcal{I}} \sup_{x \in \bar{O}} |f_i(x)| < \infty.$$

Moreover, let $\sqrt{Q} : H \to H$ be a bounded linear operator given by

$$\sqrt{Q}v = \sum_{i \in \mathcal{I}} \alpha_i f_i \langle f_i, v \rangle_H, \quad v \in H$$

for real numbers $(\alpha_i)_{i \in \mathcal{I}}$ satisfying $\sum_{i \in \mathcal{I}} \alpha_i^2 < \infty$. Then $\sqrt{Q}v : \bar{O} \to H$ is a continuous mapping with

$$\sup_{x \in \bar{O}} |(\sqrt{Q}v)(x)| \leq c\|v\|_H = c\sqrt{\int_O |v(x)|^2 \, dx}$$

for every $v \in H$, where the constant $c \in [0, \infty)$ is given by

$$c = \left(\left(\sum_{i \in I} \alpha_i^2 \right)^{1/2} \left(\sup_{i \in \mathcal{I}} \sup_{x \in \bar{O}} |f_i(x)| \right) \right) = \|\sqrt{Q}\|_{\mathrm{HS}} \left(\sup_{i \in \mathcal{I}} \sup_{x \in \bar{O}} |f_i(x)| \right).$$

Indeed, for $v \in H$, we obtain

$$\begin{aligned}
\sup_{x \in \bar{O}} |(\sqrt{Q}v)(x)| &= \sup_{x \in \bar{O}} \left| \sum_{i \in I} \alpha_i f_i(x) \langle f_i, v \rangle_H \right| \\
&\leq \sum_{i \in I} \left(|\alpha_i| \cdot |\langle f_i, v \rangle_H| \cdot \sup_{x \in \bar{O}} |f_i(x)| \right) \\
&\leq \sum_{i \in I} (|\alpha_i| \cdot |\langle f_i, v \rangle_H|) \left(\sup_{i \in \mathcal{I}} \sup_{x \in \bar{O}} |f_i(x)| \right) \\
&\leq \left(\sum_{i \in I} \alpha_i^2 \right)^{1/2} \left(\sum_{i \in I} |\langle f_i, v \rangle_H|^2 \right)^{1/2} \left(\sup_{i \in \mathcal{I}} \sup_{x \in \bar{O}} |f_i(x)| \right) \\
&\leq \left(\sum_{i \in I} \alpha_i^2 \right)^{1/2} \left(\sup_{i \in \mathcal{I}} \sup_{x \in \bar{O}} |f_i(x)| \right) \|v\|_H = c\|v\|_H.
\end{aligned}$$

In the next step, we define B by

$$B : H \to L(H, D), \quad (B(v)(w))(x) := (G(v))(x) \cdot (\sqrt{Q}w)(x) \tag{4.4}$$

for every $x \in (0, 1)^d$ and $v, w \in H$. Indeed, B is well defined, since

$$\|B(v)(w)\|_D = \left(\int_{(0,1)^d} |G(v)(x) \cdot (\sqrt{Q}w)(x)|^2 \, dx \right)^{1/2}$$

$$\leq \left(\int_{(0,1)^d} |G(v)(x)|^2 \, dx \right)^{1/2} \left(\sup_{x \in \bar{O}} |(\sqrt{Q}w)(x)| \right)$$

$$= \|G(v)\|_H \left(\sup_{x \in \bar{O}} |(\sqrt{Q}w)(x)| \right) \leq c\|G(v)\|_H \|w\|_H$$

for all $v, w \in H$. Therefore, $B(v)$ is a bounded linear operator from H to $H = D$ with the property

$$\|B(v)\|_{L(H,D)} \leq c\|G(v)\|_H$$

for all $v \in H$. In the same way, we obtain

$$\|B(v) - B(w)\|_{L(H,D)} \leq c\|G(v) - G(w)\|_H \leq c\, L\|v - w\|_H$$

for all $v, w \in H$, since G is globally Lipschitz continuous. Hence, B is also a globally Lipschitz continuous function from H to $L(H, H)$. Combining the definitions in (4.1)–(4.2) and (4.4), we obtain that the operator $B : H \to L(H, D)$ is defined by

$$(B(v)(w))(x) := g(v(x)) \cdot (\sqrt{Q}w)(x) \quad x \in (0, 1)^d$$

for every $v, w \in H$. Let $\gamma \in [0, 1)$. Hence, $(-A)^\gamma e^{At} B(v)$ is a linear bounded operator from H to H and

$$\|(-A)^\gamma e^{At} B(v)\|_{\mathrm{HS}} \leq \|(-A)^\gamma e^{At}\|_{L(H,H)} \|B(v)\|_{\mathrm{HS}} \leq t^{-\gamma} \|B(v)\|_{\mathrm{HS}}$$

$$= t^{-\gamma} \left(\sum_{i \in \mathcal{I}} \|B(v)(f_i)\|_H^2 \right)^{1/2}$$

$$= t^{-\gamma} \left(\sum_{i \in \mathcal{I}} \alpha_i^2 \left\| B(v)\left(\frac{f_i}{\alpha_i}\right) \right\|_H^2 \right)^{1/2}$$

$$\leq t^{-\gamma} \left(\sum_{i \in \mathcal{I}} \alpha_i^2 \right)^{1/2} \left(\sup_{i \in \mathcal{I}} \left\| B(v)\left(\frac{f_i}{\alpha_i}\right) \right\|_H \right)$$

$$\leq t^{-\gamma} \left(\sum_{i \in \mathcal{I}} \alpha_i^2 \right)^{1/2} \left(\sup_{i \in \mathcal{I}} \|G(v)\|_H \left(\sup_{x \in \bar{O}} \left| \left(\sqrt{Q}\frac{f_i}{\alpha_i}\right)(x) \right| \right) \right)$$

$$= t^{-\gamma} \left(\sum_{i \in \mathcal{I}} \alpha_i^2 \right)^{1/2} \|G(v)\|_H \left(\sup_{i \in \mathcal{I}} \sup_{x \in \bar{O}} |f_i(x)| \right)$$

$$= c\|G(v)\|_H t^{-\gamma}$$

for every $t > 0$ and $v \in H$ (see also [8, Remark 1]). In the same way, we obtain

$$\|(-A)^\gamma e^{At} (B(v) - B(w))\|_{\mathrm{HS}} \leq c\|G(v) - G(w)\| t^{-\gamma}$$

for every $t > 0$ and every $v, w \in H$. This yields that Assumption 2.3 is fulfilled. Finally, if the initial value satisfies Assumption 2.4, then the SPDE

$$dX_t(x) = [\Delta X_t(x) + f(X_t(x))] \, dt + g(X_t(x))\sqrt{Q} \, dW_t, \quad X_0(x) = x_0(x)$$

with $x \in (0, 1)^d$ has a unique solution by Theorem 3.1 and if the initial value also satisfies $\mathbb{E}\|(-A)^\gamma x_0\|_H^p < \infty$ with $\gamma \in [0, \frac{1}{2})$, then the solution has values in $D((-A)^\gamma)$ almost surely in the sense of Theorem 3.2. In particular, the equation with additive noise

$$dX_t(x) = [\Delta X_t(x) + f(X_t(x))] \, dt + \sqrt{Q} \, dW_t, \quad x \in (0, 1)^d$$

with $g(y) \equiv 1$ and the stochastic heat equation with linear multiplicative noise

$$dX_t(x) = \Delta X_t(x) \, dt + X_t(x)\sqrt{Q} \, dW_t, \quad x \in (0, 1)^d$$

with $g(y) = y$ and $f(y) = 0$ for all $y \in \mathbb{R}$ both have unique solutions.

5. Proofs

Let $(V, \| \cdot \|)$ be a normed vector space and denote the L^q-norm for $q \in [1, \infty)$ of a $\mathcal{F}/\mathcal{B}(V)$-measurable mapping $Z : \Omega \to V$ by $\|Z\|_{L^q} := (\mathbb{E}\|Z\|^q)^{1/q}$. We need the following version of the Burkholder–Davis–Gundy inequality in infinite dimensions (see [3, Lemma 7.7]).

LEMMA 5.1. *Let $X : [0, T] \times \Omega \to HS(U, H)$ be a predictable stochastic process with $\mathbb{E} \int_0^T \|X_s\|_{\mathrm{HS}}^2 \, ds < \infty$. Then, we obtain*

$$\left\| \int_0^t X_s \, dW_s \right\|_{L^q} \leq q \left(\int_0^t \| \, \|X_s\|_{HS}\|_{L^q}^2 \, ds \right)^{1/2}$$

for every $t \in [0, T]$ and every $q \in [2, \infty)$. Both sides could be infinite.

PROOF OF THEOREM 3.1. Let $p \geq 2$ be given by Assumption 2.4. First, we introduce the $\mathbb{R}$-vector space $\mathcal{V}_p$ of all equivalence classes of predictable stochastic processes $X : \Omega \times [0, T] \to H$ with $\sup_{0 \leq t \leq T} \|X_t\|_{L^p} < \infty$, where all stochastic processes that are modifications of each other lie in one equivalence class. Then, we equip this space with the norms $\|X\|_\mu := \sup_{0 \leq t \leq T} e^{\mu t}\|X_t\|_{L^p}$ for every $X \in \mathcal{V}_p$ and every $\mu \in \mathbb{R}$. Note that the pair $(\mathcal{V}_p, \| \cdot \|_\mu)$ is a Banach space for every $\mu \in \mathbb{R}$.

Now, we consider the mapping $\phi : \mathcal{V}_p \to \mathcal{V}_p$ given by

$$(\phi X)_t := e^{At}x_0 + \int_0^t e^{A(t-s)} F(X_s) \, ds + \int_0^t e^{A(t-s)} B(X_s) \, dW_s$$

for every $t \in [0, T]$ and $X \in \mathcal{V}_p$. First, we show that ϕ is well defined. Given $t \in [0, T]$ and $X \in \mathcal{V}_p$, the mapping from $[0, t] \times \Omega$ to $HS(U, H)$ (the space of Hilbert–Schmidt operators from U to H) defined by

$$(s, \omega) \to e^{A(t-s)} B(X_s(\omega))$$

for every $s \in [0, t]$ and $\omega \in \Omega$ is a predictable stochastic process, since e^{As} is continuous in $L(D, H)$ for $s \in (0, T]$ and since $B(\,\cdot\,)$ is strongly measurable by Assumption 2.3. Hence,

$$
\begin{aligned}
\left\| \int_0^t e^{A(t-s)} B(X_s)\, dW_s \right\|_{L^p} &\leq p \left(\int_0^t \| \, \| e^{A(t-s)} B(X_s) \|_{\mathrm{HS}} \|_{L^p}^2 \, ds \right)^{1/2} \\
&\leq p \left(\int_0^t \| L(1 + \|X_s\|_H)(t-s)^{(\varepsilon - 1/2)} \|_{L^p}^2 \, ds \right)^{1/2} \\
&\leq Lp \left(\int_0^t (1 + \|X_s\|_{L^p})^2 (t-s)^{(2\varepsilon - 1)} ds \right)^{1/2}
\end{aligned}
$$

due to Lemma 5.1 and Assumption 2.3 and, therefore, we obtain

$$
\begin{aligned}
\left\| \int_0^t e^{A(t-s)} B(X_s)\, dW_s \right\|_{L^p} &\leq Lp \left(1 + \sup_{0 \leq s \leq T} \|X_s\|_{L^p} \right) \left(\int_0^t s^{(2\varepsilon - 1)} \, ds \right)^{1/2} \\
&\leq \frac{Lp}{\sqrt{2\varepsilon}} \left(1 + \sup_{0 \leq s \leq T} \|X_s\|_{L^p} \right) t^\varepsilon < \infty
\end{aligned}
$$

for every $t \in (0, T]$. By Lebesgue's theorem one can show that $\int_0^t e^{A(t-s)} B(X_s)\, dW_s$ for $t \in [0, T]$ is mean square continuous and, therefore, it has a predictable version. The calculation for the second integral in the definition of ϕ is similar and, hence, we obtain that ϕ is well defined.

Now, we show that ϕ is a contraction with respect to $\| \cdot \|_\mu$ for an appropriate $\mu \in \mathbb{R}$. From Assumption 2.2,

$$
\|F(v) - F(w)\|_H \leq K \|v - w\|_H
$$

for all $v, w \in H$ with a constant $K > 0$. Hence, for $X, Y \in \mathcal{V}_p$ and $t \in [0, T]$ we obtain

$$
\begin{aligned}
(\phi X)_t - (\phi Y)_t = \int_0^t & e^{A(t-s)} (F(X_s) - F(Y_s)) \, ds \\
&+ \int_0^t e^{A(t-s)} (B(X_s) - B(Y_s)) \, dW_s \quad \text{almost surely,}
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\|(\phi X)_t - (\phi Y)_t\|_{L^p} &\leq \int_0^t \| e^{A(t-s)} (F(X_s) - F(Y_s)) \|_{L^p} \, ds \\
&\quad + \left\| \int_0^t e^{A(t-s)} (B(X_s) - B(Y_s)) \, dW_s \right\|_{L^p} \\
&\leq K \left(\sup_{0 \leq s \leq T} \| e^{As} \|_{L(H,H)} \right) \int_0^t \|X_s - Y_s\|_{L^p} \, ds
\end{aligned}
$$

$$+ p\left(\int_0^t \|\|e^{A(t-s)}(B(X_s) - B(Y_s))\|_{\mathrm{HS}}\|_{L^p}^2 \, ds\right)^{1/2}$$

$$\leq K\left(\int_0^t e^{-\mu s} \, ds\right)\|X - Y\|_\mu$$

$$+ p\left(\int_0^t \|L(t-s)^{(\varepsilon-1/2)}\|X_s - Y_s\|_H\|_{L^p}^2 \, ds\right)^{1/2}$$

for $\mu \in \mathbb{R}$ by Lemma 5.1. This yields

$$e^{\mu t}\|(\phi X)_t - (\phi Y)_t\|_{L^p} \leq K\left(\int_0^t e^{\mu s} \, ds\right)\|X - Y\|_\mu$$

$$+ Lp\left(\int_0^t (t-s)^{(2\varepsilon-1)}e^{2\mu t}\|X_s - Y_s\|_{L^p}^2 \, ds\right)^{1/2}$$

$$\leq \frac{K}{\mu}(e^{\mu t} - 1)\|X - Y\|_\mu$$

$$+ Lp\left(\int_0^t (t-s)^{(2\varepsilon-1)}e^{2\mu(t-s)} \, ds\right)^{1/2}\|X - Y\|_\mu$$

for every $\mu < 0$, $t \in (0, T]$ and, hence,

$$\|\phi X - \phi Y\|_\mu \leq \frac{K}{|\mu|}\|X - Y\|_\mu + Lp\left(\int_0^T s^{(2\varepsilon-1)}e^{2\mu s} \, ds\right)^{1/2}\|X - Y\|_\mu$$

$$\leq \left(\frac{K}{|\mu|} + Lp\sqrt{\int_0^T s^{(2\varepsilon-1)}e^{2\mu s} \, ds}\right)\|X - Y\|_\mu$$

for every $\mu < 0$.

Finally, for $\mu \to -\infty$, we see that ϕ is a contraction with respect to $\|\cdot\|_\mu$, so there is a unique element $X \in \mathcal{V}_p$ with $X = \phi X$. This is the assertion that was to be proved. $\square$

PROOF OF THEOREM 3.2. The solution process X satisfies

$$\mathbb{P}\left[X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}F(X_s) \, ds + \int_0^t e^{A(t-s)}B(X_s) \, dW_s\right] = 1$$

for every $t \in [0, T]$.

First,

$$\|(-A)^\gamma e^{At}x_0\|_{L^p} = \|e^{At}(-A)^\gamma x_0\|_{L^p} \leq \|(-A)^\gamma x_0\|_{L^p} < \infty$$

for every $t \in [0, T]$, where $p \geq 2$ is given in Assumption 2.4.

Next, we obtain

$$\left\| (-A)^\gamma \int_0^t e^{A(t-s)} F(X_s)\, ds \right\|_{L^p} \leq \int_0^t \|(-A)^\gamma e^{A(t-s)} F(X_s)\|_{L^p}\, ds$$

$$\leq \int_0^t \|(-A)^\gamma e^{A(t-s)}\|_{L(H,H)} \|F(X_s)\|_{L^p}\, ds$$

$$\leq \int_0^t (t-s)^{-\gamma} K(1 + \|X_s\|_{L^p})\, ds$$

for $t \in [0, T]$, where we used the fact that $\|F(v)\|_H \leq K(1 + \|v\|_H)$ for all $v \in H$ with a constant $K > 0$. This yields

$$\left\| (-A)^\gamma \int_0^t e^{A(t-s)} F(X_s)\, ds \right\|_{L^p} \leq K\Big(1 + \sup_{0 \leq s \leq T} \|X_s\|_{L^p}\Big)\Big(\int_0^t s^{-\gamma}\, ds\Big)$$

$$\leq K\Big(1 + \sup_{0 \leq s \leq T} \|X_s\|_{L^p}\Big)\frac{t^{(1-\gamma)}}{(1-\gamma)}$$

$$\leq \frac{K(T+1)}{(1-\gamma)}\Big(1 + \sup_{0 \leq s \leq T} \|X_s\|_{L^p}\Big) < \infty$$

for every $t \in [0, T]$. Finally, we obtain

$$\left\| (-A)^\gamma \int_0^t e^{A(t-s)} B(X_s)\, dW_s \right\|_{L^p} \leq p\Big(\int_0^t \big\| \|(-A)^\gamma e^{A(t-s)} B(X_s)\|_{\mathrm{HS}}\big\|_{L^p}^2\, ds\Big)^{1/2}$$

$$\leq p\Big(\int_0^t \|L(1 + \|X_s\|_H)(t-s)^{(\varepsilon-1/2)}\|_{L^p}^2\, ds\Big)^{1/2}$$

$$\leq Lp\Big(1 + \sup_{0 \leq s \leq T} \|X_s\|_{L^p}\Big)\Big(\int_0^t s^{(2\varepsilon-1)}\, ds\Big)^{1/2}$$

$$\leq Lp\Big(1 + \sup_{0 \leq s \leq T} \|X_s\|_{L^p}\Big)\frac{T^\varepsilon}{\sqrt{2\varepsilon}} < \infty$$

for all $t \in [0, T]$ due to [3, Proposition 4.15] and Lemma 5.1, which yields the assertion. $\qquad\square$

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