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RESEARCH ARTICLE

Higher Orbital Integrals, Cyclic Cocycles and Noncommutative Geometry

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Abstract
Let G be a linear real reductive Lie group. Orbital integrals define traces on the group algebra of G. We introduce a construction of higher orbital integrals in the direction of higher cyclic cocycles on the Harish-Chandra Schwartz algebra of G. We analyze these higher orbital integrals via Fourier transform by expressing them as integrals on the tempered dual of G. We obtain explicit formulas for the pairing between the higher orbital integrals and the K-theory of the reduced group C^*-algebra, and we discuss their application to K-theory.

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1. Introduction

Let G be a linear real reductive group and let f be a compactly supported smooth function on G. For $x \in G$, let $Z_G(x)$ be the centralizer of G associated with x and $d_{G/Z_G(x)}\dot{g}$ be the left invariant measure on $G/Z_G(x)$ determined by a Haar measure dg on G. The following integral

$$\Lambda_f^{Z_G(x)} := \int_{G/Z_G(x)} f(gxg^{-1}) d_{G/Z_G(x)}\dot{g}$$

is an important tool in representation theory with deep connections to number theory. Harish-Chandra showed the above integrals extend to all f in the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$, and obtained his famous Plancherel formula [9, 10, 13].

In this paper, we aim to study the noncommutative geometry of the above integral and its generalizations. Let H be a Cartan subgroup of G and K be a maximal compact subgroup of G. The Weyl group $W(H, G) = N_K(H)/Z_K(H)$ is defined as the quotient of the normalizer $N_K(H)$ by the centralizer $Z_K(H)$. Let $H^{\text{reg}} \subset H$ be the subset of regular elements. In particular, for any $x \in H^{\text{reg}}$, we have that $Z_G(x) = H$. Following Harish-Chandra, we define the orbital integral associated to H to be

$$F^H : \mathcal{C}(G) \to C^\infty(H^{\text{reg}})^{-W(H,G)},\ F_f^H(x) := \epsilon^H(x)\Delta_H^G(x) \int_{G/H} f(gxg^{-1}) d_{G/H}\dot{g}, \tag{1.1}$$

where $C^\infty(H^{\text{reg}})^{-W(H,G)}$ is the space of anti-symmetric functions with respect to the Weyl group $W(H, G)$ action on H, $\epsilon^H(h)$ is a sign function on H, and Δ_H^G is the Weyl denominator for H. Our starting point is the property that for a given $x \in H^{\text{reg}}$, the linear functional on $\mathcal{C}(G)$,

$$F^H(x) : f \mapsto F_f^H(x),$$

is a trace on $\mathcal{C}(G)$; cf. [17].

In cyclic cohomology, traces are special examples of cyclic cocycles on an algebra. In noncommutative geometry, there is a fundamental pairing between the periodic cyclic cohomology and the K-theory of an algebra. We say that a linear real reductive Lie group G is of equal rank if and only if the dimension of a Cartan subgroup of G equals the dimension of a Cartan subgroup of the maximal compact subgroup K of G. In this case, G has discrete series representations [18, Theorem 12.20]. The pairing between the orbital integrals $F^H(x)$ and $K_0(\mathcal{C}(G))$ behaves differently between the cases when G is of equal rank and nonequal rank. More explicitly, we will show in this article that when G has equal rank, F^H defines an isomorphism as abelian groups from the K-theory of $\mathcal{C}(G)$ to the representation ring of K. Nevertheless, when G has nonequal rank, F^H vanishes on K-theory of $\mathcal{C}(G)$ completely (cf. [17]). Furthermore, many numerical invariants for G-equivariant Dirac operators in the literature [1, 8, 17, 21, 31] etc., vanish when G has nonequal rank. Our main goal in this article is to introduce generalizations of orbital integrals in the sense of higher cyclic cocycles on $\mathcal{C}(G)$ which will treat equal and nonequal rank groups in a uniform way and give new interesting numerical invariants for G-equivariant Dirac operators. We

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remark that orbital integrals and cyclic (co)homology of $\mathcal{C}(G)$ were well studied in the literature (e.g., [3, 23, 24, 25, 26, 34]). Our approach here differs from prior work in its emphasis on the construction of explicit cocycles. To understand the nonequal rank case better, we start with the example of the abelian group G=$\mathbb{R}$, which turns out to be very instructive. Here, $\mathcal{C}(\mathbb{R})$ is the usual algebra of Schwartz functions on $\mathbb{R}$ with the convolution product, and it carries a nontrivial degree one cyclic cohomology. Indeed, we can define a cyclic cocycle φ on $\mathcal{C}(\mathbb{R})$ as follows (cf. [24, Prop. 1.4]:

$$\varphi(f_0, f_1) = \int_{\mathbb{R}} s \cdot f_0(-s) f_1(s)\, ds. \tag{1.2}$$

Under the Fourier transform, the convolution algebra $\mathcal{C}(\mathbb{R})$ is transformed into the Schwartz functions with pointwise multiplication. Accordingly, φ can be rewritten as a cocycle $\hat{\varphi}$ on $\mathcal{C}(\widehat{\mathbb{R}})$:

$$\hat{\varphi}(\hat{f}_0, \hat{f}_1) = \frac{1}{\sqrt{-1}} \int_{\widehat{\mathbb{R}}} \hat{f}_0 d\hat{f}_1, \tag{1.3}$$

where $\hat{f}_i \in \mathcal{C}(\hat{\mathbb{R}})$ are the Fourier transforms of f_i. Equation (1.3) is more familiar. It follows from the Connes-Hochschild-Kostant-Rosenberg theorem ([7, Theorem 46]) that $\hat{\varphi}$ generates the degree one cyclic cohomology of $\mathcal{C}(\widehat{\mathbb{R}})$, and accordingly φ generates the degree one cyclic cohomology of $\mathcal{C}(\mathbb{R})$.

It is crucial to have the identity function $s\colon \mathbb{R} \to \mathbb{R}$ in Equation (1.2) to have the integral of $\hat{f}_0 d\hat{f}_1$ on $\mathcal{C}(\widehat{\mathbb{R}})$. Our key discovery is a natural generalization of the function s on a general linear real reductive group G. Let $P = MAN$ be a cuspidal parabolic subgroup of G (sometimes we use $P = M_P A_P N_P$ to emphasize that the subgroups are associated to the parabolic subgroup P). By the Iwasawa decomposition $G = KMAN$, we can write an element $g \in G$ as

$$g = \kappa(g)\mu(g)e^{\widetilde{H}(g)}n \in KMAN = G.$$

Note that the above decomposition might not be unique, but the A-part is always unique. Let $\dim(A) = m$. By fixing a basis for the Lie algebra $\mathfrak{a}$, the map

$$\widetilde{H} = (H_1, H_2, \ldots, H_m)$$

provides us the right ingredient to generalize the cocycle φ in Equation (1.2). We introduce a generalization $\Phi_{P,x}$ for orbital integrals in Definition 3.3. For $f_0, ..., f_m \in \mathcal{C}(G)$ and semi-simple element $x \in M$, $\Phi_{P,x}$ is defined by the following integral:

$$\begin{aligned}&\Phi_{P,x}(f_0, f_1, \ldots, f_m)\\ :=& \int_{h \in M/Z_M(x)} \int_{KN} \int_{G^{\times m}} \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot H_{\tau(1)}(g_1 ... g_m k) H_{\tau(2)}(g_2 ... g_m k) \ldots H_{\tau(m)}(g_m, k)\\ &f_0\big(khxh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}\big) f_1(g_1) \ldots f_m(g_m) dg_1 \cdots dg_m dk dn dh,\end{aligned}$$

where $Z_M(x)$ is the centralizer of x in M. Though the function $\widetilde{H}$ is not a group cocycle on G, we show in Lemma 3.1 that it satisfies a kind of twisted group cocycle property, which leads us to the following theorem in Section 3.1.

Theorem I (Theorem 3.5). *Suppose that G is a linear real reductive Lie group. For any cuspidal parabolic subgroup $P = MAN$ and a semi-simple element $x \in M$, the cochain $\Phi_{P,x}$ is a continuous cyclic cocycle of degree m on $\mathcal{C}(G)$.*

Modeling on the above example of $\mathbb{R}$ (e.g., Equation (1.3)), we analyze the higher orbital integral $\Phi_{P,x}$ by computing its Fourier transform. Using Harish-Chandra's theory of orbital integrals and character formulas for parabolically induced representations, we introduce in Definition 4.14 a cyclic cocycle

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$\widehat{\Phi}_x$ defined as an integral on the tempered dual space $\widehat{G}_{\text{temp}}$. In Theorem 4.15 and 4.18, we generalize Equation (1.3) to linear connected real reductive Lie groups.

As an application of our study, we compute the pairing between the K-theory of $\mathcal{C}(G)$ and $\Phi_{P,x}$. Lafforgue showed in [19] that Harish-Chandra's Schwartz algebra $\mathcal{C}(G)$ is a subalgebra of $C_r^*(G)$, stable under holomorphic functional calculus. Therefore, the K-theory of $\mathcal{C}(G)$ is isomorphic to the K-theory of the reduced group C^*-algebra $C_r^*(G)$. The structure of $C_r^*(G)$ is studied by [4, 33]. Under the assumption that G is connected, we have that [2, (4.11)],

$$K_i\big(\mathcal{C}(G)\big) = 0, \quad \text{if} \quad i \neq \dim(G/K) \mod 2.$$

That is, the K-theory group $K_*\big(\mathcal{C}(G)\big)$ is concentrated in one degree. Replacing G by $G\times\mathbb{R}$ if necessary, we can assume that $K_1\big(\mathcal{C}(G)\big)$ is trivial and all the information is contained in $K_0\big(\mathcal{C}(G)\big)$.

In [5, 6], we are able to explicitly identify a set of generators of the K-theory of $C_r^*(G)$ as a free abelian group; cf. Theorem C.3. With wave packets (Section 4.2), we construct generators $[Q_\lambda] \in K(C_r^*(G))$ in Theorem C.3, where Q_λ are matrices with entries taking values in $\mathcal{C}(G)$. Applying Harish-Chandra's theory of orbital integrals, we compute explicitly the index pairing (Section 2.3) between $[Q_\lambda]$ and $\Phi_{P_\circ,x}$, where $P_\circ$ denotes the maximal cuspidal parabolic subgroup. This would be enough for us to distinguish elements in $K_*\big(\mathcal{C}(G)\big)$. In [16, Theorem 2.1], we show that the paring between $[Q_\lambda]$ and $\Phi_{P,x}$ vanishes for $P \neq P_\circ$ or x does not lie in a compact subgroup of M.

Theorem II (Theorem 5.4). *Suppose that G is a linear connected real reductive Lie group and $P_\circ = M_\circ A_\circ N_\circ$ is the maximal cuspidal parabolic subgroup. The index pairing between periodic cyclic cohomology and K-theory*

$$HP^{\text{even}}\big(\mathcal{C}(G)\big) \otimes K_0\big(\mathcal{C}(G)\big) \to \mathbb{C}$$

is given by the following formulas:

- *for the identity element $e \in G$,*

$$\langle \Phi_{P_\circ,e}, [Q_\lambda]\rangle = \frac{1}{|W_{M_\circ\cap K}|} \cdot \sum_{w\in W_K} m\Big(\sigma^{M_\circ}(w\cdot\lambda)\Big),$$

  where $\sigma^{M_\circ}(w\cdot\lambda)$ is the discrete series representation of $M_\circ$ with Harish-Chandra parameter $w\cdot\lambda$, and $m\big(\sigma^{M_\circ}(w\cdot\lambda)\big)$ is its Plancherel measure;
- *for any $t \in T^{reg}$,*

$$\langle \Phi_{P_\circ,t}, [Q_\lambda]\rangle = \frac{\sum_{w\in W_K}(-1)^w e^{w\cdot\lambda}(t)}{\Delta_T^{M_\circ}(t)}.$$

We refer the readers to Theorem 5.4 for the notations involved in the above formulas. For the case of equal rank, the first formula was obtained in [8], in which Connes-Moscovici used the L^2-index on homogeneous spaces to detect the Plancherel measure of discrete series representations. It is interesting to point out (cf. Remark 3.6) that the higher orbital integrals $\Phi_{P_\circ,x}$ actually extend to a family of Banach subalgebras of $C_r^*(G)$ introduced by Lafforgue, [19, Definition 4.1.1]. However, we have chosen to work with the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$, as our proofs rely crucially on Harish-Chandra's theory of orbital integrals and character formulas.

Note that the higher orbital integrals $\Phi_{P,x}$ reduce to the classical ones when G is equal rank. Nevertheless, our main result, Theorem II for higher orbital integrals, is also new in the equal rank case. For example, as a corollary to Theorem II, in Corollary 5.5, we are able to detect the character information of limit of discrete series representations using the higher orbital integrals. This allows us to identify the contribution of limit of discrete series representations in the K-theory of $C_r^*(G)$ without using geometry of the homogeneous space G/K (e.g., the Connes-Kasparov index map). As an

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application, our computation of the index pairing in Theorem II suggests a natural isomorphism $\mathcal{F}^T$, Definition 5.7 and Corollary 5.8,

$$\mathcal{F}^T : K(C_r^*(G)) \to \mathrm{Rep}(K).$$

In [5, 6], we will prove that $\mathcal{F}^T$ is the inverse of the Connes-Kasparov index map,

$$\mathrm{Ind}\colon \mathrm{Rep}(K) \to K(C_r^*(G)).$$

Given a Dirac operator D on G/K, the Connes-Kasparov index map gives an element $\mathrm{Ind}(D)$ in $K(C_r^*(G))$. In this article, Theorem II and its corollaries, we study the representation theoretic information of the index pairing

$$\langle [\Phi_{P,x}], [\mathrm{Ind}(D)] \rangle.$$

As an application of this paper, in [16], we proved a topological formula for the above pairing for a G-invariant elliptic operator on a manifold X with proper cocompact G-action, generalizing the Connes-Moscovici L^2-index theorem [8]. In [27], we extended our study to G-proper manifolds with boundary and established a generalized Atiyah-Patodi-Singer index theorem.

In this article, motivated by the applications in K-theory, we introduce $\Phi_{P,x}$ as a cyclic cocycle on $\mathcal{C}(G)$. Actually, the construction of $\Phi_{P,x}$ can be generalized to construct a larger class of Hochschild cocycles for $\mathcal{C}(G)$. Our construction is also closely related to the work in [22]. In particular, the definition of $\Phi_{P,x}$ is compatible with the localization process introduced by Nistor [22, Section 4.2]. It is an interesting question to extend the construction of $\Phi_{P,x}$ to a cyclic cocycle on $A \rtimes G$ considered in [22]. Our preliminary study on examples also suggests that the construction of $\Phi_{P,x}$ in this paper generalizes to groups beyond linear connected real reductive Lie groups (e.g., real algebraic groups with the Iwasawa theory), covering of linear connected real reductive Lie groups and some p-adic groups like $SL(2, \mathbb{Q}_p)$. A careful study of the analog of the Harish-Chandra Schwartz algebra is needed to generalize Theorem I and II to these groups. We plan to report our study in this direction in a separate publication soon.

The article is organized as follows. In Section 2, we review some basics about representation theory of linear real reductive Lie groups, Harish-Chandra's Schwartz algebra and cyclic theory. We introduce the higher orbital integral $\Phi_{P,x}$ in Section 3 and prove Theorem I. The Fourier transform of the higher orbital integral is studied in Section 4 and provides a key tool in our proof of Theorem II. And in Section 5, we compute the pairing between the higher orbital integrals $\Phi_{P,x}$ and $K(C_r^*(G))$, proving Theorem II and its corollaries. For the convenience of readers, we have included in Appendix B and C a review of background material about related topics in representation theory and $K(C_r^*(G))$.

2. Preliminaries

In this article, we shall not attempt to strive for the utmost generality in the class of groups we shall consider. Instead, we shall aim for (relative) simplicity.

2.1. Linear real reductive Lie group and Cartan subgroups

Let $G \subseteq GL(n, \mathbb{R})$ be a self-adjoint group which is also the group of real points of a connected algebraic group defined over $\mathbb{R}$ (we will additionally assume that G is connected in Section 4 and 5.). For brevity, we shall simply say that G is a *linear real reductive Lie group*. In this case, the *Cartan involution* on the Lie algebra $\mathfrak{g}$ is given by $\theta(X) = -X^T$, where X^T denotes the transpose matrix of X.

Let $K = G \cap O(n)$, which is a maximal compact subgroup of G. Let $\mathfrak{k}$ be the Lie algebra of K. We have a θ-stable Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let H be a Cartan subgroup of G. Then, H has a θ-stable decomposition $H = T \times A$, where $T = H \cap K$ is the compactly embedded part and $\exp\colon \mathfrak{a} \to A$ is a bijection. Here, the Lie algebra $\mathfrak{a}$ of A is an abelian subalgebra in $\mathfrak{s}$. Any choice of a positive $\mathfrak{a}$-root

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system defines a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ in $\mathfrak{g}$ and thus defines a cuspidal parabolic subgroup $P = MAN$ in G. We say that P is the cuspidal parabolic subgroup associated to the Cartan subgroup H and vice versa.

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be an arbitrary θ-stable Cartan subalgebra of $\mathfrak{g}$, where $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{s}$. Let $\mathcal{R}(\mathfrak{h}, \mathfrak{g})$ be the associated root system. If $\alpha \in \mathcal{R}(\mathfrak{h}, \mathfrak{g})$ is a real root (that is, $\alpha|_{\mathfrak{t}} \equiv 0$), then we can apply the Cayley transform to $\mathfrak{h}$ and obtain a new Cartan subalgebra $\mathfrak{h}' = \mathfrak{t}' \oplus \mathfrak{a}'$ such that $\dim(\mathfrak{t}') > \dim(\mathfrak{t})$. Let P and P' be the two cuspidal parabolic subgroups associated to $\mathfrak{h}$ and $\mathfrak{h}'$, respectively. We say that P' is *more compact than* P. In this way, we define a partial order on the set of all cuspidal parabolic subgroups of G.

Definition 2.1. We say that a Cartan subgroup H is *maximally compact* if $\dim T$ is maximal among all θ-stable Cartan subgroups. In other words, T is a Cartan subgroup of K. We denote by $H_\circ$ the maximally compact Cartan subgroup and $P_\circ = M_\circ A_\circ N_\circ$ its associated cuspidal parabolic subgroup. We call $P_\circ$ the *maximal cuspidal parabolic subgroup*.

2.2. *Harish-Chandra's Schwartz function*

Following Harish-Chandra [12, Lemma 2], we fix a real-valued nondegenerate bilinear symmetric form B on $\mathfrak{g}$ which is invariant under the adjoint action of G on $\mathfrak{g}$ and also the Cartan involution θ. Moreover, we can choose B such that $\langle , \rangle = -B(\cdot, \theta \cdot)$ defines a K-invariant inner product on $\mathfrak{s}$. Thus, $\langle , \rangle$ induces a G-invariant Riemannian metric on G/K. For $g \in G$, we use $\|g\|$ to denote the Riemannian distance from eK to gK in G/K. For every $n \geq 0$, $X, Y \in U(\mathfrak{g})$, and $f \in C^\infty(G)$, set

$$\nu_{X,Y,n}(f) := \sup_{g \in G} \left\{ (1 + \|g\|)^n \Xi(g)^{-1} \big| L(X) R(Y) f(g) \big| \right\},$$

where L and R denote the left and right regular representations, respectively, and Ξ is the Harish-Chandra's Ξ-function [11].

Definition 2.2. The Harish-Chandra Schwartz space $\mathcal{C}(G)$ is the space of $f \in C^\infty(G)$ such that for all $n \geq 0$ and $X, Y \in U(\mathfrak{g})$, $\nu_{X,Y,n}(f) < \infty$.

The space $\mathcal{C}(G)$is a Fréchet space in the semi- norms $\nu_{X,Y,n}$. It is closed under convolution, which is a continuous operation on this space. Moreover, if G has equal rank (thus has discrete series representations), then all K-finite matrix coefficients of discrete series representations lie in $\mathcal{C}(G)$. It is proved in [19] that $\mathcal{C}(G)$ is a $*$-subalgebra of the reduced group C^*-algebra $C_r^*(G)$ that is closed under holomorphic functional calculus.

2.3. *Cyclic cohomology*

Definition 2.3. Let A be an algebra over $\mathbb{C}$.Define the space of Hochschild cochains of degree k of A to be

$$C^k(A) : = \mathrm{Hom}_{\mathbb{C}}\big(A^{\otimes(k+1)}, \mathbb{C}\big),$$

consisting of all bounded $k+1$-linear functionals on A. Define the Hochschild codifferential $b \colon C^k(A) \to C^{k+1}(A)$ by the following formula:

$$\begin{aligned} &b\Phi(a_0 \otimes \cdots \otimes a_{k+1}) \\ &= \sum_{i=0}^{k} (-1)^i \Phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) + (-1)^{k+1} \Phi(a_{k+1} a_0 \otimes a_1 \otimes \cdots \otimes a_k). \end{aligned}$$

The Hochschild cohomology of A is the cohomology of the complex $(C^*(A), b)$.

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Definition 2.4. We call a k-cochain $\Phi \in C^k(A)$ *cyclic* if for all $a_0, \ldots, a_k \in A$, it holds that

$$\Phi(a_k, a_0, \ldots, a_{k-1}) = (-1)^k \Phi(a_0, a_1, \ldots, a_k).$$

The subspace C^k_λ of cyclic cochains is closed under the Hochschild codifferential. The cyclic cohomology $HC^*(A)$ is defined by the cohomology of the subcomplex of cyclic cochains.

One can define a periodicity map $S : HC^k(A) \to HC^{k+2}(A)$; cf. [20, Section 2.2.5]. And the periodic cyclic cohomology $HP^*(A)$ (for $*$=even or odd) is defined to be the inductive limit of $HC^{2k}(A)$ (or $HC^{2k+1}(A)$). Let $R = (R_{i,j}), i, j = 1, \ldots, m$ be an idempotent in $M_m(A)$. The following formula

$$\langle [\Phi], [R] \rangle :\ = \frac{1}{k!} \sum_{i_0, \cdots, i_{2k}=1}^{m} \Phi(R_{i_0 i_1}, R_{i_1 i_2}, \ldots, R_{i_{2k} i_0})$$

defines a natural pairing between $[\Phi] \in HP^{\text{even}}(A)$ and $K_0(A)$; that is,

$$\langle \cdot , \cdot \rangle \colon HP^{\text{even}}(A) \otimes K_0(A) \to \mathbb{C}.$$

3. Higher orbital integrals

In this section, we construct higher orbital integrals as cyclic cocycles on $\mathcal{C}(G)$.

3.1. Higher cyclic cocycles

Let $P = MAN$ be a cuspidal parabolic subgroup and denote $m = \dim A$. By the Iwasawa decomposition, we have that

$$G = KMAN.$$

We define a map $\widetilde{H} \colon G \to \mathfrak{a}$ by the decomposition

$$g = \kappa(g)\mu(g)e^{\widetilde{H}(g)}n \in KMAN = G.$$

By fixing a basis for the Lie algebra $\mathfrak{a}$, we write $\widetilde{H} = (H_1, \ldots, H_m)$.

Lemma 3.1. *For any $g_0, g_1 \in G$, the function $H_i(g_1\kappa(g_0))$ does not depend on the choice of $\kappa(g_0)$. Moreover, the following identity holds:*

$$H_i(g_0) + H_i(g_1\kappa(g_0)) = H_i(g_1 g_0).$$

Proof. Using $G = KMAN$, we write

$$g_0 = k_0 m_0 a_0 n_0, \qquad g_1 = k_1 m_1 a_1 n_1.$$

Recall that the group MA normalizes N and M commutes with A. Thus, for any $k \in K \cap M$, there exists $n_1' \in N$ such that

$$H_i(g_1 k) = H_i(k_1 m_1 a_1 n_1 k) = H_i(k_1 m_1 k a_1 n_1') = H_i(a_1) = H_i(g_1).$$

It follows that $H_i(g_1\kappa(g_0))$ is well defined. Next, by the definition of H_i,

$$H_i(g_1 g_0) = H_i(a_1 n_1 k_0 a_0) = H_i(a_1 n_1 k_0) + H_i(a_0).$$

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The lemma follows from the following identities:

$$H_i(g_1\kappa(g_0)) = H_i(a_1n_1k_0), \qquad H_i(g_0) = H_i(a_0).$$

□

Let S_m be the permutation group of m elements. For any $\tau \in S_m$, let $\operatorname{sgn}(\tau) = \pm 1$ depending on the parity of τ.

Definition 3.2. We define a function

$$C \in C^\infty\left(K \times G^{\times m}\right)$$

by

$$\begin{aligned} C(k, g_1, \ldots, g_m) :&= \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot H_{\tau(1)}(g_1k) H_{\tau(2)}(g_2k) \ldots H_{\tau(m)}(g_mk) \\ &= \det\left(\widetilde{H}(g_1k), \ldots, \widetilde{H}(g_mk)\right). \end{aligned}$$

Definition 3.3. For any $f_0, \ldots, f_m \in \mathcal{C}(G)$ and semi-simple element $x \in M$, we define a Hochschild cochain on $\mathcal{C}(G)$ by the following formula:

$$\begin{aligned} &\Phi_{P,x}(f_0, f_1, \ldots, f_m) := \int_{h \in M/Z_M(x)} \int_{KN} \int_{G^{\times m}} C(k, g_1g_2 \ldots g_m, \ldots, g_{m-1}g_m, g_m) \\ &f_0\left(khxh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}\right) f_1(g_1) \ldots f_m(g_m) dg_1 \cdots dg_m dk dn dh, \end{aligned} \tag{3.1}$$

where $Z_M(x)$ is the centralizer of x in M.

We prove in Theorem A.5 that the above integral (3.1) is convergent for all semi-simple elements $x \in M$. A similar estimate leads us to the following property.

Proposition 3.4. *For every semi-simple element $x \in M$, the integral $\Phi_{P,x}$ defines a (continuous) Hochschild cochain on the Schwartz algebra $\mathcal{C}(G)$.*

For simplicity, we omit the respective measures $dg_1, \cdots, dg_m$, dk, dn, dh, in the integral (3.1) for $\Phi_{P,x}$.

Theorem 3.5. *Let $P = MAN$ be a cuspidal parabolic subgroup of G and $x \in M$ be a semi-simple element. The cochain $\Phi_{P,x}$ is a cyclic cocycle and defines an element*

$$[\Phi_{P,x}] \in HC^m(\mathcal{C}(G)).$$

Remark 3.6. We notice that our proofs in Sections 3.2 and 3.3 also work for the algebra $\mathcal{S}_t(G)$ (for sufficiently large t) introduced in Definition A.3. And we can conclude from Theorem A.5 that $\Phi_{P,x}$ defines a continuous cyclic cocycle on $\mathcal{S}_t(G) \supset \mathcal{C}(G)$ for a sufficiently large t for every $x \in M$.

The proof of Theorem 3.5 occupies the rest of this section.

3.2. Cocycle condition

In this subsection, we prove that the cochain $\Phi_{P,x}$ introduced in Definition 3.3 is a Hochschild cocycle. We have the following expression for the codifferential of $\Phi_{P,x}$:

$$\begin{aligned} &b\Phi_{P,x}(f_0, f_1, \ldots, f_m, f_{m+1}) \\ &= \sum_{i=0}^{m} (-1)^i \Phi_{P,x}\left(f_0, \ldots, f_i * f_{i+1}, \ldots, f_{m+1}\right) + (-1)^{m+1} \Phi_{P,x}\left(f_{m+1} * f_0, f_1, \ldots, f_m\right). \end{aligned} \tag{3.2}$$

Here, $f_i * f_{i+1}$ is the convolution product given by

$$f_i * f_{i+1}(h) = \int_G f_i(g) f_{i+1}(g^{-1}h) dg.$$

Lemma 3.7. *For $i = 0$, the term on the right-hand side of (3.2) can be computed by the following integral:*

$$\Phi_{P,x}\big(f_0 * f_1, f_2, \ldots, f_{m+1}\big) = \int_{M/Z_M(x)} \int_{KN} \int_{G^{\times(m+1)}} C(k, t_2 t_3 \ldots t_{m+1}, \ldots, t_m t_{m+1}, t_{m+1})$$
$$f_0\big(khxh^{-1}nk^{-1}(t_1 \ldots t_{m+1})^{-1}\big) f_1(t_1) f_2(t_2) \ldots f_{m+1}(t_{m+1}).$$

Proof. By definition,

$$\Phi_{P,x}\big(f_0 * f_1, f_2, \ldots, f_{m+1}\big) = \int_{M/Z_M(x)} \int_{KN} \int_G \int_{G^{\times m}} C(k, g_1 g_2 \ldots g_m, \ldots, g_{m-1} g_m, g_m)$$
$$f_0(g) f_1\big(g^{-1} khxh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}\big) f_2(g_1) \ldots f_{m+1}(g_m).$$

By changing variables,

$$t_1 = g^{-1} khxh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}, \qquad t_j = g_{j-1}, \qquad j = 2, \ldots m+1,$$

we get

$$g = khxh^{-1}nk^{-1}(t_1 \ldots t_{m+1})^{-1}.$$

One can prove the lemma by replacing g_i by t_i. □

Lemma 3.8. *For $1 \le i \le m$, we have*

$$\begin{aligned} &\Phi_{P,x}\big(f_0, \ldots, f_i * f_{i+1}, \ldots, f_{m+1}\big) \\ &= \int_{M/Z_M(x)} \int_{KN} \int_{G^{\times(m+1)}} C(k, t_1 t_2 \ldots t_{m+1}, \ldots, (t_{i+1} \ldots t_{m+1})\hat{}, \ldots, t_{m+1}) \\ &\quad f_0\big(khxh^{-1}nk^{-1}(t_1 \ldots t_{m+1})^{-1}\big) f_1(t_1) \ldots f_{m+1}(t_{m+1}), \end{aligned} \tag{3.3}$$

where $(t_{i+1} \ldots t_{m+1})\hat{}$ means that the term is omitted in the expression.

Proof. The left-hand side of the above equation,

$$\begin{aligned} &\Phi_{P,x}\big(f_0, \ldots, f_i * f_{i+1}, \ldots, f_{m+1}\big) \\ &= \int_{M/Z_M(x)} \int_{KN} \int_G \int_{G^{\times m}} C(k, g_1 g_2 \ldots g_m, \ldots, g_{m-1} g_m, g_m) \\ &\quad f_0\big(khxh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}\big) f_1(g_1) \ldots f_{i-1}(g_{i-1}) \\ &\quad \big(f_i(g) f_{i+1}(g^{-1} g_i)\big) f_{i+2}(g_{i+1}) \ldots f_{m+1}(g_m). \end{aligned} \tag{3.4}$$

Let $t_j = g_j$ for $j = 1, \ldots, i-1$, and for $j = i+2, \ldots, m+1$

$$t_i = g, \qquad t_{i+1} = g^{-1} g_i, \qquad t_j = g_{j-1}.$$

□

Lemma 3.9. *The last term in the right-hand side of (3.2) can be computed by the following integral:*

$$\Phi_{P,x}(f_{m+1} * f_0, f_1, \dots, f_m) = \int_{M/Z_M(x)} \int_{KN} \int_{G^{\times(m+1)}} C(\kappa(t_{m+1}k), t_1t_2\dots t_m, \dots, t_{m-1}t_m, t_m) \\ f_0(khxh^{-1}nk^{-1}(t_1\dots t_{m+1})^{-1}) f_1(t_1)\dots f_{m+1}(t_{m+1}).$$

Proof. By definition,

$$\Phi_{P,x}(f_{m+1} * f_0, f_1, \dots, f_m) = \int_{M/Z_M(x)} \int_{KN} \int_G \int_{G^{\times m}} C(k, g_1g_2\dots g_m, \dots, g_{m-1}g_m, g_m) \\ f_{m+1}(g) f_0(g^{-1}khxh^{-1}nk^{-1}(g_1\dots g_m)^{-1}) f_1(g_1)\dots f_m(g_m). \tag{3.5}$$

As before, we write

$$t_j = g_j, \qquad j = 1, \dots, m,$$

and $t_{m+1} = g$. We can rewrite Equation (3.5) as

$$\Phi_{P,x}(f_{m+1} * f_0, f_1, \dots, f_m) = \int_{M/Z_M(x)} \int_{KN} \int_{G^{\times(m+1)}} C(k, t_1t_2\dots t_m, \dots, t_{m-1}t_m, t_m) \\ f_0(t_{m+1}^{-1}khxh^{-1}nk^{-1}(t_1\dots t_m)^{-1}) f_1(t_1)\dots f_{m+1}(t_{m+1}). \tag{3.6}$$

For all $t_{m+1} \in G$ and $k \in K$, we decompose

$$t_{m+1}^{-1}k = k_1\mu_1a_1n_1 \in KMAN.$$

It follows that $k = t_{m+1}k_1\mu_1a_1n_1$ and $k = \kappa(t_{m+1}k_1)$. We see

$$\Phi_{P,x}(f_{m+1} * f_0, f_1, \dots, f_m) = \int_{M/Z_M(x)} \int_{KN} \int_{G^{\times(m+1)}} C(k, t_1t_2\dots t_m, \dots, t_{m-1}t_m, t_m) \\ f_0(k_1\mu_1a_1n_1hxh^{-1}nn_1^{-1}a_1^{-1}\mu_1^{-1}k_1^{-1}t_{m+1}^{-1}(t_1\dots t_m)^{-1}) f_1(t_1)\dots f_{m+1}(t_{m+1}).$$

Since $\mu_1a_1 \in MA$ normalizes the nilpotent group N, we can find $\tilde{n}_1, n_1' \in N$ such that

$$f_0(k_1\mu_1a_1n_1hxh^{-1}nn_1^{-1}a_1^{-1}\mu_1^{-1}k_1^{-1}t_{m+1}^{-1}(t_1\dots t_m)^{-1}) \\ = f_0(k_1\mu_1hx(\mu_1h)^{-1}\tilde{n}_1nn_1'^{-1}k_1^{-1}(t_1\dots t_{m+1})^{-1}).$$

Renaming $\tilde{n}_1nn_1'^{-1}$ by n, we conclude that

$$\Phi_{P,x}(f_{m+1} * f_0, f_1, \dots, f_m) = \int_{M/Z_M(x)} \int_{KN} \int_{G^{\times(m+1)}} C(\kappa(t_{m+1}k_1), t_1t_2\dots t_m, \dots, t_{m-1}t_m, t_m) \\ f_0(k_1hxh^{-1}nk_1^{-1}(t_1\dots t_{m+1})^{-1}) \cdot f_1(t_1)\dots f_{m+1}(t_{m+1}).$$

This completes the proof. □

Combining Lemmas 3.7, 3.8 and 3.9, we have reached the following equation:

$$b\Phi_{P,x}(f_0, f_1, \dots, f_m, f_{m+1}) \\ = \int_{M/Z_M(x)} \int_{KN} \int_{G^{\times(m+1)}} \tilde{C}(k, t_1, \dots t_{m+1}) \cdot f_0(k_1hxh^{-1}nk_1^{-1}(t_1\dots t_{m+1})^{-1}) \\ \cdot f_1(t_1)\dots f_{m+1}(t_{m+1}), \tag{3.7}$$

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where $\tilde{C} \in C^\infty\left(K \times G^{\times m}\right)$ is given by

$$\tilde{C}(k, t_1, \ldots, t_{m+1}) = \sum_{i=0}^{m} (-1)^i C(k, t_1 t_2 \ldots t_{m+1}, \ldots, (t_{i+1} \ldots t_{m+1})\hat{}, \ldots, t_{m+1}) \\ + (-1)^{m+1} C(\kappa(t_{m+1}k), t_1 t_2 \ldots t_m, \ldots, t_{m-1}t_m, t_m).$$

Lemma 3.10. *We have that*

$$b\Phi_{P,x}(f_0, f_1, \ldots, f_m, f_{m+1}) = 0.$$

Proof. We will show that

$$\tilde{C}(k, t_1, \ldots, t_{m+1}) = 0.$$

To begin with, we notice the following expression:

$$C\big(\kappa(t_{m+1}k), t_1 t_2 \ldots t_m, \ldots, t_{m-1}t_m, t_m\big) \\ = \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot H_{\tau(1)}\big(t_1 \ldots t_m \kappa(t_{m+1}k)\big) H_{\tau(2)}\big(t_2 \ldots t_m \kappa(t_{m+1}k)\big) \ldots H_{\tau(m)}\big(t_m \kappa(t_{m+1}k)\big).$$

By Lemma 3.1, we have

$$H_{\tau(i)}\big(t_i \ldots t_m \kappa(t_{m+1}k)\big) = H_{\tau(i)}\big(t_i \ldots t_{m+1}k\big) - H_{\tau(i)}(t_{m+1}k).$$

It follows that

$$C\big(\kappa(t_{m+1}k), t_1 t_2 \ldots t_m, \ldots, t_{m-1}t_m, t_m\big) \\ = \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot \Big(H_{\tau(1)}(t_1 \ldots t_{m+1}k) - H_{\tau(1)}(t_{m+1}k)\Big) \\ \cdot \Big(H_{\tau(2)}(t_2 \ldots t_{m+1}k) - H_{\tau(2)}(t_{m+1}k)\Big) \ldots \Big(H_{\tau(m)}(t_m t_{m+1}k) - H_{\tau(m)}(t_{m+1}k)\Big).$$

As the above sum is invariant with respect to the permutation group S_m, the terms containing more than one factor $H_{\tau(i)}(t_{m+1}k)$ vanish. Thus,

$$C\big(\kappa(t_{m+1}k), t_1 t_2 \ldots t_m, \ldots, t_{m-1}t_m, t_m\big) \\ = \sum_{i=1}^{m} \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot H_{\tau(1)}(t_1 \ldots t_{m+1}k) \ldots \Big(-H_{\tau(i)}(t_{m+1}k)\Big) \ldots H_{\tau(m)}(t_m t_{m+1}k) \\ + \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot H_{\tau(1)}(t_1 \ldots t_{m+1}k) \ldots H_{\tau(m)}(t_m t_{m+1}k).$$

In the above expression, by changing the permutations

$$\big(\tau(1), \ldots, \tau(m)\big) \mapsto \big(\tau(1), \ldots, \tau(i-1), \tau(m), \tau(i), \ldots, \tau(m-1)\big),$$

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we get

$$
\begin{aligned}
&\sum_{\tau\in S_m} \operatorname{sgn}(\tau)\cdot H_{\tau(1)}(t_1\dots t_{m+1}k)\dots H_{\tau(i-1)}(t_{i-1}\dots t_{m+1}k)\\
&\Big(-H_{\tau(i)}(t_{m+1}k)\Big)H_{\tau(i+1)}(t_{i+1}\dots t_{m+1}k)\dots H_{\tau(m)}(t_m t_{m+1}k)\\
=&(-1)^{n-i}\sum_{\tau\in S_m} \operatorname{sgn}(\tau)\cdot H_{\tau(1)}(t_1\dots t_{m+1}k)\dots H_{\tau(i-1)}(t_{i-1}\dots t_m k)\\
&H_{\tau(m)}(t_{m+1}k)H_{\tau(i)}(t_{i+1}\dots t_{m+1}k)\dots H_{\tau(m-1)}(t_m t_{m+1}k).
\end{aligned}
$$

Putting all the above together, we have

$$
\begin{aligned}
&(-1)^{m+1}C\big(\kappa(t_{m+1}k), t_1t_2\dots t_m,\dots,t_{m-1}t_m,t_m\big)\\
&\quad=\sum_{i=0}^{m}(-1)^{i+1}\cdot C(k,t_1t_2\dots t_{m+1},\dots,(t_{i+1}\dots t_{m+1})\hat{},\dots,t_{m+1}),
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{C}(k,t_1,\dots t_{m+1}) = &\sum_{i=0}^{m}(-1)^{i}C(k,t_1t_2\dots t_{m+1},\dots,(t_{i+1}\dots t_{m+1})\hat{},\dots,t_{m+1})\\
&+(-1)^{m+1}C(\kappa(t_{m+1}k),t_1t_2\dots t_m,\dots,t_{m-1}t_m,t_m)=0.
\end{aligned}
$$

□

We conclude from Lemmas 3.7–3.10 that $\Phi_{P,x}$ is a Hochschild cocycle. We will prove that $\Phi_{P,x}$ is cyclic in the next subsection.

3.3. Cyclic condition

In this subsection, we prove that the cocycle $\Phi_{P,x}$ is cyclic. Recall

$$
\begin{aligned}
\Phi_{P,x}(f_1,\dots,f_m,f_0) = &\int_{M/Z_M(x)}\int_{KN}\int_{G^{\times m}} C(k,g_1g_2\dots g_m,\dots,g_{m-1}g_m,g_m)\\
&f_1\big(khxh^{-1}nk^{-1}(g_1\dots g_m)^{-1}\big)f_2(g_1)\dots f_m(g_{m-1})f_0(g_m).
\end{aligned}
\tag{3.8}
$$

By changing the variables,

$$t_1 = khxh^{-1}nk^{-1}(g_1\dots g_m)^{-1},$$

and $t_j = g_{j-1}$ for $j = 2,\dots,m$. We have

$$g_m = (t_1\dots t_m)^{-1}khxh^{-1}nk^{-1},$$

and

$$g_i\dots g_m = (t_1\dots t_i)^{-1}khxh^{-1}nk^{-1}.$$

It follows that

$$\begin{aligned}
&\Phi_{P,x}(f_1,\ldots,f_m,f_0) = \int_{M/Z_M(x)}\int_{KN}\int_{G^{\times m}}\\
&C\big(k, t_1^{-1}khxh^{-1}nk^{-1},\ldots,(t_1\ldots t_{m-1})^{-1}khxh^{-1}nk^{-1},(t_1\ldots t_m)^{-1}khxh^{-1}nk^{-1}\big)\\
&f_0\big((t_1\ldots t_m)^{-1}khxh^{-1}nk^{-1}\big)f_1(t_1)f_2(t_2)\ldots f_m(t_m)\\
=&\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot\int_{M/Z_M(x)}\int_{KN}\int_{G^{\times m}} H_{\tau(1)}(t_1^{-1}khxh^{-1}n)\ldots H_{\tau(m)}((t_1\ldots t_m)^{-1}khxh^{-1}n)\\
&f_0\big((t_1\ldots t_m)^{-1}khxh^{-1}nk^{-1}\big)f_1(t_1)f_2(t_2)\ldots f_m(t_m).
\end{aligned}$$

We write

$$(t_1\ldots t_m)^{-1}k = k_1\mu_1a_1n_1 \in KMAN.$$

Then,

$$k = (t_1\ldots t_m)k_1\mu_1a_1n_1, \tag{3.9}$$

and

$$\begin{aligned}
f_0\big((t_1\ldots t_m)^{-1}khxh^{-1}nk^{-1}\big) &= f_0\big(k_1\mu_1a_1n_1hxh^{-1}nn_1^{-1}a_1^{-1}\mu_1^{-1}k_1^{-1}(t_1\ldots t_m)^{-1}\big)\\
&= f_0\big(k_1h'xh'^{-1}n'k_1^{-1}(t_1\ldots t_m)^{-1}\big).
\end{aligned}$$

Thus, we rewrite the right-hand side of (3.8)

$$\begin{aligned}
\Phi_{P,x}(f_1,\ldots,f_m,f_0) = \sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot\int_{M/Z_M(x)}\int_{KN}\int_{G^{\times m}} H_{\tau(1)}(t_1^{-1}k)\ldots H_{\tau(m)}((t_1\ldots t_m)^{-1}k)\\
f_0\big(k_1hxh^{-1}nk_1^{-1}(t_1\ldots t_m)^{-1}\big)f_1(t_1)f_2(t_2)\ldots f_m(t_m).
\end{aligned} \tag{3.10}$$

By Lemma 3.1 and (3.9), we have, for $1\le i\le m-1$,

$$\begin{aligned}
H_{\tau(i)}\big((t_1\ldots t_i)^{-1}k\big) &= -H_{\tau(i)}\big(t_1\ldots t_i\kappa((t_1\ldots t_i)^{-1}k)\big)\\
&= -H_{\tau(i)}\big(t_1\ldots t_i\kappa(t_{i+1}\ldots t_mk_1)\big)\\
&= H_{\tau(i)}\big(t_{i+1}\ldots t_mk_1\big) - H_{\tau(i)}\big(t_1\ldots t_mk_1\big)\big),
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
H_{\tau(m)}\big((t_1\ldots t_m)^{-1}k\big) &= -\,H_{\tau(m)}\big(t_1\ldots t_m\kappa((t_1\ldots t_m)^{-1}k)\big)\\
&= -\,H_{\tau(m)}\big(t_1\ldots t_mk_1\big).
\end{aligned} \tag{3.12}$$

Putting (3.10), (3.11) and (3.12) together, we see that

$$\begin{aligned}
&\Phi_{P,x}(f_1,\ldots,f_m,f_0)\\
=&\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot\int_{M/Z_M(m)}\int_{KN}\int_{G^{\times m}}\prod_{i=1}^{m-1}\Big(H_{\tau(i)}\big(t_{i+1}\ldots t_mk_1\big) - H_{\tau(i)}\big(t_1\ldots t_mk_1\big)\Big)\\
&\big(-H_{\tau(m)}(t_1\ldots t_mk_1)\big)\cdot f_0\big(k_1hxh^{-1}nk_1^{-1}(t_1\ldots t_m)^{-1}\big)f_1(t_1)f_2(t_2)\ldots f_m(t_m).
\end{aligned}$$

By symmetry, one can check that

$$\begin{aligned}&\sum_{\tau\in S_m}\operatorname{sgn}(\tau)\cdot\prod_{i=1}^{m-1}\Big(H_{\tau(i)}\big(t_{i+1}\dots t_mk_1\big)-H_{\tau(i)}\big(t_1\dots t_mk_1\big)\Big)\cdot H_{\tau(m)}(t_1\dots t_mk_1)\\&=\sum_{\tau\in S_m}\operatorname{sgn}(\tau)\cdot\prod_{i=1}^{m-1}H_{\tau(i)}\big(t_{i+1}\dots t_mk_1\big)\cdot H_{\tau(m)}(t_1\dots t_mk_1).\end{aligned}\tag{3.13}$$

In the above expression, by changing the permutation

$$\big(\tau(1),\dots,\tau(m)\big)\mapsto\big(\tau(2),\dots,\tau(m),\tau(1)\big),$$

we can simplify Equation (3.13) to the following one:

$$\begin{aligned}&\sum_{\tau\in S_m}\operatorname{sgn}(\tau)\cdot\prod_{i=1}^{m-1}\Big(H_{\tau(i)}\big(t_{i+1}\dots t_mk_1\big)-H_{\tau(i)}\big(t_1\dots t_mk_1\big)\Big)\cdot H_{\tau(m)}(t_1\dots t_mk_1)\\&=(-1)^{m-1}\cdot\sum_{\tau\in S_m}\operatorname{sgn}(\tau)\cdot\prod_{i=1}^{m}H_{\tau(i)}\big(t_i\dots t_mk_1\big).\end{aligned}$$

Finally, we have obtained the following identity:

$$\begin{aligned}\Phi_{P,x}(f_1,\dots,f_m,f_0)&=(-1)^m\cdot\sum_{\tau\in S_m}\operatorname{sgn}(\tau)\cdot\int_{M/Z_M(x)}\int_{KN}\int_{G^{\times m}}\prod_{i=1}^{m}H_{\tau(i)}\big(t_i\dots t_mk\big)\\&\quad\cdot f_0\big(khxh^{-1}nk^{-1}(t_1\dots t_m)^{-1}\big)f_1(t_1)f_2(t_2)\dots f_m(t_m)\\&=(-1)^m\cdot\Phi_{P,x}(f_0,\dots,f_m).\end{aligned}$$

Hence, we conclude that $\Phi_{P,x}$ is a cyclic cocycle, and we have completed the proof of Theorem 3.5.

4. The Fourier transform of $\Phi_{P_\circ,x}$

In this section, we study the Fourier transform of the cyclic cocycle $\Phi_{P,x}$ introduced in Section 3. From now on, we additionally assume that G is connected following Knapp's book [18]. For the reader's convenience, we start with recalling the basic material on parabolic induction and the Plancherel formula in Section 4.1 and 4.2.

4.1. Parabolic induction

A brief introduction to discrete series representations can be found in Appendix B. In this section, we review the construction of parabolic induction. Let H be a θ-stable Cartan subgroup of G with Lie algebra $\mathfrak{h}$. Let $P=M_PA_PN_P$ be a cuspidal parabolic subgroup associated to H as in subsection 2.1.

Definition 4.1. Let η be a unitary representation of M_P and φ a unitary representation of A_P. The product $\sigma\otimes\varphi$ defines a unitary representation of $L_P=M_PA_P$. A *basic representation* of G is a representation by extending $\sigma\otimes\varphi$ to P trivially across N_P then inducing to G:

$$\pi_{\eta,\varphi}=\operatorname{Ind}_P^G(\eta\otimes\varphi).$$

If $\eta=\sigma$ is a discrete series representation, then $\operatorname{Ind}_P^G(\sigma\otimes\varphi)$ will be called a *basic representation induced from the discrete series representation of M_P and unitary representation of A_P*. This construction is known as *parabolic induction*.

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The character of $\pi_{\sigma,\varphi}$ is given in Theorem B.5, Equation (B.3) and Corollary B.6. Note that basic representations might not be irreducible. Knapp and Zuckerman completed the classification of tempered representations by showing which basic representations are irreducible and proved that every tempered representation of G is basic and every basic representation is tempered.

Now consider a single cuspidal parabolic subgroup $P \subseteq G$ with $L_P = M_P A_P$, and form the group

$$W(A_P, G) = N_K(\mathfrak{a}_P)/Z_K(\mathfrak{a}_P),$$

where $N_K(\mathfrak{a}_P)$ and $Z_K(\mathfrak{a}_P)$ are the normalizer and centralizer of $\mathfrak{a}_P$ in K, respectively. The group $W(A_P, G)$ acts as an outer automorphism of M_P, and hence on the set of equivalence classes of representations of M_P. For any discrete series representation σ of M_P, we define

$$W_\sigma = \left\{w \in N_K(\mathfrak{a}_P) \colon\ \mathrm{Ad}_w^*\, \sigma \cong \sigma\right\}/Z_K(\mathfrak{a}_P).$$

Then, the above Weyl group acts on the family of induced representations

$$\left\{\mathrm{Ind}_P^G(\sigma \otimes \varphi)\right\}_{\varphi \in \widehat{A}_P}.$$

Definition 4.2. Let P_1 and P_2 be two cuspidal parabolic subgroups of G with $L_{P_i} = M_{P_i} A_{P_i}$. Let σ_1 and σ_2 be two discrete series representations of M_{P_i}. We say that

$$(P_1, \sigma_1) \sim (P_2, \sigma_2) \tag{4.1}$$

if there exists an element w in G that conjugates L_{P_1} of P_1 to L_{P_2} of P_2, and conjugates σ_1 to a representation unitarily equivalent to σ_2. In this case, there is a unitary G-equivariant isomorphism

$$\mathrm{Ind}_{P_1}^G(\sigma_1 \otimes \varphi) \cong \mathrm{Ind}_{P_2}^G(\sigma_2 \otimes (\mathrm{Ad}_w^*\, \varphi))$$

as Hilbert $C_0(\widehat{A}_{P_j})$-modules that covers the isomorphism

$$\mathrm{Ad}_w^* : C_0(\widehat{A}_{P_1}) \to C_0(\widehat{A}_{P_2}).$$

We denote by $[P, \sigma]$ the equivalence class of (4.1), and $\mathcal{P}(G)$ the set of all equivalence classes.

Finally, we recall the functoriality of parabolic induction.

Lemma 4.3. *If $S = M_S A_S N_S$ is any cuspidal parabolic subgroup of L, then the unipotent radical of SN_P is $N_S N_P$, and the product*

$$Q = M_Q A_Q N_Q = M_S(A_s a : P)(N_S N_P)$$

is a cuspidal parabolic subgroup of G.

Proof. See [29, Lemma 4.1.1]. □

Theorem 4.4 (Induction in stages). *Let η be a unitary representation (not necessarily a discrete series representation) of M_S. We decompose*

$$\varphi = (\varphi_1, \varphi_2) \in \widehat{A}_S \times \widehat{A}_P.$$

There is a canonical equivalence

$$\mathrm{Ind}_P^G\left(\mathrm{Ind}_S^{M_P}(\eta \otimes \varphi_1) \otimes \varphi_2\right) \cong \mathrm{Ind}_Q^G\left(\eta \otimes (\varphi_1, \varphi_2)\right).$$

Proof. See [18, p. 170]. □

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4.2. Wave packets

Let $\widehat{G}_{\text{temp}}$ be the set of equivalence classes of irreducible unitary tempered representations of G. For a Schwartz function f on G, its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\pi) = \int_G f(g)\pi(g)dg, \qquad \pi \in \widehat{G}_{\text{temp}}.$$

Thus, the Fourier transform assigns to f a family of operators on different Hilbert spaces (tempered representations of G) indexed by π.

The group A_P, which consists entirely of positive definite matrices, is isomorphic to its Lie algebra via the exponential map. So A_P carries the structure of a vector space, and we can speak of its space of Schwartz functions in the ordinary sense of harmonic analysis. The same goes for the unitary (Pontryagin) dual $\widehat{A}_P$. By a tempered measure on A_P, we mean a smooth measure for which integration extends to a continuous linear functional on the Schwartz space. Recall Harish-Chandra's Plancherel formula for G [13].

Theorem 4.5. *There is a unique smooth, tempered, W_σ-invariant function $m_{P,\sigma}$ on the spaces $\widehat{A}_P$ such that*

$$\|f\|^2_{L^2(G)} = \sum_{[P,\sigma]\in\mathcal{P}(G)} \int_{\widehat{A}_P} \left\|\widehat{f}(\pi_{\sigma,\varphi})\right\|^2_{HS} m_{P,\sigma}(\varphi)d\varphi$$

for every Schwartz function $f \in \mathcal{C}(G)$. We call $m_{P,\sigma}(\varphi)$ the Plancherel density of the representation $\operatorname{Ind}_P^G(\sigma\otimes\varphi)$.

As $\varphi \in \widehat{A}_P$ varies, the induced G-representations

$$\pi_{\sigma,\varphi} = \operatorname{Ind}_P^G(\sigma\otimes\varphi)$$

can be identified with one another as representations of K. Denote by $\operatorname{Ind}_P^G(\sigma)$ this common Hilbert space, and $\mathcal{L}(\operatorname{Ind}_P^G(\sigma))$ the space of K-finite Hilbert-Schmidt operators on $\operatorname{Ind}_P^G(\sigma)$. We shall discuss the adjoint to the Fourier transform.

Definition 4.6. Let h be a Schwartz-class function from $\widehat{A}_P$ into operators on $\operatorname{Ind}_P^G(\sigma)$ that is invariant under the W_σ-action. That is,

$$h \in \left[\mathcal{C}(\widehat{A}_P)\otimes\mathcal{L}^2(\operatorname{Ind}_P^G(\sigma))\right]^{W_\sigma}.$$

The *wave packet* associated to h is the scalar function defined by the following formula:

$$\check{h}(g) = \int_{\widehat{A}_P} \operatorname{Trace}\big(\pi_{\sigma,\varphi}(g^{-1})\cdot h(\varphi)\big)\cdot m_{P,\sigma}(\varphi)d\varphi.$$

A fundamental theorem of Harish-Chandra asserts that wave packets are Schwartz functions on G.

Theorem 4.7. *The wave packets associated to the Schwartz-class functions from $\widehat{A}_P$ into $\mathcal{L}(\operatorname{Ind}_P^G(\sigma))$ all belong to the Harish-Chandra Schwartz space $\mathcal{C}(G)$. Moreover, the wave packet operator $h \to \check{h}$ is adjoint to the Fourier transform.*

Proof. See [30, Theorems 12.7.1 and 13.4.1] and [4, Corollary 9.8]. □

4.3. Derivatives of Fourier transform

Let $P = MAN$ be a cuspidal parabolic subgroup. Here, P does not have to be maximal. Thus, we can decompose $A_P = A_\circ \times A_S$ (see Lemma 4.3). Suppose that $\pi = \operatorname{Ind}_P^G(\eta^M\otimes\varphi)$, where η^M is an

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irreducible tempered representation (does not have to be a discrete series representation) of M with character denoted by $\Theta^M(\eta^M)$ and

$$\varphi \in \widehat{A}_P = \widehat{A}_\circ \times \widehat{A}_S.$$

We denote $r = \dim(\widehat{A}_S)$ and $m = \dim(\widehat{A}_\circ)$. As a vector space, let

$$x_1, \ldots, x_{\dim A_\circ}, x_{\dim A_\circ+1}, \ldots, x_{\dim A_P}$$

be the coordinates for $\widehat{A}_P$. For $i = 0, \ldots, m$, let $h_i \in \mathcal{C}(\widehat{A}_P)$, and v_i, w_i be unit K-finite vectors in $\mathrm{Ind}_P^G(\eta^M)$. We denote by $\frac{\partial h_i}{\partial j}$ the partial derivative of h_i with respect to x_j, $j = 1, \ldots, m$.

Definition 4.8. Suppose that $f_i \in \mathcal{C}(G)$ are wave packets associated to

$$h_i \cdot v_i \otimes w_i^* \in \mathcal{C}\big(\widehat{A}_P, \mathcal{L}(\mathrm{Ind}_P^G(\eta^M))\big).$$

We define an $(m+1)$-linear map T_π with an image in $\mathcal{C}(\widehat{A}_P)$ by

$$\begin{aligned} &T_\pi(\widehat{f_0}, \ldots, \widehat{f_m}) \\ &= \begin{cases} \sum_{\tau \in S_m} \mathrm{sgn}(\tau) \cdot h_0(\varphi) \cdot \prod_{i=1}^m \frac{\partial h_i(\varphi)}{\partial_{\tau(i)}} & \text{if } v_i = w_{i+1}, i = 0, \ldots, m-1, \text{ and } v_m = w_0; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2}$$

Next we want to generalize the above definition to the Fourier transforms of all $f \in \mathcal{C}(G)$. The induced space $\pi = \mathrm{Ind}_P^G(\eta^M \otimes \varphi)$ has a dense subspace:

$$\left\{ s \colon K \to V^{\eta^M} \text{ continuous} \middle| s(km) = \eta^M(m)^{-1} s(k) \text{ for } k \in K, m \in K \cap M \right\}, \tag{4.3}$$

where V^{η^M} is the Hilbert space of M-representation η^M. The group G action on π is given by the formula

$$(\pi(g)s)(k) = e^{-\langle \log \varphi + \rho, H(g^{-1}k) \rangle} \cdot \eta^M(\mu(g^{-1}k))^{-1} \cdot s(\kappa(g^{-1}k)), \tag{4.4}$$

where ρ denotes the half sum of positive roots. By Equation (4.4), the Fourier transform

$$\begin{aligned} (\pi(f)s)(k) &= \left(\widehat{f}(\pi)s\right)(k) \\ &= \int_G (e^{-\langle \log \varphi + \rho, H(g^{-1}k) \rangle} \cdot \eta^M(\mu(g^{-1}k))^{-1} f(g) \cdot s(\kappa(g^{-1}k)) dg. \end{aligned}$$

Suppose now that $f_0, \ldots, f_m$ are arbitrary Schwartz functions on G and $\widehat{f_0}, \ldots, \widehat{f_m}$ are their Fourier transforms.

Definition 4.9. For any $1 \leq i \leq n$, we define a bounded operator $\frac{\partial \widehat{f}}{\partial_i}$ from $\pi = \mathrm{Ind}_P^G(\eta^M \otimes \varphi)$ to itself by the following formula:

$$\left(\left(\frac{\partial \widehat{f}(\pi)}{\partial_i}\right) s\right)(k) := \int_G H_i(g^{-1}k) \cdot (e^{-\langle \log \varphi + \rho, H(g^{-1}k) \rangle} \cdot \eta^M(\mu(g^{-1}k))^{-1} \cdot f(g) \cdot s(\kappa(g^{-1}k)). \tag{4.5}$$

We define an $(m+1)$-linear map

$$T_\pi \colon \underbrace{\mathcal{C}(G) \times \cdots \times \mathcal{C}(G)}_{m+1} \to \mathbb{C}$$

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by

$$T_\pi(\widehat{f_0},\dots,\widehat{f_m}) := \sum_{\tau\in S_m} \operatorname{sgn}(\tau)\cdot \operatorname{Trace}\Big(\widehat{f_0}(\pi)\cdot\prod_{i=1}^m \frac{\partial\widehat{f_i}(\pi)}{\partial_{\tau(i)}}\Big).$$

The above definition generalizes (4.2).

Proposition 4.10. *For any* $\pi = \operatorname{Ind}_P^G(\eta^M\otimes\varphi)$*, we have the following identity:*

$$\begin{aligned} T_\pi(\widehat{f_0},\dots,\widehat{f_m}) = (-1)^m \sum_{\tau\in S_m}\operatorname{sgn}(\tau)\int_{KMAN}\int_{G^{\times m}} H_{\tau(1)}(g_1\dots g_m k)\dots H_{\tau(m)}(g_m k)\\ e^{\langle\log\varphi+\rho,\log a\rangle}\cdot\Theta^M(\eta^M)(m)\cdot f_0(kmank^{-1}(g_1g_2\dots g_m)^{-1})f_1(g_1)\dots f_m(g_m).\end{aligned}\tag{4.6}$$

Proof. By definition, for any $\tau\in S_m$,

$$\begin{aligned}&\Big(\widehat{f_0}(\pi)\cdot\prod_{i=1}^m\frac{\partial\widehat{f_i}(\pi)}{\partial_{\tau(i)}}\Big)s(k)\\ &=\int_{G^{\times(k+1)}} H_{\tau(1)}(g_1^{-1}\kappa(g_0^{-1}k))H_{\tau(2)}\big(g_2^{-1}\kappa((g_0g_1)^{-1}k)\big)\\ &H_{\tau(m)}\big(g_m^{-1}\kappa((g_0g_1\dots g_{m-1})^{-1}k)\big)\cdot e^{-\langle\log\varphi+\rho,H((g_0g_1\dots g_m)^{-1}k)\rangle}\\ &\eta^M(\mu((g_0g_1\dots g_m)^{-1}k))^{-1}\cdot f_0(g_0)f_1(g_1)\dots f_m(g_m)s\big(\kappa((g_0g_1\dots g_m)^{-1}k)\big).\end{aligned}$$

By setting $g=(g_0g_1\dots g_m)^{-1}k$, we have

$$g_0 = kg^{-1}(g_1g_2\dots g_m)^{-1},$$

and

$$(g_0g_1\dots g_j)^{-1}k = g_{j+1}g_{j+2}\dots g_m g.$$

Recall that recall MA normalizes N and M centralizes A. We denote

$$g^{-1}=\mu ank'^{-1}\in MANK=G.$$

Thus,

$$\begin{aligned}&\left(\widehat{f_0}(\pi)\cdot\prod_{i=1}^m\frac{\partial\widehat{f_i}(\pi)}{\partial_{\tau(i)}}\right)s(k)\\ &=\int_{KMAN}\int_{G^{\times k}} H_{\tau(1)}\Big(g_1^{-1}\kappa(g_1\dots g_mk)\Big)\dots H_{\tau(m)}\Big(g_m^{-1}\kappa(g_mk)\Big)\\ &e^{\langle\log\varphi+\rho,\log a\rangle}\cdot\eta^M(\mu)\cdot f_0\Big(k\mu ank'^{-1}(g_1g_2\dots g_m)^{-1}\Big)f_1(g_1)\dots f_m(g_m)s(k').\end{aligned}\tag{4.7}$$

By Lemma 3.1,

$$H_{\tau(i)}\big(g_i^{-1}\kappa(g_i\dots g_mk)\big) = H_{\tau(i)}\big(g_{i+1}\dots g_mk\big) - H_{\tau(i)}\big(g_i\dots g_mk\big).$$

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Thus,

$$\begin{aligned}
&\sum_{\tau\in S_m}\mathrm{sgn}(\tau)H_{\tau(1)}\big(g_1^{-1}\kappa(g_1\dots g_mk)\big)H_{\tau(2)}\big(g_2^{-1}\kappa((g_2g_3\dots g_mk)\big)\dots H_{\tau(m)}\big(g_m^{-1}\kappa(g_mk)\big)\\
&=\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\big(H_{\tau(1)}(g_2\dots g_mk)-H_{\tau(1)}(g_1g_2\dots g_mk)\big)\big(H_{\tau(2)}(g_3\dots g_mk)-H_{\tau(2)}(g_2\dots g_mk)\big)\\
&\quad\dots\big(H_{\tau(m-1)}(g_mk)-H_{\tau(m-1)}(g_{m-1}g_mk)\big)\big(-H_{\tau(m)}(g_mk)\big).
\end{aligned}\tag{4.8}$$

By induction on m, we can prove that the right-hand side of Equation (4.8) equals

$$(-1)^m\sum_{\tau\in S_m}\mathrm{sgn}(\tau)H_{\tau(1)}\big(g_1\dots g_mk\big)H_{\tau(2)}\big(g_2g_3\dots g_mk\big)\cdot H_{\tau(m)}\big(g_mk\big).$$

By (4.7) and (4.8), we conclude that

$$\begin{aligned}
&\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\Big(\widehat{f_0}(\pi)\cdot\prod_{i=1}^m\frac{\partial\widehat{f_i}(\pi)}{\partial_{\tau(i)}}\Big)s(k)\\
&\quad=(-1)^m\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\int_{KMAN}\int_{G^{\times k}}H_{\tau(1)}\big(g_1\dots g_mk\big)H_{\tau(2)}\big(g_2g_3\dots g_mk\big)\cdot H_{\tau(m)}\big(g_mk\big)\\
&\quad\quad e^{\langle\log\varphi+\rho,\log a\rangle}\cdot\eta^M(\mu)\cdot f_0\big(k\mu ank'^{-1}(g_1g_2\dots g_m)^{-1}\big)f_1(g_1)\dots f_m(g_m)s(k').
\end{aligned}$$

Expressing it as a kernel operator, we have

$$\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot\Big(\widehat{f_0}(\pi)\cdot\prod_{i=1}^m\frac{\partial\widehat{f_i}(\pi)}{\partial_{\tau(i)}}\Big)s(k)=\int_K L(k,k')s(k')dk',$$

where

$$\begin{aligned}
L(k,k')=(-1)^m&\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\int_{MAN}\int_{G^{\times k}}H_{\tau(1)}\big(g_1\dots g_mk\big)H_{\tau(2)}\big(g_2g_3\dots g_mk\big)\cdot H_{\tau(m)}\big(g_mk\big)\\
&e^{\langle\log\varphi+\rho,\log a\rangle}\cdot\eta^M(\mu)\cdot f_0(k\mu ank'^{-1}(g_1g_2\dots g_m)^{-1})f_1(g_1)\dots f_m(g_m).
\end{aligned}$$

The proposition follows from the fact that $T_\pi=\int_K L(k,k)dk$. □

Suppose that $P_1=M_1A_1N_1$ and $P_2=M_2A_2N_2$ are two cuspidal parabolic subgroups such that P_1 is more noncompact than P_2. Moreover, we assume that the induced representation $\mathrm{Ind}_{P_1}^G\big(\sigma_1\otimes\varphi_1\big)$ is reducible and decomposes into

$$\mathrm{Ind}_{P_1}^G\big(\sigma_1\otimes\varphi_1\big)=\bigoplus_k\mathrm{Ind}_{P_2}^G\big(\delta_k\otimes\varphi_2\big),$$

where σ_1 is a discrete series representation of M_1 and δ_k are different limit of discrete series representations of M_2. We decompose

$$\widehat{A}_2=\widehat{A}_\circ\times\widehat{A}_S,\quad \widehat{A}_1=\widehat{A}_2\times\widehat{A}_{12}=\widehat{A}_\circ\times\widehat{A}_S\times\widehat{A}_{12}.$$

Let $h_i\in\mathcal{C}(\widehat{A}_1)$, and v_i,w_i be unit K-finite vectors in $\mathrm{Ind}_{P_1}^G(\sigma_1)$ for $i=0,\dots,m$. We put

$$\widehat{f_i}=h_i\cdot v_i\otimes w_i^*\in\mathcal{C}\big(\widehat{A}_1,\mathcal{L}(\mathrm{Ind}_{P_1}^G(\sigma_1))\big).$$

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The following lemma follows from Definition 4.9.

Lemma 4.11. *Suppose that* $\pi = \mathrm{Ind}_{P_2}^{G}(\delta_k \otimes \varphi_2)$. *If*

$$v_i = w_{i+1}, \quad i = 0, \ldots, m-1,$$

and $v_m = w_0$, *then*

$$T_\pi(\widehat{f}_0, \ldots, \widehat{f}_m) = \sum_{\tau \in S_m} \mathrm{sgn}(\tau) \cdot h_0\big((\varphi_2, 0)\big) \cdot \prod_{i=1}^{m} \frac{\partial h_i\big((\varphi_2, 0)\big)}{\partial_{\tau(i)}}.$$

Otherwise, $T_\pi(\widehat{f}_0, \ldots, \widehat{f}_m) = 0$.

4.4. Cocycles on $\widehat{G}_{\mathrm{temp}}$

Let $P_\circ = M_\circ A_\circ N_\circ$ be a maximal cuspidal parabolic subgroup (cf. Definition 2.1) and T be the maximal torus of K. In particular, the Lie algebra $\mathfrak{t}$ of T gives a Cartan subalgebra of $\mathfrak{m}_\circ$ and $T \subseteq M_\circ$.

Definition 4.12. For an irreducible tempered representation π of G, we define

$$\mathcal{A}(\pi) = \left\{ \eta^{M_\circ} \otimes \varphi \in (\widehat{M_\circ A_\circ})_{\mathrm{temp}} \,\middle|\, \mathrm{Ind}_{P_\circ}^{G}(\eta^{M_\circ} \otimes \varphi) = \pi \right\}.$$

Definition 4.13. Let $m(\eta^{M_\circ})$ be the Plancherel density for the irreducible tempered representations $\eta^{M_\circ}$ of $M_\circ$. We put

$$\mu(\pi) = \sum_{\eta^{M_\circ} \otimes \varphi \in \mathcal{A}(\pi)} m(\eta^{M_\circ}).$$

Recall the Plancherel formula

$$f(e) = \int_{\pi \in \widehat{G}_{\mathrm{temp}}} \mathrm{Trace}\big(\widehat{f}(\pi)\big) \cdot m(\pi) d\pi, \tag{4.9}$$

where $m(\pi)$ is the Plancherel density for the G-representation π.

Definition 4.14. We define $\widehat{\Phi}_e$ by the following formula:

$$\widehat{\Phi}_e(\widehat{f}_0, \ldots, \widehat{f}_m) = \int_{\pi \in \widehat{G}_{\mathrm{temp}}} T_\pi(\widehat{f}_0, \ldots, \widehat{f}_m) \cdot \mu(\pi) \cdot d\pi.$$

Theorem 4.15. *For any* $f_0, \ldots, f_m \in \mathcal{C}(G)$,

$$\Phi_{P_\circ, e}(f_0, \ldots, f_m) = (-1)^m \widehat{\Phi}_e(\widehat{f}_0, \ldots, \widehat{f}_m).$$

The proof of Theorem 4.15 is presented in Section 4.5.

Example 4.16. Suppose that $G = \mathbb{R}^m$. Let

$$x^i = (x^i_1, \ldots x^i_m) \in \mathbb{R}^m$$

be the coordinates of $\mathbb{R}^m$. On $\mathcal{C}(\mathbb{R}^m)$, the cocycle $\Phi_{P_\circ, e}$ is given as follows:

$$\begin{aligned} &\Phi_{P_\circ, e}(f_0, \ldots, f_m) \\ &= \sum_{\tau \in S_m} \mathrm{sgn}(\tau) \int_{x^1 \in \mathbb{R}^m} \cdots \int_{x^m \in \mathbb{R}^m} x^1_{\tau(1)} \ldots x^m_{\tau(m)} f_0\big(-(x^1 + \cdots + x^m)\big) f_1(x^1) \ldots f_m(x^m). \end{aligned}$$

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However, the cocycle $\widehat{\Phi}_e$ on $\mathcal{C}(\widehat{\mathbb{R}}^m)$ is given as follows:

$$\widehat{\Phi}_e(\widehat{f}_0, \dots, \widehat{f}_m) = (-\sqrt{-1})^m \int_{\mathbb{R}^m} \widehat{f}_0 d\widehat{f}_1 \dots d\widehat{f}_m.$$

To see they are equal, we can compute

$$\begin{aligned}
\Phi_{P_\circ, e}(f_0, \dots, f_m) &= \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot \Big(f_0 * (x_{\tau(1)} f_1) * \dots * (x_{\tau(m)} f_m)\Big)(0). \\
&= \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot \int_{\mathbb{R}^m} \Big(\widehat{f_0 * (x_{\tau(1)} f_1) * \dots * (x_{\tau(m)} f_m)}\Big) \\
&= \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot \int_{\mathbb{R}^m} \Big(\widehat{f}_0 \cdot \widehat{(x_{\tau(1)} f_1)} \dots \widehat{(x_{\tau(m)} f_m)}\Big) \\
&= \sum_{\tau \in S_m} \operatorname{sgn}(\tau) \cdot \int_{\mathbb{R}^m} (\frac{1}{-\sqrt{-1}})^m \Big(\widehat{f}_0 \cdot \frac{\partial \widehat{f}_1}{\partial_{\tau(1)}} \dots \frac{\partial \widehat{f}_m}{\partial_{\tau(m)}}\Big) \\
&= (\sqrt{-1})^m \int_{\mathbb{R}^m} \widehat{f}_0 d\widehat{f}_1 \dots d\widehat{f}_m.
\end{aligned}$$

To introduce the cocycle $\widehat{\Phi}_t$ for any $t \in T^{\mathrm{reg}}$, we first recall the formula (C.7) for orbital integrals splits into three parts:

$$\text{regular part} + \text{singular part} + \text{higher part}.$$

Accordingly, for any $t \in T^{\mathrm{reg}}$, we define

- regular part: for regular $\lambda \in \Lambda_K^* + \rho_c$ (see Definition B.1), we define

$$\Big[\widehat{\Phi}_t(\widehat{f}_0, \dots, \widehat{f}_m)\Big]_\lambda = \left(\sum_{w \in W_K} (-1)^w \cdot e^{w \cdot \lambda}(t) \right) \cdot \int_{\varphi \in \widehat{A}_\circ} T_{\mathrm{Ind}_{P_\circ}^G (\sigma^{M_\circ}(\lambda) \otimes \varphi)}(\widehat{f}_0, \dots, \widehat{f}_m) \cdot d\varphi,$$

  where $\sigma^{M_\circ}(\lambda)$ is the discrete series representation of $M_\circ$ with Harish-Chandra parameter λ.
- singular part: for any singular $\lambda \in \Lambda_K^* + \rho_c$, we define

$$\Big[\widehat{\Phi}_t(\widehat{f}_0, \dots, \widehat{f}_m)\Big]_\lambda = \frac{\sum_{w \in W_K} (-1)^w \cdot e^{w \cdot \lambda}(t)}{n(\lambda)} \cdot \sum_{i=1}^{n(\lambda)} \int_{\varphi \in \widehat{A}_\circ} \epsilon(i) \cdot T_{\mathrm{Ind}_{P_\circ}^G (\sigma_i^{M_\circ}(\lambda) \otimes \varphi)}(\widehat{f}_0, \dots, \widehat{f}_m) \cdot d\varphi,$$

  where $\sigma_i^{M_\circ}$ are limit of discrete series representations of $M_\circ$ with Harish-Chandra parameter λ and $n(\lambda)$ is the number of different limit of discrete series representations with Harish-Chandra parameter λ, and $\epsilon(i) = 1$ for $i = 1, \dots \frac{n(\lambda)}{2}$ and $\epsilon(i) = -1$ for $i = \frac{n(\lambda)}{2} + 1, \dots n(\lambda)$ (compare with the notations in Theorem C.7).
- higher part:

$$\Big[\widehat{\Phi}_t(\widehat{f}_0, \dots, \widehat{f}_m)\Big]_{\mathrm{high}} = \int_{\pi \in \widehat{G}_{\mathrm{temp}}^{\mathrm{high}}} T_\pi(\widehat{f}_0, \dots, \widehat{f}_m) \cdot \left(\sum_{\eta^{M_\circ} \otimes \varphi \in \mathcal{A}(\pi)} \kappa^{M_\circ}(\eta^{M_\circ}, t) \right) \cdot d\varphi,$$

  where the functions $\kappa^{M_\circ}(\eta^{M_\circ}, t)$ are defined in Subsection C.3, and

$$\widehat{G}_{\mathrm{temp}}^{\mathrm{high}} = \{\pi \in \widehat{G}_{\mathrm{temp}} | \pi = \mathrm{Ind}_{P_\circ}^G(\eta^{M_\circ} \otimes \varphi), \eta^{M_\circ} \in (\widehat{M}_\circ)_{\mathrm{temp}}^{\mathrm{high}}\},$$

where $(\widehat{M_\circ})^{\text{high}}_{\text{temp}}$ is the set of irreducible tempered representations of $M_\circ$ which are not (limit of) discrete series representations.

Definition 4.17. For any element $t \in T^{\text{reg}}$, we define

$$\begin{aligned}&\widehat{\Phi}_t(\widehat{f_0}, \dots, \widehat{f_m})\\ &= \sum_{\text{regular } \lambda \in \Lambda^*_K + \rho_c} \left[\widehat{\Phi}_h(\widehat{f_0}, \dots, \widehat{f_m})\right]_\lambda + \sum_{\text{singular } \lambda \in \Lambda^*_K + \rho_c} \left[\widehat{\Phi}_t(\widehat{f_0}, \dots, \widehat{f_m})\right]_\lambda + \left[\widehat{\Phi}_t(\widehat{f_0}, \dots, \widehat{f_m})\right]_{\text{high}}.\end{aligned}$$

Theorem 4.18. *For any $t \in T^{reg}$, and $f_0, \dots, f_m \in \mathcal{C}(G)$,*

$$\Delta^{M_\circ}_T(t)\Phi_{P_\circ, t}(f_0, \dots, f_m) = (-1)^m \widehat{\Phi}_t(\widehat{f_0}, \dots, \widehat{f_m}).$$

The proof of Theorem 4.18 is presented in Section 4.6.

4.5. *Proof of Theorem 4.15*

We split the proof into several steps:

Step 1: Change the integral from $\widehat{G}_{\text{temp}}$ to $(\widehat{M_\circ A_\circ})_{\text{temp}}$:

$$\begin{aligned}\widehat{\Phi}_e(\widehat{f_0}, \dots, \widehat{f_m}) &= \int_{\pi \in \widehat{G}_{\text{temp}}} T_\pi(\widehat{f_0}, \dots, \widehat{f_m}) \cdot \mu(\pi) \cdot d\pi\\ &= \int_{\eta^{M_\circ} \otimes \varphi \in (\widehat{M_\circ A_\circ})_{\text{temp}}} T_{\text{Ind}^G_{P_\circ}(\eta^{M_\circ} \otimes \varphi)}(\widehat{f_0}, \dots, \widehat{f_m}) \cdot m(\eta^{M_\circ}).\end{aligned}$$

Step 2: Replace $T_{\text{Ind}^G_{P_\circ}(\eta^{M_\circ} \otimes \varphi)}$ in the above expression of $\widehat{\Phi}_e$ by Equation (4.6):

$$\begin{aligned}&\widehat{\Phi}_e(\widehat{f_0}, \dots, \widehat{f_m})\\ &= (-1)^m \sum_{\tau \in S_m} \text{sgn}(\tau) \int_{\eta^{M_\circ} \otimes \varphi \in (\widehat{M_\circ A_\circ})_{\text{temp}}} \int_{K M_\circ A_\circ N_\circ} \int_{G^{\times m}} H_{\tau(1)}(g_1 \dots g_m k) \dots H_{\tau(m)}(g_m k)\\ &\quad e^{\langle \log \varphi + \rho, \log a \rangle} \cdot \Theta^{M_\circ}(\eta^{M_\circ})(m) \cdot f_0\Big(kmank^{-1}(g_1 g_2 \dots g_m)^{-1}\Big) f_1(g_1) \dots f_m(g_m) \cdot m(\eta^{M_\circ}).\end{aligned}$$

Step 3: Simplify $\widehat{\Phi}_e$ by Harish-Chandra's Plancherel formula. We write

$$\begin{aligned}&\widehat{\Phi}_e(\widehat{f_0}, \dots, \widehat{f_m})\\ &= (-1)^m \sum_{\tau \in S_m} \text{sgn}(\tau) \int_{K N_\circ} \int_{G^{\times m}} H_{\tau(1)}(g_1 \dots g_m k) \dots H_{\tau(m)}(g_m k) \cdot f' \cdot f_1(g_1) \dots f_m(g_m),\end{aligned}$$

where the function f' is defined by the following formula:

$$\begin{aligned}f'(k, n, g_1, \dots, g_m) = \int_{\eta^{M_\circ} \otimes \varphi \in (\widehat{M_\circ A_\circ})_{\text{temp}}} \int_{M_\circ A_\circ} & e^{\langle \log \varphi + \rho, \log a \rangle}\\ &\Theta^{M_\circ}(\eta^{M_\circ})(m) \cdot f_0(kmank^{-1}(g_1 g_2 \dots g_m)^{-1}) \cdot m(\eta^{M_\circ}).\end{aligned}$$

If we put

$$c(k, m, a, n, g_1, \dots, g_m) = e^{\langle \rho, \log a \rangle} \cdot f_0\Big(kmank^{-1}(g_1 g_2 \dots g_m)^{-1}\Big),$$

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then

$$f'(k,n,g_1,\ldots,g_m)=\int_{\eta^{M_\circ}\otimes\varphi\in(\widehat{M_\circ A_\circ})_{\text{temp}}}\Big(\Theta^{M_\circ}(\eta^{M_\circ})\otimes\varphi\Big)(c)\cdot m(\eta^{M_\circ}).$$

By (4.9),

$$f'=c(k,e,e,n,g_1,\ldots,g_m)=f_0\Big(knk^{-1}(g_1g_2\ldots g_m)^{-1}\Big).$$

This completes the proof.

4.6. Proof of Theorem 4.18

Our proof strategy for Theorem 4.18 is similar to the one used to prove Theorem 4.15. We split its proof into 3 steps as before.

Step 1: Let Λ_T^* be the weight lattice for T and $\Lambda^*_{K\cap M_\circ}$ be the intersection of Λ_T^* and the positive Weyl chamber for the group $M_\circ\cap K$. We denote by $\rho_c^{M_\circ\cap K}$ the half sum of positive roots for $M_\circ\cap K$. For any $\lambda\in\Lambda^*_{K\cap M_\circ}$, we can find an element $w\in W_K/W_{K\cap M_\circ}$ such that $w\cdot\lambda\in\Lambda_K^*$. Moreover, for any $w\in W_K/W_{K\cap M_\circ}$,

$$\operatorname{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda)\otimes\varphi)\cong\operatorname{Ind}_{P_\circ}^G(\sigma^{M_\circ}(w\cdot\lambda)\otimes\varphi).\tag{4.10}$$

For the regular part,

$$\begin{aligned}
&\sum_{\text{regular }\lambda\in\Lambda_K^*+\rho_c}\Big[\widehat{\Phi}_t(\widehat{f}_0,\ldots,\widehat{f}_m)\Big]_\lambda\\
&=\sum_{\text{regular }\lambda\in\Lambda_K^*+\rho_c}\left(\sum_{w\in W_K}(-1)^w\cdot e^{w\cdot\lambda}(t)\right)\cdot\int_{\varphi\in\widehat{A}_\circ}T_{\operatorname{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda)\otimes\varphi)}(\widehat{f}_0,\ldots,\widehat{f}_m)\cdot d\varphi\\
&=\sum_{\text{regular }\lambda\in\Lambda^*_{K\cap M_\circ}+\rho_c}\left(\sum_{w\in W_{K\cap M_\circ}}(-1)^w\cdot e^{w\cdot\lambda}(t)\right)\cdot\int_{\varphi\in\widehat{A}_\circ}T_{\operatorname{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda)\otimes\varphi)}(\widehat{f}_0,\ldots,\widehat{f}_m)\cdot d\varphi.
\end{aligned}\tag{4.11}$$

Here, the last equation follows from (4.10). Remembering that the above is anti-invariant under the W_K-action, we can replace ρ_c by $\rho_c^{M_\circ\cap K}$. That is, (4.11) equals

$$\sum_{\text{regular }\lambda\in\Lambda^*_{K\cap M_\circ}+\rho_c^{M_\circ\cap K}}\left(\sum_{w\in W_{K\cap M_\circ}}(-1)^w\cdot e^{w\cdot\lambda}(t)\right)\cdot\int_{\varphi\in\widehat{A}_\circ}T_{\operatorname{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda)\otimes\varphi)}(\widehat{f}_0,\ldots,\widehat{f}_m)\cdot d\varphi.$$

Similarly, for the singular part,

$$\begin{aligned}
&\sum_{\text{singular }\lambda\in\Lambda_K^*+\rho_c}\Big[\widehat{\Phi}_t(\widehat{f}_0,\ldots,\widehat{f}_m)\Big]_\lambda=\sum_{\text{singular }\lambda\in\Lambda^*_{K\cap M_\circ}+\rho_c^{M_\circ\cap K}}\left(\sum_{w\in W_{K\cap M_\circ}}(-1)^w\cdot e^{w\cdot\lambda}(t)\right)\\
&\times\left(\sum_{i=1}^{n(\lambda)}\frac{\epsilon(i)}{n(\lambda)}\cdot\int_{\varphi\in\widehat{A}_\circ}T_{\operatorname{Ind}_{P_\circ}^G(\sigma_i^{M_\circ}(\lambda)\otimes\varphi)}(\widehat{f}_0,\ldots,\widehat{f}_m)\right).
\end{aligned}$$

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Finally, for the higher part,

$$\begin{aligned}\left[\widehat{\Phi}_t(\widehat{f}_0,\dots,\widehat{f}_m)\right]_{\text{high}} &= \int_{\pi\in\widehat{G}^{\text{high}}_{\text{temp}}} T_\pi(\widehat{f}_0,\dots,\widehat{f}_m)\left(\sum_{\eta^{M_\circ}\otimes\varphi\in\mathcal{A}(\pi)}\kappa^{M_\circ}(\eta^{M_\circ},t)\right)\cdot d\varphi\\ &= \int_{\eta^{M_\circ}\otimes\varphi\in\widehat{M}^{\text{high}}_{\text{temp}}\times\widehat{A}_\circ} T_{\text{Ind}^G_{P_\circ}(\eta^{M_\circ}\otimes\varphi)}(\widehat{f}_0,\dots,\widehat{f}_m)\cdot\kappa^{M_\circ}(\eta^{M_\circ},t).\end{aligned}$$

Step 2: We apply Proposition 4.10 and obtain the following.

∘ regular part:

$$\begin{aligned}&\sum_{\text{regular }\lambda\in\Lambda^*_K+\rho_c}\left[\widehat{\Phi}_t(\widehat{f}_0,\dots,\widehat{f}_m)\right]_\lambda\\ &= (-1)^m\sum_{\tau\in S_m}\text{sgn}(\tau)\sum_{\text{regular }\lambda\in\Lambda^*_{K\cap M_\circ}+\rho_c^{M_\circ\cap K}}\left(\sum_{w\in W_{K\cap M_\circ}}(-1)^w\cdot e^{w\cdot\lambda}(t)\right)\\ &\quad\int_{\varphi\in\widehat{A}_\circ}\int_{KM_\circ A_\circ N_\circ}\int_{G^{\times m}} H_{\tau(1)}(g_1\dots g_mk)\dots H_{\tau(m)}(g_mk)\\ &\quad e^{\langle\log\varphi+\rho,\log a\rangle}\cdot\Theta^{M_\circ}(\lambda)(m)\cdot f_0\big(kmank^{-1}(g_1g_2\dots g_m)^{-1}\big)f_1(g_1)\dots f_m(g_m).\end{aligned}$$

∘ singular part:

$$\begin{aligned}&\sum_{\text{singular }\lambda\in\Lambda^*_K+\rho_c}\left[\widehat{\Phi}_t(\widehat{f}_0,\dots,\widehat{f}_m)\right]_\lambda\\ &= (-1)^m\sum_{\tau\in S_m}\text{sgn}(\tau)\sum_{\text{singular }\lambda\in\Lambda^*_{K\cap M_\circ}+\rho_c^{M_\circ\cap K}}\sum_{i=1}^{n(\lambda)}\left(\frac{\epsilon(i)}{n(\lambda)}\sum_{w\in W_{K\cap M_\circ}}(-1)^w\cdot e^{w\cdot\lambda}(t)\right)\\ &\quad\int_{\varphi\in\widehat{A}_\circ}\int_{KM_\circ A_\circ N_\circ}\int_{G^{\times m}} H_{\tau(1)}(g_1\dots g_mk)\dots H_{\tau(m)}(g_mk)\\ &\quad e^{\langle\log\varphi+\rho,\log a\rangle}\cdot\Theta_i^{M_\circ}(\lambda)(m)\cdot f_0\Big(kmank^{-1}(g_1g_2\dots g_m)^{-1}\Big)f_1(g_1)\dots f_m(g_m).\end{aligned}$$

∘ higher part:

$$\begin{aligned}&\left[\widehat{\Phi}_t(\widehat{f}_0,\dots,\widehat{f}_m)\right]_{\text{high}}\\ &= (-1)^m\int_{\eta^{M_\circ}\otimes\varphi\in\widehat{M}^{\text{high}}_{\text{temp}}\times\widehat{A}_\circ}\int_{KM_\circ A_\circ N_\circ}\int_{G^{\times m}} H_{\tau(1)}(g_1\dots g_mk)\dots H_{\tau(m)}(g_mk)\\ &\quad e^{\langle\log\varphi+\rho,\log a\rangle}\cdot\Theta^{M_\circ}(\eta^{M_\circ})(m)\cdot f_0\big(kmank^{-1}(g_1g_2\dots g_m)^{-1}\big)\\ &\quad f_1(g_1)\dots f_m(g_m)\cdot\kappa^{M_\circ}(\eta^{M_\circ},t).\end{aligned}$$

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Step 3: All the above computations imply that

$$\begin{aligned}
&\widehat{\Phi}_t(\widehat{f_0},\dots,\widehat{f_m})\\
&=\sum_{\text{regular }\lambda\in\Lambda_K^*+\rho_c}\left[\widehat{\Phi}_t(\widehat{f_0},\dots,\widehat{f_m})\right]_\lambda+\sum_{\text{singular }\lambda\in\Lambda_K^*+\rho_c}\left[\widehat{\Phi}_t(\widehat{f_0},\dots,\widehat{f_m})\right]_\lambda+\left[\widehat{\Phi}_t(\widehat{f_0},\dots,\widehat{f_m})\right]_{\text{high}}\\
&=(-1)^m\int_{KN_\circ}\int_{G^{\times m}}f'\cdot\left(\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\cdot H_{\tau(1)}\big(g_1\dots g_mk\big)\dots H_{\tau(m)}\big(g_mk\big)\cdot f_1(g_1)\dots f_m(g_m)\right),
\end{aligned}\tag{4.12}$$

where

$$\begin{aligned}
&f'(t,k,n,g_1,\dots,g_m)\\
&=\sum_{\text{regular }\lambda\in\Lambda^*_{K\cap M_\circ}+\rho_c^{M_\circ\cap K}}\left(\sum_{w\in W_{K\cap M_\circ}}(-1)^w\cdot e^{w\cdot\lambda}(t)\right)\cdot\int_{\varphi\in\widehat{A}_\circ}\Big(\Theta^{M_\circ}(\lambda)\otimes\varphi\Big)(c)\\
&+\sum_{\text{singular }\lambda\in\Lambda^*_{K\cap M_\circ}+\rho_c^{M_\circ\cap K}}\left(\frac{\sum_{w\in W_{K\cap M_\circ}}(-1)^w\cdot e^{w\cdot\lambda}(t)}{n(\lambda)}\right)\cdot\sum_{i=1}^{n(\lambda)}\epsilon(i)\cdot\int_{\varphi\in\widehat{A}_\circ}\Big(\Theta_i^{M_\circ}(\lambda)\otimes\varphi\Big)(c)\\
&+\int_{\eta^{M_\circ}\otimes\varphi\in\widehat{M}^{\text{high}}_{\text{temp}}\otimes\widehat{A}_\circ}\Big(\Theta^{M_\circ}(\eta^{M_\circ})\otimes\varphi\Big)(c)\cdot\kappa^{M_\circ}(\eta^{M_\circ},t),
\end{aligned}$$

where

$$c(k,m,a,n,g_1,\dots,g_m)=e^{\langle\rho,\log a\rangle}\cdot f_0\big(kmank^{-1}(g_1g_2\dots g_m)^{-1}\big).$$

Because T is a compact Cartan subgroup of $M_\circ$, the sign function in the definition of Harish-Chandra's orbital integral (1.1) is trivial [32, Section 8.1.1]. Hence, we apply Theorem C.7 to the function c and obtain

$$f'=F_c^T(t)=\Delta_T^{M_\circ}(t)\cdot\int_{h\in M_\circ/T_\circ}f_0\Big(khth^{-1}nk^{-1}(g_1g_2\dots g_m)^{-1}\Big).\tag{4.13}$$

By (4.12) and (4.13), we conclude that

$$\begin{aligned}
&\widehat{\Phi}_t(\widehat{f_0},\dots,\widehat{f_m})\\
&=\Delta_T^{M_\circ}(t)\cdot(-1)^m\sum_{\tau\in S_m}\mathrm{sgn}(\tau)\int_{h\in M_\circ/T_\circ}\int_{KN_\circ}\int_{G^{\times m}}H_{\tau(1)}\big(g_1\dots g_mk\big)\dots H_{\tau(m)}\big(g_mk\big)\\
&\quad f_0\big(khth^{-1}nk^{-1}(g_1g_2\dots g_m)^{-1}\big)f_1(g_1)\dots f_m(g_m)\\
&=(-1)^m\Delta_T^{M_\circ}(t)\cdot\Phi_{P_\circ,t}(f_0,\dots,f_m).
\end{aligned}$$

This completes the proof of Theorem 4.18.

5. Higher Index Pairing

In this section, we study the K-theory of the reduced group C^*-algebra of G by computing its pairing with $\Phi_{P_\circ,t}$ for $t\in T^{\text{reg}}\cap M_\circ$ and $\Phi_{P_\circ,e}$. Moreover, we construct a group isomorphism

$$\mathcal{F}\colon K_*(C_r^*(G))\to\mathrm{Rep}(K),$$

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where $\mathrm{Rep}(K)$ is the character ring of the compact Lie group K. By replacing G with $G\times\mathbb{R}$ if necessary, we may assume that $\dim(A_\circ)$ is even.

5.1. Generators of $K_0(C_r^*(G))$

In Theorem C.4, we explain that the K-theory group of $C_r^*(G)$ is a free abelian group generated by the following components:

$$\begin{aligned} K_0(C_r^*(G)) &\cong \bigoplus_{[P,\sigma]^{\mathrm{ess}}} K_0\big(\mathcal{K}(C_r^*(G)_{[P,\sigma]})\big) \\ &\cong \bigoplus_{\lambda\in\Lambda_K^*+\rho_c} \mathbb{Z}. \end{aligned} \tag{5.1}$$

Let $[P,\sigma]\in\mathcal{P}(G)$ be an essential class corresponding to $\lambda\in\Lambda_K^*+\rho_c$. In this subsection, we construct a generator of $K_0(C_r^*(G))$ associated to λ. We decompose $\widehat{A}_P=\widehat{A}_S\times\widehat{A}_\circ$ and denote $r=\dim\widehat{A}_S$ and $m=\dim\widehat{A}_\circ$. Let V be an r-dimensional complex vector space and W an m-dimensional Euclidean space. Take

$$z=(x_1,\cdots,x_r,y_1,\dots y_m),\quad x_i\in\mathbb{C},y_j\in\mathbb{R}$$

to be coordinates on $V\oplus W$. Assume that the finite group $(\mathbb{Z}_2)^r$ acts on V by simple reflections. In terms of coordinates,

$$(x_1,\cdots,x_r,y_1,\dots,y_m)\mapsto(\pm x_1,\cdots,\pm x_r,y_1,\dots,y_m).$$

Let us consider the Clifford algebra

$$\mathrm{Clifford}(V)\otimes\mathrm{Clifford}(W)$$

together with the spinor module $S=S_V\otimes S_W$. Here, the spinor modules are equipped with a $\mathbb{Z}_2$-grading:

$$S^+=S_V^+\otimes S_W^+\oplus S_V^-\otimes S_W^-,\quad S^-=S_V^+\otimes S_W^-\oplus S_V^-\otimes S_W^+.$$

Let $\mathcal{C}(V),\mathcal{C}(W)$ and $\mathcal{C}(V\oplus W)$ be the algebra of Schwartz functions on V, W and $V\oplus W$. For any $z\in V\oplus W$, the Clifford action $c(z)\colon S^\pm\longrightarrow S^\mp$ is defined as follows.

Let $e_1,\dots e_{2^{r-1}}$ be a basis for S_V^+, let $e_{2^{r-1}+1},\dots e_{2^r}$ be a basis for S_V^-, and let $f_1,\dots f_{2^{\frac{m}{2}}}$ be a basis for S_W. We write

$$c_{i,j,k,l}(z)=\langle c(z)e_i\otimes f_l,e_j\otimes f_k\rangle,\quad 1\le i,j\le 2^r,1\le k,l\le 2^{\frac{m}{2}}$$

and define

$$T:=\begin{pmatrix} e^{-|z|^2}\cdot\mathrm{id}_{S^+} & e^{-\frac{|z|^2}{2}}(1-e^{-|z|^2})\cdot\frac{c(z)}{|z|^2} \\ e^{-\frac{|z|^2}{2}}c(z) & (1-e^{-|z|^2})\cdot\mathrm{id}_{S^-}\end{pmatrix}-\begin{pmatrix}0&0\\0&\mathrm{id}_{S^-}\end{pmatrix},$$

which is a $2^{r+\frac{m}{2}}\times 2^{r+\frac{m}{2}}$ matrix:

$$\big(t_{i,j,k,l}\big),\quad 1\le i,j\le 2^r,1\le k,l\le 2^{\frac{m}{2}},$$

with $t_{i,j,k,l}\in\mathcal{C}(V\oplus W)$.

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Definition 5.1. On the m-dimensional Euclidean space W, we can define

$$B^m = \begin{pmatrix} e^{-|y|^2} \cdot \mathrm{id}_{S_W^+} & e^{-\frac{|y|^2}{2}}(1-e^{-|y|^2}) \cdot \frac{c(y)}{|y|^2} \\ e^{-\frac{|y|^2}{2}} c(y) & (1-e^{-|y|^2}) \cdot \mathrm{id}_{S_W^-} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_{S_W^-} \end{pmatrix},$$

which is a $2^{\frac{m}{2}} \times 2^{\frac{m}{2}}$ matrix:

$$(b_{k,l}), \quad 1 \le k, l \le 2^{\frac{m}{2}},$$

with $b_{k,l} \in \mathcal{C}(W)$. By straightforward computation, one can check that both the two matrices

$$\begin{pmatrix} e^{-|y|^2} \cdot \mathrm{id}_{S_W^+} & e^{-\frac{|y|^2}{2}}(1-e^{-|y|^2}) \cdot \frac{c(y)}{|y|^2} \\ e^{-\frac{|y|^2}{2}} c(y) & (1-e^{-|y|^2}) \cdot \mathrm{id}_{S_W^-} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_{S_W^-} \end{pmatrix}$$

are idempotents. In fact, B^m is the *Bott generator* in $K_0(C_0(W)) \cong \mathbb{Z}$.

Lemma 5.2. *If we restrict to $W \subset V \oplus W$ (that is, $x = 0$), then*

$$T\big|_{x=0} = \begin{pmatrix} \mathrm{id}_{S_V^+} & 0 \\ 0 & -\mathrm{id}_{S_V^-} \end{pmatrix} \otimes B^m.$$

Proof. By definition, we have that

- $t_{i,j,k,l} = e^{-z^2}$ when $e_i, e_j, f_k, f_l \in S^+$;
- $t_{i,j,k,l} = -e^{-z^2}$ when $e_i, e_j, f_k, f_l \in S^-$;
- $t_{i,j,k,l} = e^{-\frac{|z|^2}{2}}(1-e^{-|z|^2}) \cdot \frac{c_{i,j,k,l}(z)}{|z|^2}$ when $e_i, f_k, \in S^+$ and $e_j, f_l, \in S^-$;
- $t_{i,j,k,l} = e^{-|z|^2} \cdot c_{i,j,k,l}(z)$ when $e_i, f_k, \in S^-$ and $e_j, f_l, \in S^+$.

Moreover, the Clifford action $c(z)$ equals

$$c(x) \otimes 1 + 1 \otimes c(y) \in \mathrm{End}(S_V) \otimes \mathrm{End}(S_W)$$

for $z = (x, y) \in V \oplus W$. Thus, $c_{i,j,k,l}(z)\big|_{x=0} = c_{k,l}(y)$. This completes the proof. □

Let σ be a discrete series representation of M_P and $\varphi \in \widehat{A}_\circ$. Then,

$$\varphi \otimes 1 \in \widehat{A}_\circ \times \widehat{A}_S = \widehat{A}_P.$$

Because $[P, \sigma]$ is essential, the induced representation decomposes as

$$\mathrm{Ind}_P^G(\sigma \otimes \varphi \otimes 1) = \bigoplus_{i=1}^{2^r} \mathrm{Ind}_{P_\circ}^G\left(\delta_i \otimes \varphi\right),$$

where δ_i are limit of discrete series representations of $M_\circ$. By Equation (B.1), the characters of the limit of discrete series representations of δ_i are all the same up to a sign after restricting to a compact Cartan subgroup of M_P. We can organize the numbering so that

$$\delta_i, \quad i = 1, \ldots 2^{r-1}$$

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have the same character after restriction and

$$\delta_i, \quad i = 2^{r-1}+1, \dots 2^r$$

have the same character. In particular, δ_i with $1 \leq i \leq 2^{r-1}$ and δ_j with $2^{r-1}+1 \leq j \leq 2^r$ have the opposite characters after restriction.

We fix 2^r unit K-finite vectors $v_i \in \mathrm{Ind}_P^G(\delta_i)$ and define

$$S_\lambda := \left(t_{i,j,k,l} \cdot v_i \otimes v_j^*\right). \tag{5.2}$$

The matrix

$$S_\lambda \in \left[\mathcal{C}\left(\widehat{A}_P, \mathcal{L}(\mathrm{Ind}_P^G \sigma)\right)\right]^{W_\sigma},$$

and it is an idempotent. By the Morita equivalence (C.3),

$$\mathcal{K}\left(\mathrm{Ind}_P^G \sigma\right)^{W_\sigma} \sim \left(C_0(\mathbb{R}) \rtimes \mathbb{Z}_2\right)^r \otimes C_0(\mathbb{R}^m).$$

Definition 5.3. We define

$$Q_\lambda \in M_{2^{r+m}}(\mathcal{C}(G))$$

to be the wave packet associated to S_λ. Then, $[Q_\lambda]$ is the generator in $K_0\left(C_r^*(G)_{[P,\sigma]}\right)$ for essential class $[P,\sigma] \in \mathcal{P}(G)$.

5.2. The main results

Let G be a linear connected real reductive Lie group with maximal compact subgroup K. We choose a maximal torus T of K, and $P_\circ$ a maximal cuspidal parabolic subgroup of G. It follows from Appendix C that for any $\lambda \in \Lambda_K^* + \rho_c$, there is a generator

$$[Q_\lambda] \in K\left(C_r^*(G)\right).$$

In Section 3, we defined a family of cyclic cocycles

$$\Phi_{P_\circ,e}, \qquad \Phi_{P_\circ,t} \in HC\left(\mathcal{C}(G)\right)$$

for all $t \in T^{\mathrm{reg}}$ and the maximal compact cuspidal parabolic subgroup $P_\circ$.

Theorem 5.4. *The index pairing between periodic cyclic cohomology and K-theory*

$$HP^{even}\left(\mathcal{C}(G)\right) \otimes K_0\left(\mathcal{C}(G)\right) \to \mathbb{C}$$

is given by

- *we have*

$$\langle \Phi_{P_\circ,e}, [Q_\lambda] \rangle = \frac{1}{|W_{M_\circ \cap K}|} \cdot \sum_{w \in W_K} m\left(\sigma^{M_\circ}(w \cdot \lambda)\right),$$

  where $\sigma^{M_\circ}(w\cdot\lambda)$ is the discrete series representation with Harish-Chandra parameter $w \cdot \lambda$, and $m\left(\sigma^{M_\circ}(w\cdot\lambda)\right)$ is its Plancherel measure;

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∘ *for any* $t \in T^{reg}$*,*

$$\langle \Phi_{P_\circ,t}, [Q_\lambda] \rangle = \frac{\sum_{w \in W_K} (-1)^w e^{w\cdot\lambda}(t)}{\Delta_T^{M_\circ}(t)}. \tag{5.3}$$

The proof of Theorem 5.4 is presented in Sections 5.3 and 5.4.

Corollary 5.5. *The index paring of* $[Q_\lambda]$ *and normalized higher orbital integral equals the character of the representation* $\mathrm{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda) \otimes \varphi)$ *at* $\varphi = 1$*. That is,*

$$\left\langle \frac{\Delta_T^{M_\circ}}{\Delta_T^G} \cdot \Phi_{P_\circ,t}, [Q_\lambda] \right\rangle = \Theta(P_\circ, \sigma^{M_\circ}(\lambda), 1)(t).$$

Proof. It follows from applying the character formula, Corollary B.6, to the right side of Equation (5.3). □

Remark 5.6. If the group G is of equal rank, then the normalization factor is trivial. And the above corollary says that the orbital integral equals the character of a (limit of) discrete series representations. This result in the equal rank case is also obtained by Hochs-Wang in [17] using a fixed point theorem and the Connes-Kasparov isomorphism. In contrast to the Hochs-Wang approach, our proof is based on representation theory and does not use any geometry of the homogenous space G/K or the Connes-Kasparov theory.

We notice that though the cocycles $\Phi_{P_\circ,t}$ introduced in Definition 3.3 are only defined for regular elements in T, Theorem 5.4 suggests that the pairing $\Delta_T^{M_\circ}(t)\langle \Phi_{P_\circ,t}, [Q_\lambda] \rangle$ is a well-defined smooth function on T. This inspires us to introduce the following map.

Definition 5.7. Define a map $\mathcal{F}^T : K_0(C_r^*(G)) \to C^\infty(T)$ by

$$\mathcal{F}^T([Q_\lambda])(t) : = \Delta_T^{M_\circ} \cdot \langle \Phi_{P_\circ,t}, [Q_\lambda] \rangle, \quad \lambda \in \Lambda_K^* + \rho_c.$$

The map $\mathcal{F}^T$ is first defined on the regular part T^{reg} but can be extended smoothly to all elements in T as the right-hand side of the above equation extends to a smooth function on T.

By the Weyl character formula, for any irreducible K-representation V_λ with highest weight $\lambda \in \Lambda_K^*$, its character is given by

$$\Theta_\lambda(t) = \frac{\sum_{w \in W_K} (-1)^w e^{w\cdot(\lambda+\rho_c)}(t)}{\Delta_T^K(t)}.$$

Multiplying by Δ_T^K, we can identify $\mathrm{Rep}(K)$ with the following subset of $C^\infty(T)$:

$$\left\{ f \in C^\infty(T) \middle| f(t) = \sum_{\lambda \in \Lambda_K^* + \rho_c} n_\lambda \cdot \left(\sum_{w \in W_K} (-1)^w e^{w\cdot(\lambda)}(t) \right), \quad n_\lambda \in \mathbb{Z} \right\}.$$

Under the above identification, we have the following corollary.

Corollary 5.8. *The map* $\mathcal{F}^T : K_0(C_r^*(G)) \to Rep(K)$ *is an isomorphism of abelian groups.*

In [5, 6], we use the above property of $\mathcal{F}^T$ to show that $\mathcal{F}^T$ is actually the inverse of the Connes-Kasparov Dirac index map, index : $\mathrm{Rep}(K) \to K(C_r^*(G))$.

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Remark 5.9. The cyclic homology of the algebra $\mathcal{C}(G)$ was studied by Wassermann [34]. Wassermann's result and the unpublished description of the decomposition of $\mathcal{C}(G)$ analogous to Equation (C.2) implies that the Connes-Chern character

$$\mathrm{ch} : K_0\big(\mathcal{C}(G)\big) \to HP_{\mathrm{even}}\big(\mathcal{C}(G)\big)$$

induces an isomorphism

$$K_0\big(\mathcal{C}(G)\big) \otimes_{\mathbb{Z}} \mathbb{C} \cong HP_{\mathrm{even}}\big(\mathcal{C}(G)\big).$$

Corollary 5.8 shows that higher orbital integrals $\Phi_{P_\circ,t}, t \in T^{\mathrm{reg}}$ distinguish $K_0(\mathcal{C}(G))$. We can conclude from this fact that $\Phi_{P_\circ,t}, t \in T^{\mathrm{reg}}$ actually spans $HP^{\mathrm{even}}\big(\mathcal{C}(G)\big)$. As this outline of arguments involve some nontrivial unpublished works, we will not state this result as a 'theorem'.

5.3. *Regular case*

Suppose that $\lambda \in \Lambda_K^* + \rho_c$ is regular and $\sigma^{M_\circ}(\lambda)$ is the discrete series representation of $M_\circ$ with Harish-Chandra parameter λ. We consider the generator $[Q_\lambda]$, the wave packet associated to the matrix S_λ introduced in (5.2), corresponding to

$$\mathrm{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda) \otimes \varphi), \qquad \varphi \in \widehat{A}_\circ.$$

According to Theorem 4.15,

$$\begin{aligned}(-1)^m\langle \Phi_{P_\circ,e}, Q_\lambda\rangle &= \int_{\pi \in \widehat{G}_{\mathrm{temp}}} T_\pi\Bigg(\mathrm{Trace}\Bigg(\underbrace{S_\lambda \otimes \cdots \otimes S_\lambda}_{m+1}\Bigg)\Bigg) \cdot \mu(\pi)\\ &= \int_{\widehat{A}_\circ} T_{\mathrm{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda)\otimes\varphi)} \cdot \Bigg(\mathrm{Trace}\Bigg(\underbrace{S_\lambda \otimes \cdots \otimes S_\lambda}_{m+1}\Bigg)\Bigg) \cdot \mu\Big(\mathrm{Ind}_P^G(\sigma^{M_\circ}(\lambda) \otimes \varphi)\Big).\end{aligned}$$

By Definition 4.13,

$$\begin{aligned}\mu\Big(\mathrm{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda) \otimes \varphi)\Big) &= \mu\Big(\mathrm{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda) \otimes 1\Big)\\ &= \sum_{w \in W_K/W_{K\cap M_\circ}} m\Big(\sigma^{M_\circ}(w \cdot \lambda)\Big)\\ &= \frac{1}{|W_{K\cap M_\circ}|} \cdot \sum_{w \in W_K} m\Big(\sigma^{M_\circ}(w \cdot \lambda)\Big).\end{aligned}$$

Moreover, in the case of regular λ, $S_\lambda = [B^m \cdot (v \otimes v^*)]$, where B^m is the Bott generator for $K_0(\mathcal{C}(\widehat{A}_\circ))$ and v is a unit K-finite vector in $\mathrm{Ind}_{P_\circ}^G(\sigma^{M_\circ}(\lambda))$. By (4.2),

$$\int_{\widehat{A}_\circ} T_{\mathrm{Ind}_{P_\circ}^G(\sigma(\lambda)\otimes\varphi)}\Bigg(\mathrm{Trace}\big(\underbrace{S_\lambda \otimes \cdots \otimes S_\lambda}_{m+1}\big)\Bigg) = \langle B^m, b_m\rangle = 1,$$

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where $[b_m] \in HC^m(\mathcal{C}(\mathbb{R}^m))$ is the cyclic cocycle on $\mathcal{C}(\mathbb{R}^m)$ of degree m; cf. Example 4.16. We conclude that

$$\langle \Phi_{P_\circ,e}, [Q_\lambda] \rangle = \frac{(-1)^m}{|W_{K\cap M_\circ}|} \cdot \sum_{w\in W_K} m\Big(\sigma^{M_\circ}(w\cdot\lambda)\Big).$$

For the orbital integral $\Phi_{P_\circ,t}$, only the regular part will contribute. The computation is similar as above, and we conclude that

$$\begin{aligned}\langle \Phi_{P_\circ,t}, Q_\lambda \rangle &= (-1)^m \sum_{w\in W_K} (-1)^w e^{w\cdot\lambda}(t) \cdot \int_{\widehat{A}_\circ} T_{\mathrm{Ind}^G_{P_\circ}(\sigma^{M_\circ}(\lambda)\otimes\varphi)}\Bigg(\mathrm{Trace}\Bigg(\underbrace{S_\lambda\otimes\cdots\otimes S_\lambda}_{m+1}\Bigg)\Bigg)\\ &= (-1)^m \sum_{w\in W_K} (-1)^w e^{w\cdot\lambda}(t).\end{aligned}$$

5.4. *Singular case*

Suppose now that $\lambda \in \Lambda^*_K + \rho_c$ is singular. We decompose

$$\widehat{A}_P = \widehat{A}_\circ \times \widehat{A}_S, \qquad \varphi = (\varphi_1, \varphi_2).$$

We denote $r = \dim(A_S)$ and $m = \dim(A_\circ)$ as before. In this case, we have that

$$\mathrm{Ind}^G_P(\sigma^M \otimes \varphi_1 \otimes 1) = \bigoplus_{i=1}^{2^r} \mathrm{Ind}^G_{P_\circ}\left(\sigma^{M_\circ}_i \otimes \varphi_1\right),$$

where σ^M is a discrete series representation of M and $\sigma^{M_\circ}_i$, $i = 1, \ldots, 2^r$, are limit of discrete series representations of $M_\circ$ with Harish-Chandra parameter λ.

Recall that the generator Q_λ is the wave packet associated to S_λ. The index paring equals

$$(-1)^m \langle \Phi_{P_\circ,e}, Q_\lambda \rangle = \int_{\widehat{A}_P} T_{\mathrm{Ind}^G_P(\sigma^M\otimes\varphi)}\Bigg(\mathrm{Trace}\Bigg(\underbrace{S_\lambda\otimes\cdots\otimes S_\lambda}_{m+1}\Bigg)\Bigg) \cdot \mu\Big(\mathrm{Ind}^G_P(\sigma^M\otimes\varphi)\Big).$$

By the definition of μ,

$$\mu\Big(\mathrm{Ind}^G_P(\sigma^M\otimes\varphi)\Big) = \sum_{\eta^{M_\circ}\otimes\varphi_1 \in \mathcal{A}\big((\mathrm{Ind}^G_P(\sigma^M\otimes\varphi)\big)} m(\eta^{M_\circ}).$$

Thus, the function $\mu\big(\mathrm{Ind}^G_P(\sigma^M \otimes \varphi_1 \otimes \varphi_2)\big)$ is constant with respect to $\varphi_1 \in \widehat{A}_\circ$. It follows from (4.2) that

$$\begin{aligned}&(-1)^m \langle \Phi_{P_\circ,e}, Q_\lambda \rangle\\ &= \int_{\widehat{A}_S}\Bigg(\mu\Big(\mathrm{Ind}^G_P(\sigma^M\otimes\varphi)\Big) \cdot \int_{\widehat{A}_\circ} T_{\mathrm{Ind}^G_P(\sigma^M,\varphi)}\Bigg(\mathrm{Trace}(\underbrace{S_\lambda\otimes\cdots\otimes S_\lambda}_{m+1})\Bigg)\Bigg)\\ &= \sum_{\tau\in S_m} \sum_{j_0=i_1,\ldots j_{m-1}=i_m, j_m=i_0}\\ &\quad \int_{\widehat{A}_S}\Bigg(\mu\big(\mathrm{Ind}^G_P(\sigma^M\otimes\varphi\big) \int_{\widehat{A}_\circ} (-1)^\tau \cdot s_{i_0,j_0}(\varphi) \frac{\partial s_{i_1,j_1}(\varphi)}{\partial_{\tau(1)}} \cdots \frac{\partial s_{i_m,j_m}(\varphi)}{\partial_{\tau(m)}}\Bigg),\end{aligned}$$

where $s_{i,j} \in \mathcal{C}(\widehat{A}_\circ \times \widehat{A}_S)$ with $1 \le i, j \le 2^{r+\frac{m}{2}}$ is the coefficient of the (i, j)-th entry in the matrix S_λ. We notice that the dimension of $\widehat{A}_P$ is $m+r$. It follows from the Connes-Hochschild-Kostant-Rosenberg theorem ([7, Theorem 46]) that the periodic cyclic cohomology of the algebra $\mathcal{C}(\widehat{A}_P)$ is spanned by a cyclic cocycle of degree $m + r$. Accordingly, we conclude that

$$\langle [\Phi_{P_\circ,e}], Q_\lambda \rangle = 0$$

because it equals the pairing of the Bott element $B^{m+r} \in K(\widehat{A}_P)$ and a cyclic cocycle in $HC(\widehat{A}_P)$ with degree only $m < m + r$.

Next, we turn to the index pairing of orbital integrals $\Phi_{P_\circ,t}$ for $t \in T^{\mathrm{reg}}$. In this singular case, it is clear that the regular part of higher orbital integrals will not contribute. For the higher part,

$$\begin{aligned}(-1)^m &\langle [\Phi_{P_\circ,t}]_{\mathrm{high}}, Q_\lambda \rangle \\ &= \int_{\varphi \in \widehat{A}_P} T_{\mathrm{Ind}_P^G(\sigma^M \otimes \varphi)} \Bigg(\mathrm{Trace} \Bigg(\underbrace{S_\lambda \otimes \cdots \otimes S_\lambda}_{m+1} \Bigg) \Bigg) \cdot \Bigg(\sum_{\eta^{M_\circ} \otimes \varphi_1 \in \mathcal{A}(\mathrm{Ind}_P^G(\sigma^M \otimes \varphi))} \kappa^{M_\circ}(\eta^{M_\circ}, t) \Bigg).\end{aligned}$$

Note that the function

$$\sum_{\eta^{M_\circ} \otimes \varphi_1 \in \mathcal{A}(\mathrm{Ind}_P^G(\sigma^M \otimes \varphi))} \kappa^{M_\circ}(\eta^{M_\circ}, t)$$

is constant in $\varphi_1 \in \widehat{A}_\circ$. By (4.2), we see that

$$\begin{aligned}&\int_{\widehat{A}_S} \Bigg(\sum_{\eta^{M_\circ} \otimes \varphi_1 \in \mathcal{A}\left(\mathrm{Ind}_P^G(\sigma^M \otimes \varphi)\right)} \kappa^{M_\circ}\left(\eta^{M_\circ}, t\right) \cdot \int_{\widehat{A}_\circ} T_{\mathrm{Ind}_P^G(\sigma^M \otimes \varphi)} \mathrm{Trace} \Bigg(\underbrace{S_\lambda \otimes \cdots \otimes S_\lambda}_{m+1} \Bigg) \Bigg) \\ &= \sum_{\tau \in S_m} \sum_{\eta^{M_\circ} \otimes \varphi_1 \in \mathcal{A}(\mathrm{Ind}_P^G(\sigma^M \otimes \varphi))} \sum_{j_0 = i_1, \ldots j_{m-1} = i_m, j_m = i_0} \\ &\quad \int_{\widehat{A}_S} \int_{\widehat{A}_\circ} \Bigg(\kappa^{M_\circ}(\eta^{M_\circ}, t)(-1)^\tau \cdot s_{i_0, j_0}(\varphi) \frac{\partial s_{i_1, j_1}(\varphi)}{\partial_{\tau(1)}} \ldots \frac{\partial s_{i_m, j_m}(\varphi)}{\partial_{\tau(m)}} \Bigg).\end{aligned}$$

We conclude that

$$\langle [\Phi_{P_\circ,t}]_{\mathrm{high}}, Q_\lambda \rangle = 0$$

because it equals the paring of the Bott element $B^{m+r} \in K(\widehat{A}_P)$ and a cyclic cocycle on $\mathcal{C}(\widehat{A}_P)$ of degree m, which is trivial in $HP^{\mathrm{even}}(\mathcal{C}(\widehat{A}_P))$.

For the singular part, by the Schur's orthogonality, we have $\langle [\Phi_{P_\circ,t}]_{\lambda'}, Q_\lambda \rangle = 0$ unless $\lambda' = \lambda$. When $\lambda' = \lambda$, Theorem 4.18 gives us the following computation:

$$\begin{aligned}(-1)^m &\langle [\Phi_{P_\circ,t}]_\lambda, Q_\lambda \rangle \\ &= \Bigg(\frac{1}{2^r} \sum_{w \in W_K} (-1)^w e^{w \cdot \lambda}(t) \Bigg) \cdot \sum_{k=1}^{2^r} \int_{\varphi \in \widehat{A}_P} \epsilon(k) \cdot T_{\mathrm{Ind}_{P_\circ}^G(\sigma_k^{M_\circ} \otimes \varphi_1)} \Bigg(\mathrm{Trace} \Bigg(\underbrace{S_\lambda \otimes \cdots \otimes S_\lambda}_{m+1} \Bigg) \Bigg).\end{aligned}$$

For each fixed k, it follows from Lemma 5.2 and Lemma 4.11 that

$$\int_{\varphi\in\widehat{A}_P} T_{\mathrm{Ind}_{P_\circ}^G(\sigma_k^{M_\circ}\otimes\varphi_1)}\left(\mathrm{Trace}\left(\underbrace{S_\lambda\otimes\cdots\otimes S_\lambda}_{m+1}\right)\right)$$
$$=\int_{\varphi\in\widehat{A}_P} T_{\mathrm{Ind}_{P_\circ}^G(\sigma_k^{M_\circ}\otimes\varphi_1)}\left(\mathrm{Trace}\left(\underbrace{S_\lambda\otimes\cdots\otimes S_\lambda}_{m+1}\right)\Big|_{\varphi_2=0}\right)$$
$$=\sum_{j_0=i_1,\dots j_{m-1}=i_m, j_m=i_0=k}\ \sum_{\tau\in S_m}\int_{\varphi_1\in\widehat{A}_\circ}(-1)^\tau\cdot s_{i_0,j_0}(\varphi_1)\frac{\partial s_{i_1,j_1}(\varphi_1)}{\partial_{\tau(1)}}\dots\frac{\partial s_{i_m,j_m}(\varphi_1)}{\partial_{\tau(m)}}$$
$$=\begin{cases}\langle B^m,b_m\rangle=1 & \text{if } k=1,\dots 2^{r-1}\\ -\langle B^m,b_m\rangle=1 & \text{if } k=2^{r-1}+1,\dots 2^r.\end{cases}$$

Combining all the above together and the fact that $\epsilon(k)=1$ for $k=1,\dots 2^{r-1}$ and $\epsilon(k)=-1$ for $k=2^{r-1}+1,\dots 2^r$, we conclude that

$$\langle[\Phi_{P_\circ,t}],Q_\lambda\rangle=\langle[\Phi_{P_\circ,t}]_\lambda,Q_\lambda\rangle=(-1)^m\sum_{w\in W_K}(-1)^w e^{w\cdot\lambda}(t).$$

Appendix

A. Integration of Schwartz functions

Let $\mathfrak{a}\subseteq\mathfrak{s}$ be the maximal abelian subalgebra of $\mathfrak{s}$ and $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{a}$ be the most noncompact Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{u}=\mathfrak{k}\oplus i\mathfrak{s}$ and U be the compact Lie group with Lie algebra $\mathfrak{u}$. Take $\nu\in\mathfrak{a}^*$ an integral weight. Let $\tilde\nu\in\mathfrak{t}^*\oplus i\mathfrak{a}^*$ be an integral weight so that its restriction $\tilde\nu\big|_{i\mathfrak{a}^*}=i\cdot\nu$. Let $G^{\mathbb{C}}$ be the complexification of G. Suppose that V is a finite-dimensional irreducible holomorphic representation of $G^{\mathbb{C}}$ with highest weight $\tilde\nu$. Introduce a Hermitian inner product V so that U acts on V unitarily.

We take u_ν to be a unit vector in the sum of the weight spaces for weights that restrict to ν on $\mathfrak{a}$.

Lemma A.1. *For any $g\in G$, we have that*

$$e^{\langle\nu,\widetilde{H}(g)\rangle}=\|g\cdot u_\nu\|.$$

Proof. The proof is borrowed from [18, Proposition 7.17]. By the Iwasawa decomposition, we write $g=kan$ with $a=\exp(X)$ and $X\in\mathfrak{a}$. Since u_ν is the highest vector for the action of $\mathfrak{a}$, $\mathfrak{n}$ annihilates u_ν. Thus,

$$\|gu_\nu\|=\|kau_\nu\|=e^{\langle\nu,X\rangle}\|ku_\nu\|=e^{\langle\nu,X\rangle}.$$

The last equation follows from the fact that $K\subseteq U$ acts on V in a unitary way. However, we have that $H(g)=X$. This completes the proof. □

Proposition A.2. *There exists a constant $C_\nu>0$ such that*

$$\langle\nu,\widetilde{H}(g)\rangle\le C_\nu\cdot\|g\|,$$

where $\|g\|$ is the distance from $g\cdot K$ to $e\cdot K$ on G/K.

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Proof. Since $G = K\exp(\mathfrak{a}^+)K$, we write $g = k'\exp(X)k$ with $X \in \mathfrak{a}^+$. By definition,

$$\|g\| = \|X\|, \quad \text{and} \quad \widetilde{H}(g) = \widetilde{H}(ak).$$

By the above lemma, we have that

$$e^{\langle \nu, \widetilde{H}(ak)\rangle} = \|ak \cdot u_\nu\|.$$

We decompose $k \cdot u_\nu$ into the weight spaces of $\mathfrak{a}$-action. That is,

$$k \cdot u_\nu = \sum_{i=1}^{n} c_i \cdot u_i,$$

where $c_i \in \mathbb{C}$, $\|c_i\| \leq 1$ and u_i is a unit vector in the weight spaces for weights that restricts to $\lambda_i \in \mathfrak{a}^*$. It follows that

$$\begin{aligned} \|ak \cdot u_\nu\| &= \| \sum_{i=1}^{n} c_i \cdot a \cdot u_i \| \\ &\leq \sum_{i=1}^{n} \|a \cdot u_i\| = \sum_{i=1}^{n} e^{\langle \lambda_i, X\rangle} \|u_i\| \leq e^{C_\nu \cdot \|X\|}, \end{aligned} \tag{A.1}$$

where

$$C_\nu = n \cdot \sup_{Y \in \mathfrak{a}, \text{with } \|Y\|=1} \left\{ \langle \lambda_i, Y\rangle \middle| 1 \leq i \leq n \right\}.$$

This completes the proof. □

Now let us fix a cuspidal parabolic subgroup $P = MAN$. To prove the integral in the definition of $\Phi_{P,x}$ defines a continuous cochain on $\mathcal{C}(G)$, we consider a family of Banach subalgebras $\mathcal{S}_t(G)$, $t \in [0, \infty]$, of $C_r^*(G)$, which was introduced and studied by Lafforgue, [19, Definition 4.1.1].

Definition A.3. For $t \in [0, \infty]$, let $\mathcal{S}_t(G)$ be the completion of $C_c(G)$ with respect to the norm ν_t defined as follows:

$$\nu_t(f) := \sup_{g \in G} \left\{ (1 + \|g\|)^t \Xi(g)^{-1} \left| f(g) \right| \right\}.$$

Proposition A.4. *The family of Banach spaces $\{\mathcal{S}_t(G)\}_{t \geq 0}$ satisfies the following properties.*

1. *For every $t \in [0, \infty)$, $\mathcal{S}_t(G)$ is a dense subalgebra of $C_r^*(G)$ stable under holomorphic functional calculus.*
2. *For $0 \leq t_1 < t_2 < \infty$, $\|f\|_{t_1} \leq \|f\|_{t_2}$, for $f \in \mathcal{S}_{t_2}(G)$. Therefore, $\mathcal{C}(G) \subset \mathcal{S}_{t_2}(G) \subset \mathcal{S}_{t_1}(G)$.*
3. *There exists a number $d_0 > 0$ such that the integral*

$$f \mapsto f^P(xa) := \int_{KN} f(kxank^{-1}), \quad x \in M, a \in A$$

   is a continuous linear map from $\mathcal{S}_{t+d_0}(G)$ to $\mathcal{S}_t(MA)$ for $t \in [0, \infty)$.
4. *There exists $T_0 > 0$ such that the orbital integral*

$$f \mapsto \int_{G/Z_G(x)} f(gxg^{-1})$$

   is a continuous linear functional on $\mathcal{S}_t(G)$ for $t \geq T_0$, $\forall x \in G$.

Proof. Property 1 is from [19, Proposition 4.1.2]; Property 2 follows from the definition of the norm ν_t; Property 3 follows from [10, Lemma 21]; Property 4 follows from [10, Theorem 6]. □

Theorem A.5. *For any $f_0, \dots f_m \in \mathcal{S}_{T_0+d_0+1}(G)$ for $t \geq T$, and $x \in M$, the following integral*

$$\int_{h \in M/Z_M(x)} \int_{KN} \int_{G^{\times m}} H_1(g_1 k) \dots H_m(g_m k)$$
$$f_0\Big(khxh^{-1}nk^{-1}(g_1 \dots g_m)^{-1}\Big) \cdot f_1(g_1) \dots f_m(g_m)$$

is finite and defines a continuous n-linear functional on $\mathcal{S}_{d_0+T_0+1}(G)$.

Proof. We put

$$\tilde{f}_i(g_i) = \sup_{k \in K} \left\{ \left| H_i(g_i k) f_i(g_i) \right| \right\}.$$

By Proposition A.2, we find constants $C_i > 0$ so that

$$|H_i(g_i k)| \leq C_i \|g_i k\| = C_i \|g_i\|.$$

It shows from Definition A.3 that $\tilde{f}_i$ belongs to $\mathcal{S}_{d_0+T_0}(G)$, $i = 1, \dots n$. Thus, the integration in (A.1) is bounded by the following:

$$\int_{h \in M/Z_M(x)} \int_{KN} \int_{G^{\times m}} \left| f_0\Big(khxh^{-1}nk^{-1}(g_1 \dots g_m)^{-1}\Big) \cdot \tilde{f}_1(g_1) \dots \tilde{f}_m(g_m) \right|$$
$$= \int_{h \in M/Z_M(x)} \int_{KN} F(khxh^{-1}nk^{-1}), \tag{A.2}$$

where by Proposition A.4.2,

$$F = \left| f_0 * \tilde{f}_1 * \dots * \tilde{f}_m \right| \in \mathcal{S}_{d_0+T_0}(G).$$

For any $x \in M, a \in A$, we introduce

$$F^{(P)}(xa) = \int_{KN} F(kxank^{-1}).$$

By Proposition A.4.3, we have that $F^{(P)}$ belongs to $\mathcal{S}_{T_0}(MA)$. Applying Proposition A.4.4 to the group MA, we conclude the orbital integral

$$\int_{M/Z_M(x)} F^{(P)}(hxh^{-1}) < +\infty,$$

from which we obtain the desired finiteness of the integral (A.2). Furthermore, with the continuity of the above maps,

$$f_i \mapsto \tilde{f}_i, \qquad f_0 \otimes \tilde{f}_1 \otimes \dots \otimes \tilde{f}_m \mapsto F, \qquad F \mapsto F^{(P)}, \qquad F^{(P)} \mapsto \int_{M/Z_M(x)} F^{(P)}(hxh^{-1}),$$

and we conclude that the integral (A.2) is a continuous n-linear functional on $\mathcal{S}_{d_0+T_0+1}(G)$. □

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B. Characters of representations of G

B.1. Discrete series representation of G

Suppose that rank G = rank K. Then, G has a compact Cartan subgroup T with Lie algebra denoted by $\mathfrak{t}$. Moreover, $\dim(G/K)$ and $\dim(A_\circ)$ are automatically even. We can decompose the roots into compact roots and noncompact roots; that is,

$$\mathcal{R}(\mathfrak{t},\mathfrak{g}) = \mathcal{R}_c(\mathfrak{t},\mathfrak{g}) \cup \mathcal{R}_n(\mathfrak{t},\mathfrak{g}).$$

We choose a set of positive roots $\mathcal{R}^+(\mathfrak{t},\mathfrak{g})$ and define

$$\rho_c = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_c^+(\mathfrak{t},\mathfrak{g})}\alpha, \qquad \rho_n = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_n^+(\mathfrak{t},\mathfrak{g})}\alpha, \qquad \rho = \rho_c + \rho_n.$$

The choice of $\mathcal{R}_c^+(\mathfrak{t},\mathfrak{k})$ determines a positive Weyl chamber $\mathfrak{t}_+^*$. Let Λ_T^* be the weight lattice in $\mathfrak{t}^*$. Then, the set

$$\Lambda_K^* = \Lambda_T^* \cap \mathfrak{t}_+^*$$

parametrizes the set of irreducible K-representations. In addition, we denote by W_K the Weyl group of the compact subgroup K. For any $w \in W_K$, let $l(w)$ be the length of w, and we denote by $(-1)^w = (-1)^{l(w)}$.

Definition B.1. Let $\lambda \in \Lambda_K^* + \rho_c$. We say that λ is *regular* if

$$\langle \lambda, \alpha\rangle \neq 0$$

for all $\alpha \in \mathcal{R}_n(\mathfrak{t},\mathfrak{g})$. Otherwise, we say λ is *singular*.

Assume that $q = \frac{\dim G/K}{2}$ and $T^{\text{reg}} \subset T$ the set of regular elements in T.

Theorem B.2 (Harish-Chandra). *For any regular $\lambda \in \Lambda_K^* + \rho_c$, there is a discrete series representation $\sigma(\lambda)$ of G with Harish-Chandra parameter λ. Its character is given by the following formula:*

$$\Theta(\lambda)\big|_{T^{\text{reg}}} = (-1)^q \cdot \frac{\sum_{w\in W_K}(-1)^w e^{w\lambda}}{\Delta_T^G},$$

where

$$\Delta_T^G = \prod_{\alpha\in\mathcal{R}^+(\mathfrak{t},\mathfrak{g})}\left(e^{\frac{\alpha}{2}} - e^{\frac{-\alpha}{2}}\right).$$

Next, we consider the case when $\lambda \in \Lambda_K^* + \rho_c$ is singular. That is, there exists at least one noncompact root α so that $\langle\lambda,\alpha\rangle = 0$. Choose a positive root system $\mathcal{R}^+(\mathfrak{t},\mathfrak{g})$ that makes λ dominant; the choices of $\mathcal{R}^+(\mathfrak{t},\mathfrak{g})$ are not unique when λ is singular. For every choice of $\mathcal{R}^+(\mathfrak{t},\mathfrak{g})$, we can associate it with a representation, denoted by $\sigma(\lambda,\mathcal{R}^+)$. We call $\sigma(\lambda,\mathcal{R}^+)$ a *limit of discrete series representation* of G. Distinct choices of $\mathcal{R}^+(\mathfrak{t},\mathfrak{g})$ lead to infinitesimally equivalent versions of $\sigma(\lambda,\mathcal{R}^+)$. Let $\Theta(\lambda,\mathcal{R}^+)$ be the character of $\sigma(\lambda,\mathcal{R}^+)$. Then,

$$\Theta(\lambda,\mathcal{R}^+)\big|_{T^{\text{reg}}} = (-1)^{\pm}\frac{\sum_{w\in W_K}(-1)^w e^{w\lambda}}{\Delta_T^G}.$$

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Moreover, for any $w \in W_K$ which fixes λ, we have that

$$\Theta\big(\lambda, w \cdot \mathcal{R}^+\big)\big|_{T^{\mathrm{reg}}} = (-1)^w \cdot \Theta\big(\lambda, w \cdot \mathcal{R}^+\big)\big|_{T^{\mathrm{reg}}}. \tag{B.1}$$

See [18, P. 460] for more detailed discussion.

B.2. Discrete series representations of M

Let $P = MAN$ be a cuspidal parabolic subgroup. The subgroup M might not be connected in general. We denote by M_0 the connected component of M and set

$$M^\sharp = M_0 Z_M,$$

where Z_M is the center for M.

Let σ_0 be a discrete series representation (or limit of discrete series representation) of the connected group M_0 and χ be a unitary character of Z_M. If σ_0 has a Harish-Chandra parameter λ, then we assume that

$$\chi\big|_{T_M \cap Z_M} = e^{\lambda - \rho_M}\big|_{T_M \cap Z_M}.$$

We have the well-defined representation $\sigma_0 \boxtimes \chi$ of $M^\sharp$, given by

$$\sigma_0 \boxtimes \chi(gz) = \sigma(g)\chi(z),$$

for $g \in M_0$ and $z \in Z_M$.

Definition B.3. The discrete series representation or limit of discrete series representation σ for the possibly disconnected group M induced from $\sigma_0 \boxtimes \chi$ is defined as

$$\sigma = \mathrm{Ind}_{M^\sharp}^{M}\big(\sigma_0 \boxtimes \chi\big).$$

Discrete series representations of M are parametrized by a pair of Harish-Chandra parameter λ and unitary character χ. Next, we show that χ is redundant for the case of $M_\circ$. Denote

- $\mathfrak{a}$ = the Lie algebra of A;
- $\mathfrak{t}_M$ = the Lie algebra of the compact Cartan subgroup of M;
- $\mathfrak{a}_M$ = the maximal abelian subalgebra of $\mathfrak{s} \cap \mathfrak{m}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$;

Then, $\mathfrak{t}_M \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{a}_\mathfrak{s} = \mathfrak{a}_M \oplus \mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{s}$.

Let α be a real root in $\mathcal{R}(\mathfrak{g}, \mathfrak{t}_M \oplus \mathfrak{a})$. Restrict α to $\mathfrak{a}$ and extend it by 0 on $\mathfrak{a}_M$ to obtain a restricted root in $\mathcal{R}(\mathfrak{g}, \mathfrak{a}_\mathfrak{s})$. Form an element $H_\alpha \in \mathfrak{a}_\mathfrak{s}$ by the following:

$$\alpha(H) = \langle H, H_\alpha \rangle, \qquad H \in \mathfrak{a}_\mathfrak{s}.$$

It is direct to check that

$$\gamma_\alpha = \exp\left(\frac{2\pi i H_\alpha}{|\alpha|^2}\right)$$

is a member of the center of M. Denote by F_M the finite group generated by all γ_α induced from real roots of $\Delta(\mathfrak{g}, \mathfrak{t}_M \oplus \mathfrak{a})$. It follows from Lemma 12.30 in [18] that

$$M^\sharp = M_0 F_M. \tag{B.2}$$

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Lemma B.4. *For the maximal cuspidal parabolic subgroup $P_\circ = M_\circ A_\circ N_\circ$, we have that*

$$Z_{M_\circ} \subseteq (M_\circ)_0.$$

Proof. There is no real root in $\mathcal{R}(\mathfrak{h}_\circ, \mathfrak{g})$ since the Cartan subgroup $H_\circ$ is maximally compact. The lemma follows from (B.2). □

It follows that discrete series or limit of discrete series representations of $M_\circ$ are para-metrized by Harish-Chandra parameter λ. We denote them by $\sigma(\lambda)$ or $\sigma(\lambda, \mathcal{R}^+)$.

B.3. Induced representations of G

Let $P = MAN$ be a cuspidal parabolic subgroup of G and $L = MA$ as before. For any Cartan subgroup J of L, let $\{J_1, J_2, \ldots, J_k\}$ be a complete set of representatives for distinct conjugacy classes of Cartan subgroups of L for which J_i is conjugate to J in G. Suppose that $x_i \in G$ satisfy $J_i = x_i J x_i^{-1}$, and for $j \in J$, write $j_i = x_i j x_i^{-1}$.

Theorem B.5. *Let $\Theta(P, \sigma, \varphi)$ be the character of the basic representation $\mathrm{Ind}_P^G(\sigma \otimes \varphi)$. Then,*

- *$\Theta(P, \sigma, \varphi)$ is a locally integrable function.*
- *$\Theta(P, \sigma, \varphi)$ is nonvanishing only on Cartan subgroups of G that are G-conjugate to Cartan subgroups of L.*
- *For any $j \in J$, we have*

$$\Theta(P, \sigma, \varphi)(j) = \sum_{i=1}^{k} |W(J_i, L)|^{-1} |\Delta_{J_i}^G(j_i)|^{-1} \Big(\sum_{w \in W(J_i, G)} |\Delta_{J_i}^L(wj_i)| \cdot \Theta_\sigma^M \left(wj_i\big|_M\right) \varphi(wj_i|_{H_p}) \Big), \tag{B.3}$$

where Θ_σ^M is the character for the M_P representation σ, and the definition of $\Delta_{J_i}^G$ (and $\Delta_{J_i}^L$) is explained in Theorem B.2.

Proof. The first two properties of $\Theta(P, \sigma, \varphi)$ can be found in [18, Proposition 10.19], and the last formula has been given in [15, Equation (2.9)]. □

Corollary B.6. *Suppose that $P_\circ$ is the maximal cuspidal parabolic subgroup of G and $\sigma^{M_\circ}(\lambda)$ is a (limit of) discrete series representation with Harish-Chandra parameter λ. We have that*

$$\Theta\Big(P_\circ, \sigma^{M_\circ}(\lambda), \varphi\Big)(h) = \frac{\sum_{w \in K} (-1)^w e^{w\lambda}(h_k) \cdot \varphi(h_p)}{\Delta_{H_\circ}^G(h)},$$

for any $h \in H_\circ^{reg}$.

Proof. The corollary follows from (B.3) and Theorem B.2. □

C. Description of $K(C_r^*(G))$

Without loss of generality, we assume that $\dim A_\circ = m$ is even. Otherwise, we can replace G by $G \times \mathbb{R}$.

C.1. Generalized Schmid identity

Suppose that $P = MAN$ is a cuspidal parabolic subgroup of G and $H = TA$ is its associated Cartan subgroup. We assume that P is not maximal, and thus, H is not the most compact. By Cayley transform, we can obtain a more compact Cartan subgroup $H' = T'A'$. We denote by $P' = M'A'N'$ the corresponding cuspidal parabolic subgroup. Here, $A = A' \times \mathbb{R}$.

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Let σ be a (limit of) discrete series representation of M, and

$$\nu \otimes 1 \in \widehat{A} = \widehat{A'} \times \widehat{\mathbb{R}}.$$

Suppose that

$$\pi = \mathrm{Ind}_P^G \left(\sigma \otimes (\nu \otimes 1)\right)$$

is a basic representation. Then, π is either irreducible or decomposes as follows:

$$\mathrm{Ind}_P^G \left(\sigma \otimes (\nu \otimes 1)\right) = \mathrm{Ind}_{P'}^G(\delta_1 \otimes \nu) \oplus \mathrm{Ind}_{P'}^G(\delta_2 \otimes \nu).$$

Here, δ_1 and δ_2 are limit of discrete series representations of M'. Moreover, they share the same Harish-Chandra parameter but correspond to different choices of positive roots. On the right-hand side of the above equation, if P' is not maximal, then one can continue the decomposition for $\mathrm{Ind}_{P'}^G(\sigma_i' \otimes \nu), i = 1, 2$. Eventually, we get

$$\mathrm{Ind}_P^G \left(\sigma \otimes (\varphi \otimes 1)\right) = \bigoplus_i \mathrm{Ind}_{P_\circ}^G(\delta_i \otimes \varphi), \tag{C.1}$$

where

$$\varphi \otimes 1 \in \widehat{A}_P = \widehat{A}_\circ \times \widehat{A}_S.$$

The number of components in the above decomposition is closely related to the R-group which we will discuss below. We refer to [18, Corollary 14.72] for detailed discussion.

As a consequence, we obtain the following lemma immediately.

Lemma C.1. *Let $P_\circ = M_\circ A_\circ N_\circ$ be the maximal cuspidal parabolic subgroup. If $\sigma \otimes \varphi$ is an irreducible representation of $M_\circ A_\circ$, then the induced representation*

$$\mathrm{Ind}_{P_\circ}^G \left(\sigma \otimes \varphi\right)$$

is also irreducible.

C.2. Essential representations

Clare-Crisp-Higson proved in [4, Section 6] that the group C^*-algebra $C_r^*(G)$ has the following decomposition:

$$C_r^*(G) \cong \bigoplus_{[P,\sigma] \in \mathcal{P}(G)} C_r^*(G)_{[P,\sigma]}, \tag{C.2}$$

where

$$C_r^*(G)_{[P,\sigma]} \cong \mathcal{K}\left(\mathrm{Ind}_P^G(\sigma)\right)^{W_\sigma}.$$

For principal series representations $\mathrm{Ind}_P^G(\sigma \otimes \varphi)$, Knapp and Stein [18, Chapter 9] showed that the stabilizer W_σ admits a semidirect product decomposition

$$W_\sigma = W'_\sigma \rtimes R_\sigma,$$

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where the *R-group* R_σ consists of those elements that actually contribute nontrivially to the intertwining algebra of $\mathrm{Ind}_P^G(\sigma\otimes\varphi)$. Wassermann notes the following Morita equivalence:

$$\mathcal{K}\big(\mathrm{Ind}_P^G(\sigma)\big)^{W_\sigma} \sim C_0(\widehat{A}_P/W'_\sigma)\rtimes R_\sigma. \tag{C.3}$$

Definition C.2. We say that an equivalence class $[P,\sigma]$ is *essential* if $W_\sigma = R_\sigma$. We denote it by $[P,\sigma]_{\mathrm{ess}}$. In this case,

$$W_\sigma = R_\sigma \cong (\mathbb{Z}_2)^r$$

is obtained by application of all combinations of $r = \dim(A_P) - \dim(A_\circ)$ commuting reflections in simple noncompact roots.

As before, let T be the maximal torus of K. We denote by Λ_T^* and Λ_K^* the weight lattice and its intersection with the positive Weyl chamber of K. The following results can be found in [5, 6].

Theorem C.3 [6]. *There is a bijection between the set of $[P,\sigma]_{\mathrm{ess}}$ and the set $\Lambda_K^* + \rho_c$ such that*

- *for regular $\lambda\in\Lambda_K^*+\rho_c$ – that is,*

$$\langle\lambda,\alpha\rangle \neq 0$$

for all noncompact roots $\alpha\in\mathcal{R}_n$ – then the correspondent essential class $[P,\sigma]$ satisfies that W_σ is trivial, $P = P_\circ$, and σ is the discrete series representation of $M_\circ$ with Harish-Chandra parameter λ. In addition,

$$\mathrm{Ind}_{P_\circ}^G(\sigma\otimes\varphi)$$

are irreducible for all $\varphi\in\widehat{A}_\circ$.

- *Otherwise, if $\langle\lambda,\alpha\rangle = 0$ for some $\alpha\in\mathcal{R}_n$, then*

$$\mathrm{Ind}_P^G(\sigma\otimes\varphi\otimes 1) = \bigoplus_{i=1}^{2^r}\mathrm{Ind}_{P_\circ}^G\big(\delta_i\otimes\varphi\big), \tag{C.4}$$

where δ_i is a limit of discrete series representation of $M_\circ$ with Harish-Chandra parameter λ, $\varphi\in\widehat{A}_\circ$ and $\varphi\otimes 1\in\widehat{A}_P$.

The computation of K-theory group of $C_r^*(G)$ can be summarized as follows.

Theorem C.4 [6]. *The K-theory group of $C_r^*(G)$ is a free abelian group generated by the following components; that is,*

$$\begin{aligned} K_0(C_r^*(G)) &\cong \bigoplus_{[P,\sigma]^{\mathrm{ess}}} K_0\big(\mathcal{K}(C_r^*(G)_{[P,\sigma]})\big)\\ &\cong \bigoplus_{[P,\sigma]^{\mathrm{ess}}} K_0\Big(\mathcal{K}\big(\mathrm{Ind}_P^G\,\sigma\big)^{W_\sigma}\Big)\\ &\cong \bigoplus_{regular\ \ part} K_0\big(C_0(\mathbb{R}^m)\big)\oplus\bigoplus_{singular\ \ part} K_0\Big(\big(C_0(\mathbb{R})\rtimes\mathbb{Z}_2\big)^r\otimes C_0(\mathbb{R}^m)\Big)\\ &\cong \bigoplus_{\lambda\in\Lambda_K^*+\rho_c}\mathbb{Z}. \end{aligned} \tag{C.5}$$

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Example C.5. Let $G = SL(2,\mathbb{R})$. The principal series representations of $SL(2,\mathbb{R})$ are para-metrized by characters

$$(\sigma,\lambda) \in \widehat{MA} \cong \{\pm 1\} \times \mathbb{R}$$

modulo the action of the Weyl group $\mathbb{Z}_2$. One family of principal series representations is irreducible at 0, while the other decomposes as a sum of two limit of discrete series representations. At the level of $C^*_r(G)$, this can be explained as

$$\begin{aligned}\widehat{MA}/\mathbb{Z}_2 &\cong \{+1\} \times [0,\infty) \cup \{-1\} \times [0,\infty)\\ &\cong \{+1\} \times \mathbb{R}/\mathbb{Z}_2 \cup \{-1\} \times \mathbb{R}/\mathbb{Z}_2,\end{aligned}$$

and the principal series representations contribute summands to $C^*_r(SL(2,\mathbb{R}))$ of the form

$$C_0(\mathbb{R}/\mathbb{Z}_2) \quad \text{and} \quad C_0(\mathbb{R}) \rtimes \mathbb{Z}_2$$

up to Morita equivalence. In addition, $SL(2,\mathbb{R})$ has discrete series representations each of which contributes a summand of $\mathbb{C}$ to $C^*_r(SL(2,\mathbb{R}))$, up to Morita equivalence. We obtain

$$C^*_r(SL(2,\mathbb{R})) \sim C_0(\mathbb{R}/\mathbb{Z}_2) \oplus C_0(\mathbb{R}) \rtimes \mathbb{Z}_2 \oplus \bigoplus_{n\in\mathbb{Z}\setminus\{0\}} \mathbb{C}.$$

Here, the part $C_0(\mathbb{R}/\mathbb{Z}_2)$ corresponds to the family of spherical principal series representations, which are not essential. Then, (C.5) can be read as follows:

$$K_0(C^*_r(SL(2,\mathbb{R}))) \cong K_0\Big(\big(C_0(\mathbb{R}) \rtimes \mathbb{Z}_2\big)\Big) \oplus \bigoplus_{n\neq 0} K_0(\mathbb{C}).$$

C.3. *The formula for orbital integrals*

In this subsection, we summarize the formulas and results in [14, 15]. If P is the minimal parabolic subgroup with the most noncompact Cartan subgroup H, then the Fourier transform of orbital integral equals the character of representation. That is, for any $h \in H^{\mathrm{reg}}$,

$$\widehat{F}^H_f(\chi) = \int_{h\in H} \chi(h) \cdot F^H_f(h) \cdot dh = \Theta(P,\chi)(f)$$

or equivalently,

$$F^H_f(h) = \int_{\chi\in\widehat{H}} \Theta(P,\chi)(f) \cdot \overline{\chi(h)} \cdot d\chi.$$

For any arbitrary cuspidal parabolic subgroup P, the formula for orbital integral is much more complicated, given as follows:

$$F^H_f(h) = \sum_{Q\in \mathrm{Par}(G,P)} \int_{\chi\in\widehat{J}} \Theta(Q,\chi)(f) \cdot \kappa^G(Q,\chi,h) d\chi. \tag{C.6}$$

Remark C.6. In the above formula,

- the sum ranges over the set

$$\mathrm{Par}(G,P) = \big\{\text{cuspidal parabolic subgroup } Q \text{ of } G \big| Q \text{ is no more compact than } P\big\}.$$

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- J is the Cartan subgroup associated to the cuspidal parabolic subgroup Q.
- χ is a unitary character of J, and $\Theta(Q,\chi)$ is a tempered invariant eigen-distribution defined in [14]. In particular, $\Theta(Q,\chi)$ is the character of parabolic induced representation or an alternating sum of characters which can be embedded in a reducible unitary principal series representation associated to a different parabolic subgroup.
- The function κ^G is rather complicated to compute. Nevertheless, for the purpose of this paper, we only need to know the existence of functions κ^G, which has been verified in [28].

In a special case when $P = G$ and $H = T$, the formula (C.6) has the following more explicit form.

Theorem C.7. *For any $t \in T^{reg}$, the orbital integral*

$$\begin{aligned} F_f^T(t) = &\sum_{\text{regular } \lambda\in\Lambda_K^*+\rho_c}\ \sum_{w\in W_K} (-1)^w \cdot e^{w\cdot\lambda}(t)\cdot\Theta(\lambda)(f) \\ &+ \sum_{\text{singular } \lambda\in\Lambda_K^*+\rho_c}\ \sum_{w\in W_K} (-1)^w \cdot e^{w\cdot\lambda}(t)\cdot\Theta(\lambda)(f) \\ &+ \int_{\pi\in\widehat{G}_{\text{temp}}^{\text{high}}} \Theta(\pi)(f)\cdot\kappa^G(\pi,t)d\chi. \end{aligned} \tag{C.7}$$

In the above formula, there are three parts:

- *regular part: $\Theta(\lambda)$ is the character of the discrete series representation with Harish-Chandra parameter λ;*
- *singular part: for singular $\lambda \in \Lambda_K^* + \rho_c$, we denote by $n(\lambda)$ the number of different limit of discrete series representations with Harish-Chandra parameter λ. By (B.1), we can organize them so that*

$$\Theta_1(\lambda)\big|_{T^{reg}} = \cdots = \Theta_{\frac{n(\lambda)}{2}}(\lambda)\big|_{T^{reg}} = -\Theta_{\frac{n(\lambda)}{2}+1}(\lambda)\big|_{T^{reg}} = \cdots = -\Theta_{n(\lambda)}(\lambda)\big|_{T^{reg}}.$$

  We put

$$\Theta(\lambda) := \frac{1}{n(\lambda)}\cdot\Big(\sum_{i=1}^{\frac{n(\lambda)}{2}}\Theta_i(\lambda) - \sum_{i=\frac{n(\lambda)}{2}+1}^{n(\lambda)}\Theta_i(\lambda)\Big).$$

- *higher part: $\widehat{G}_{\text{temp}}^{\text{high}}$ is a subset of $\widehat{G}_{\text{temp}}$ consisting of irreducible tempered representations which are not (limit of) discrete series representations.*

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