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Marcinkiewicz Multipliers and Lipschitz Spaces on Heisenberg Groups

Yanchang Han, Yongsheng Han, Ji Li, and Chaoqiang Tan

Abstract. The Marcinkiewicz multipliers are L^p bounded for $1 < p < \infty$ on the Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$ (Müller, Ricci, and Stein). This is surprising in the sense that these multipliers are invariant under a two parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, while there is no two parameter group of automorphic dilations on $\mathbb{H}^n$. The purpose of this paper is to establish a theory of the flag Lipschitz space on the Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$ that is, in a sense, intermediate between that of the classical Lipschitz space on the Heisenberg group $\mathbb{H}^n$ and the product Lipschitz space on $\mathbb{C}^n \times \mathbb{R}$. We characterize this flag Lipschitz space via the Littlewood–Paley theory and prove that flag singular integral operators, which include the Marcinkiewicz multipliers, are bounded on these flag Lipschitz spaces.

1 Introduction

Classical Calderón–Zygmund singular integrals commute with the one parameter dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta x_1, \dots, \delta x_n)$ for $\delta > 0$, while the *product* Calderón–Zygmund singular integrals commute with the multi-parameter dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta_1 x_1, \dots, \delta_n x_n)$ for $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n_+$.

In the product Calderón–Zygmund theory, the product singular integral operators are of the form $Tf = K * f$, where K is homogeneous; that is, $\delta_1 \cdots \delta_n K(\delta \cdot x) = K(x)$, or, more generally, $K(x)$ and $\delta_1 \cdots \delta_n K(\delta \cdot x)$ satisfy the same size, smoothness, and cancellation conditions. Such operators have been studied for example in Gundy and Stein [13], Fefferman and Stein [11], Fefferman [8–10], Chang [3], Chang and Fefferman [4–6], Journé [17, 18] and Pipher [29]. More precisely, Fefferman and Stein [11] studied the L^p boundedness ($1 < p < \infty$) for the product convolution singular integral operators. Journé in [17, 19] introduced a non-convolution product singular integral operators and established the product $T1$ theorem and proved the $L^\infty \to BMO$ boundedness for such operators. The product Hardy space $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ was first introduced by Gundy and Stein [13]. Chang and Fefferman [4–6] developed the theory of atomic decomposition and established the dual space of Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$,

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namely the product $BMO(\mathbb{R}^n \times \mathbb{R}^m)$ space. Carleson [1] disproved by a counter-example a conjecture that the product atomic Hardy space on $\mathbb{R}^n \times \mathbb{R}^m$ could be defined by rectangle atoms. This motivated Chang and R. Fefferman to replace the role of cubes in the classical atomic decomposition of $H^p(\mathbb{R}^n)$ by arbitrary open sets of finite measures in the product $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. Subsequently, R. Fefferman in [9] established the criterion of the $H^p \to L^p$ boundedness of singular integral operators in Journé's class by considering its action only on rectangle atoms via Journé lemma. However, R. Fefferman's criterion cannot be extended to three or more parameters without further assumptions on the nature of T as shown in Journé [17, 19]. In fact, Journé provided a counter-example in the three-parameter setting of singular integral operators such that R. Fefferman's criterion breaks down. Subsequently, the H^p to L^p boundedness for Journé's class of singular integral operators with arbitrary number of parameters was established by J. Pipher [29] by considering directly the action of the operator on (non-rectangle) atoms and an extension of Journé's geometric lemma to higher dimensions.

On the other hand, multi-parameter analysis has only recently been developed for L^p theory with $1 < p < \infty$ when the underlying multi-parameter structure is not explicit, but implicit, as in the flag multi-parameter structure studied on the Heisenberg group $\mathbb{H}^n$ by Müller, Ricci, and Stein in [23, 24]. See also Phong and Stein in [28] and Nagel, Ricci, and Stein in [25]. In [23, 24], the authors obtained the surprising result that certain Marcinkiewicz multipliers, invariant under a two-parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, are bounded on $L^p(\mathbb{H}^n)$, despite the absence of a two-parameter automorphic group of dilations on $\mathbb{H}^n$. To be precise, the Heisenberg group $\mathbb{H}^n$ is the one consisting of the set

$$\mathbb{C}^n \times \mathbb{R} = \big\{ [z, t] : z \in \mathbb{C}^n, t \in \mathbb{R} \big\}$$

with the multiplication law

$$[z, t] \circ [z', t'] = \big[z + z', t + t' + 2Im(z\overline{z'}) \big],$$

where identity is the origin [0,0] and the inverse is given by $[z, t]^{-1} = [-z, -t]$.

In addition to the Heisenberg group multiplication law, nonisotropic dilations of $\mathbb{H}^n$ are given by

$$\delta_r : \mathbb{H}^n \longrightarrow \mathbb{H}^n, \quad \delta_r\big([z, t]\big) = [rz, r^2 t].$$

A trivial computation shows that δ_r is an automorphism of $\mathbb{H}^n$ for every $r > 0$. However, the standard isotropic dilations of $\mathbb{R}^{2n+1}$ are not automorphisms of $\mathbb{H}^n$.

The "norm" function ρ on $\mathbb{H}^n$ is defined by

$$\rho([z, t]) := (|z|^2 + |t|)^{1/2}.$$

It is easy to see that $\rho([z, t]^{-1}) = \rho([-z, -t]) = \rho([z, t])$, $\rho(\delta_r([z, t])) = r\rho([z, t])$, $\rho([z, t]) = 0$ if and only if $[z, t] = [0, 0]$, and $\rho([z, t] \circ [z', t']) \le \gamma\big(\rho([z, t]) + \rho([z', t'])\big)$, where $\gamma > 1$ is a constant.

The Haar measure on $\mathbb{H}^n$ is known to coincide with the Lebesgue measure on $\mathbb{R}^{2n+1}$. The vector fields

$$T := \frac{\partial}{\partial t}, \quad X_j := \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t}, \quad Y_j := \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n,$$

form a natural basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$. The standard sub-Laplacian $\mathcal{L}$ on the Heisenberg group is defined by

$$\mathcal{L} \equiv -\sum_{j=1}^{n} \left(X_j^2 + Y_j^2 \right).$$

The operators $\mathcal{L}$ and $T = \frac{\partial}{\partial t}$ commute, and so do their spectral measures $dE_1(\xi)$ and $dE_2(\eta)$. Given a bounded function $m(\xi,\eta)$ on $\mathbb{R}_+ \times \mathbb{R}$, define the multiplier operator $m(\mathcal{L}, iT)$ on $L^2(\mathbb{H}^n)$ by

$$m(\mathcal{L}, iT) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} m(\xi,\eta) dE_1(\xi) dE_2(\eta).$$

Then $m(\mathcal{L}, iT)$ is automatically bounded on $L^2(\mathbb{H}^n)$, and if one imposes Marcinkiewicz conditions on the multiplier $m(\xi,\eta)$, namely,

$$\left| (\xi\partial_\xi)^\alpha (\eta\partial_\eta)^\beta m(\xi,\eta) \right| \le C_{\alpha,\beta},$$

for all $|\alpha|, \beta \ge 0$, then Müller, Ricci and Stein [23] proved that the Marcinkiewicz multiplier operator $m(\mathcal{L}, iT)$ is a bounded operator on $L^p(\mathbb{H}^n)$ for $1 < p < \infty$. This is surprising, since these multipliers are invariant under a two parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, while there is *no* two parameter group of automorphic dilations on $\mathbb{H}^n$. Moreover, they showed that Marcinkiewicz multiplier can be characterized by convolution operator with the form $f * K$ where, however, K is a flag convolution kernel. A flag convolution kernel on $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a distribution K on $\mathbb{H}^n$ which coincides with a C^∞ function away from the coordinate subspace $\{(0,u)\} \subset \mathbb{H}^n$, where $0 \in \mathbb{C}^n$ and $u \in \mathbb{R}$, and satisfies the following.

- (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\left| \partial_z^\alpha \partial_u^\beta K(z,u) \right| \le C_{\alpha,\beta} |z|^{-2n-|\alpha|} \cdot \left(|z|^2 + |u| \right)^{-1-\beta}$$

  for all $(z,u) \in \mathbb{H}^n$ with $z \ne 0$ and all $|\alpha|, \beta \ge 0$.
- (Cancellation Condition)

$$\left| \int_{\mathbb{R}} \partial_z^\alpha K(z,u) \phi_1(\delta u) du \right| \le C_\alpha |z|^{-2n-|\alpha|}$$

  for every multi-index α and every normalized bump function ϕ_1 on $\mathbb{R}$ and every $\delta > 0$;

$$\left| \int_{\mathbb{C}^n} \partial_u^\beta K(z,u) \phi_2(\delta z) dz \right| \le C_\gamma |u|^{-1-\beta}$$

  for every index $\beta \ge 0$ and every normalized bump function ϕ_2 on $\mathbb{C}^n$ and every $\delta > 0$; and

$$\left| \int_{\mathbb{H}^n} K(z,u) \phi_3(\delta_1 z, \delta_2 u) dz du \right| \le C$$

  for every normalized bump function ϕ_3 on $\mathbb{H}^n$ and every $\delta_1 > 0$ and $\delta_2 > 0$.

They also proved that flag singular integral operators are bounded on $L^p(\mathbb{H}^n)$ for $1 < p < \infty$. See [26] and [27] for more about flag singular integral operators on homogeneous groups.

At the endpoint estimates, it is natural to expect that the boundedness on Hardy and BMO spaces are available. However, the lack of automorphic dilations underlies the failure of such multipliers to be in general bounded on the classical Hardy space H^1 and also precludes a pure product Hardy space theory on the Heisenberg group. This was the original motivation in [14] to develop a theory of *flag* Hardy spaces H^p_{flag} on the Heisenberg group, $0 < p \le 1$, that is, in a sense 'intermediate' between the classical Hardy spaces $H^p(\mathbb{H}^n)$ and the product Hardy spaces $H^p_{\text{product}}(\mathbb{C}^n \times \mathbb{R})$ (see A. Chang and R. Fefferman [4, 5, 8–10]). They showed that singular integrals with flag kernels, which include the aforementioned Marcinkiewicz multipliers, are bounded on H^p_{flag}, as well as from H^p_{flag} to L^p, for $0 < p \le 1$. Moreover, they constructed a singular integral with a flag kernel on the Heisenberg group that was not bounded on the classical Hardy spaces $H^1(\mathbb{H}^n)$. Since, as pointed out in [14], the flag Hardy space $H^p_{\text{flag}}(\mathbb{H}^n)$ is contained in the classical Hardy space $H^p(\mathbb{H}^n)$, this counterexample implies that $H^1_{\text{flag}}(\mathbb{H}^n) \subsetneqq H^1(\mathbb{H}^n)$.

A basic question arises: Can one establish the endpoint estimates of flag singular integral operators on the Lipschitz space? The purpose of this paper is to answer this question. More precisely, we will establish a theory of the flag Lipschitz space on Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$ that is, in a sense, intermediate between those of the classical Lipschitz space on Heisenberg group $\mathbb{H}^n$ and the product Lipschitz space on $\mathbb{C}^n \times \mathbb{R}$. For more about the classical Lipschitz spaces, see [2, 15, 16, 20–22, 30, 31].

We will characterize the flag Lipschitz space via the Littlewood–Paley theory and prove that flag singular integral operators, which include the Marcinkiewicz multipliers, are bounded on the flag Lipschitz space on Heisenberg group $\mathbb{H}^n$.

Now we introduce the following notation:

$$\Delta^1_{(u,v)}(f)(z,r) = f((z,r)\circ(u,v)^{-1}) - f(z,r),$$
$$\Delta^{1,Z}_{(u,v)}(f)(z,r) = f((z,r)\circ(u,v)) + f((z,r)\circ(u,v)^{-1}) - 2f(z,r),$$

and

$$\Delta^2_w(f)(z,r) = f(z,r-w) - f(z,r),$$
$$\Delta^{2,Z}_w(f)(z,r) = f(z,r+w) + f(z,r-w) - 2f(z,r).$$

The flag Lipschitz space on Heisenberg group is defined as follows.

Definition 1.1 A continuous function $f(z,r)$ defined on $\mathbb{H}^n$ belongs to the flag Lipschitz space $\text{Lip}^{\alpha}_{\text{flag}}$, where $\alpha = (\alpha_1, \alpha_2)$ and $\alpha_1, \alpha_2 > 0$, if and only if

(i) when $0 < \alpha_1, \alpha_2 < 1$,

$$(1.1)\qquad \begin{aligned} &|\Delta^2_w \Delta^1_{(u,v)}(f)(z,r)| \\ &\quad = \Big| f\big((z,r)\circ(u,v+w)^{-1}\big) - f\big((z,r)\circ(u,v)^{-1}\big) \\ &\qquad\qquad - f\big((z,r)\circ(0,w)^{-1}\big) + f(z,r)\Big| \\ &\quad \le C|(u,v)|^{\alpha_1}|w|^{\alpha_2}, \end{aligned}$$

where $|(u,v)|^2 = |u|^2 + |v|$;

(ii) when $\alpha_1 = 1, 0 < \alpha_2 < 1$,

$$
\begin{aligned}
(1.2)\qquad & |\Delta_w^2 \Delta_{(u,v)}^{1,Z}(f)(z,r)| \\
&= \Big| [f((z,r)\circ(u,v)) + f((z,r)\circ(u,v)^{-1}) - 2f(z,r)] \\
&\qquad - [f((z,r)\circ(u,v-w)) + f((z,r)\circ(u,v+w)^{-1}) \\
&\qquad\qquad - 2f((z,r)\circ(0,w)^{-1})] \Big| \\
&\le C|(u,v)||w|^{\alpha_2};
\end{aligned}
$$

(iii) when $0 < \alpha_1 < 1, \alpha_2 = 1$,

$$
\begin{aligned}
(1.3)\qquad & |\Delta_w^{2,Z} \Delta_{(u,v)}^{1}(f)(z,r)| \\
&= \Big| [f((z,r)\circ(0,w)) + f((z,r)\circ(0,w)^{-1}) - 2f(z,r)] \\
&\qquad - [f((z,r)\circ(u,-v+w)) + f((z,r)\circ(u,v+w)^{-1}) \\
&\qquad\qquad - 2f((z,r)\circ(u,v)^{-1})] \Big| \\
&\le C|(u,v)|^{\alpha_1}|w|;
\end{aligned}
$$

(iv) when $\alpha_1 = \alpha_2 = 1$,

$$
\begin{aligned}
(1.4)\qquad & |\Delta_w^{2,Z} \Delta_{(u,v)}^{1,Z}(f)(z,r)| \\
&= \Big| [f((z,r)\circ(u,v+w)) + f((z,r)\circ(u,v-w)^{-1}) \\
&\qquad\qquad - 2f((z,r)\circ(0,w))] \\
&\qquad + [f((z,r)\circ(u,v-w)) + f((z,r)\circ(u,v+w)^{-1}) \\
&\qquad\qquad - 2f((z,r)\circ(0,w)^{-1})] \\
&\qquad - 2[f((z,r)\circ(u,v)) + f((z,r)\circ(u,v)^{-1}) - 2f(z,r))] \Big| \\
&\le C|(u,v)||w|;
\end{aligned}
$$

for all (z,r) and $(u,v) \in \mathbb{H}^n$, $w \in \mathbb{R}$ and the constant C is independent of (z,r), (u,v) and w.

When $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 > 1$, we write $\alpha_1 = m_1 + r_1$ and $\alpha_2 = m_2 + r_2$ where m_1, m_2 are integers and $0 < r_1, r_2 \le 1$. Then $f(z,u) \in \mathrm{Lip}_{\mathrm{flag}}^{\alpha}$ means that $f(z,u)$ is a $C^{m_1+m_2}$ function, modulo polynomials of degree not exceeding $m_1 + m_2$, such that all the partial derivatives $\partial_z^{m_1}\partial_u^{m_2} f(z,u)$ belong to $\mathrm{Lip}_{\mathrm{flag}}^{r}$ with $r = (r_1, r_2)$.

If $f \in \mathrm{Lip}_{\mathrm{flag}}^{\alpha}$, then $\|f\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}}$, the norm of f, is defined as the smallest constant C such that (1.1) to (1.4) hold.

Note that the flag structure is involved in the the definition of $\mathrm{Lip}_{\mathrm{flag}}^{\alpha}(\mathbb{H}^n)$ with $\alpha = (\alpha_1, \alpha_2)$, and Zygmund conditions are used when $\alpha_1 = 1$ or $\alpha_2 = 1$, or both $\alpha_1 = \alpha_2 = 1$.

In order to obtain the boundedness of flag singular integral operators on the flag Lipschitz space $\mathrm{Lip}_{\mathrm{flag}}^{\alpha}$, we characterize $\mathrm{Lip}_{\mathrm{flag}}^{\alpha}$ via the Littlewood–Paley theory. We begin by recalling the standard Calderón reproducing formula on the Heisenberg group. Note that in [24], spectral theory was used in place of the Calderón reproducing formula.

Theorem 1.2 ([12, Corollary 1]) *There is a radial function $\psi \in C^\infty(\mathbb{H}^n)$ such that $\psi \in \mathcal{S}(\mathbb{H}^n)$ and all moments of ψ vanish, and such that the following Calderón reproducing formula holds, where $*$ is Heisenberg convolution and $\psi_s(z,t) = s^{-2n-2}\psi(\frac{z}{s}, \frac{t}{s^2})$ for $s > 0$:*

$$f = \int_0^\infty \psi_s * \psi_s * f \frac{ds}{s}, \quad f \in L^2(\mathbb{H}^n).$$

We now wish to extend this formula in two ways: (1) encompass the flag structure on the Heisenberg group $\mathbb{H}^n$; (2) converges in the distribution space. For this purpose, following [24], we construct a Littlewood–Paley *component* function ψ defined on $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$, given by the partial convolution $*_2$ in the second variable only,

$$\psi(z,u) = \psi^{(1)} *_2 \psi^{(2)}(z,u) = \int_{\mathbb{R}} \psi^{(1)}(z, u-v)\psi^{(2)}(v)dv, \quad (z,u) \in \mathbb{C}^n \times \mathbb{R},$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{H}^n)$ is as in Theorem 1.2, and $\psi^{(2)}$ is a real even function defined in $\mathcal{S}(\mathbb{R})$ that satisfies

$$\int_0^\infty |\widehat{\psi^{(2)}}(t\eta)|^2 \frac{dt}{t} = 1$$

for all $\eta \in \mathbb{R}\backslash\{0\}$, along with the moment conditions

$$\int_{\mathbb{H}^n} z^\alpha u^\beta \psi^{(1)}(z,u)dzdu = 0, \qquad |\alpha|, \beta \geq 0,$$
$$\int_{\mathbb{R}} v^\gamma \psi^{(2)}(v)dv = 0, \qquad \gamma \geq 0.$$

Thus, we have

$$f(z,u) = \int_0^\infty \int_0^\infty \psi_{s,t} * \psi_{s,t} * f(z,u) \frac{ds}{s}\frac{dt}{t}, \quad f \in L^2(\mathbb{H}^n), \tag{1.5}$$

where the functions $\psi_{s,t}$ are given by

$$\psi_{s,t}(z,u) = \psi_s^{(1)} *_2 \psi_t^{(2)}(z,u),$$

with

$$\psi_s^{(1)}(z,u) = s^{-2n-2}\psi^{(1)}\Big(\frac{z}{s}, \frac{u}{s^2}\Big) \quad \text{and} \quad \psi_t^{(2)}(v) = t^{-1}\psi^{(2)}\Big(\frac{v}{t}\Big),$$

and where the integrals in (1.5) converge in $L^2(\mathbb{H}^n)$. Indeed,

$$\begin{aligned}
&\psi_{s,t} *_{\mathbb{H}^n} \psi_{s,t} *_{\mathbb{H}^n} f(z,u)\\
&\quad = (\psi_s^{(1)} *_2 \psi_t^{(2)}) *_{\mathbb{H}^n} (\psi_s^{(1)} *_2 \psi_t^{(2)}) *_{\mathbb{H}^n} f(z,u)\\
&\quad = (\psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)}) *_{\mathbb{H}^n} \big((\psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)}) *_2 f(z,u)\big)
\end{aligned}$$

yields (1.5) upon invoking the standard Calderón reproducing formula on $\mathbb{R}$ and then Theorem 1.2 on $\mathbb{H}^n$:

$$\begin{aligned}\int_0^\infty \int_0^\infty \psi_{s,t} *_{\mathbb{H}^n} \psi_{s,t} *_{\mathbb{H}^n} f(z,u)\frac{ds}{s}\frac{dt}{t} \\ &= \int_0^\infty \psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)} *_{\mathbb{H}^n} \Big\{ \int_0^\infty \psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)} *_2 f(z,u)\frac{dt}{t}\Big\}\frac{ds}{s} \\ &= \int_0^\infty \psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)} *_{\mathbb{H}^n} f(z,u)\frac{ds}{s} = f(z,u).\end{aligned}$$

Note that the Littlewood–Paley component function ψ satisfies the *flag* moment conditions, so-called because they include only half of the product moment conditions associated with the product $\mathbb{C}^n \times \mathbb{R}$:

$$\int_{\mathbb{R}} u^\alpha \psi(z,u)du = 0, \quad \text{for all } \alpha \geq 0 \text{ and } z \in \mathbb{C}^n.$$

Indeed, with the change of variable $u' = u - v$ and the binomial theorem

$$(u'+v)^\beta = \sum_{\beta=\gamma+\delta} c_{\gamma,\delta}(u')^\gamma v^\delta,$$

we have

$$\begin{aligned}\int_{\mathbb{R}} u^\alpha \psi(z,u)du &= \int_{\mathbb{R}} u^\alpha \Big\{ \int_{\mathbb{R}} \psi^{(2)}(u-v)\psi^{(1)}(z,v)dv\Big\} du \\ &= \int_{\mathbb{R}} \Big\{ \int_{\mathbb{R}} (u'+v)^\alpha \psi^{(2)}(u')du\Big\} \psi^{(1)}(z,v)dv \\ &= \sum_{\alpha=\gamma+\delta} c_{\gamma,\delta} \int_{\mathbb{R}} \Big\{ \int_{\mathbb{R}} (u')^\gamma \psi^{(2)}(u')du'\Big\} v^\delta \psi^{(1)}(z,v)dv \\ &= \sum_{\alpha=\gamma+\delta} c_{\gamma,\delta} \int_{\mathbb{R}} \{0\} v^\delta \psi^{(1)}(z,v)dv = 0,\end{aligned}$$

for all $\alpha \geq 0$ and each $z \in \mathbb{C}^n$. Note that as a consequence, the full moments $\int_{\mathbb{H}^n} z^\alpha u^\beta \psi(z,u)dzdu$ all vanish, but that in general the partial moments $\int_{\mathbb{C}^n} z^\alpha \psi(z,u)dz$ do not vanish.

In order to introduce the flag test function space on the Heisenberg $\mathbb{H}^n$, we first introduce the product test function space on $\mathbb{H}^n \times \mathbb{R}$ as follows.

Definition 1.3 The product test function space on $\mathbb{H}^n \times \mathbb{R}$ is the collection of all Schwartz functions F on $\mathbb{H}^n \times \mathbb{R}$ such that

$$\int_{\mathbb{H}^n} z^\alpha u^\beta F(z,u,v)dzdu = 0$$

for all $|\alpha|, \beta \geq 0, v \in \mathbb{R}$ and

$$\int_{\mathbb{R}} v^\gamma F(z,u,v)dv = 0$$

for all $\gamma \geq 0, (z,u) \in \mathbb{H}^n$.

If F is a product test function on $\mathbb{H}^n \times \mathbb{R}$, we write $F \in \mathcal{S}_\infty(\mathbb{H}^n \times \mathbb{R})$. The norm of F is defined by the Schwartz norm of F on $\mathbb{H}^n \times \mathbb{R}$ and is denoted by $\|F\|_{\mathcal{S}_\infty}$.

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Applying the projection π defined on $\mathbb{H}^n$ as introduced in [23]:

$$f(z,u)=(\pi F)(z,u)\equiv\int_{\mathbb{R}}F\big((z,u-v),v\big)\,dv,$$

we now define the *flag* test function space $\mathcal{S}_{\text{flag}}(\mathbb{H}^n)$.

Definition 1.4 The *flag* test function space $\mathcal{S}_{\text{flag}}(\mathbb{H}^n)$ consists of all functions $f(z,u)$ defined on $\mathbb{H}^n$ satisfying

$$f(z,u)=(\pi F)(z,u)\equiv\int_{\mathbb{R}}F\big((z,u-v),v\big)\,dv$$

for some $F\in\mathcal{S}_\infty(\mathbb{H}^n\times\mathbb{R})$. The norm of f is defined by

$$\|f\|_{\mathcal{S}_{\text{flag}}}=\inf\{\|F\|_{\mathcal{S}_\infty}:f=\pi F\}.$$

The distribution space, $\mathcal{S}'_{\text{flag}}(\mathbb{H}^n)$, is defined by the dual of $\mathcal{S}_{\text{flag}}(\mathbb{H}^n)$.

Observing that $\psi_{s,t}(z,u)=\psi_s^{(1)}*_2\psi_t^{(2)}(z,u)\in\mathcal{S}_{\text{flag}}$, we will show that the Carlderón reproducing formula (1.5) also converges in both flag test function space $\mathcal{S}_{\text{flag}}(\mathbb{H}^n)$ and distribution space $\mathcal{S}'_{\text{flag}}(\mathbb{H}^n)$ as follows.

Theorem 1.5 *Let $\psi_{s,t}(z,u)=\psi_s^{(1)}*_2\psi_t^{(2)}(z,u)$. Then*

$$f(z,u)=\int_0^\infty\int_0^\infty\psi_{s,t}*\psi_{s,t}*f(z,u)\frac{ds}{s}\frac{dt}{t},\tag{1.6}$$

where the integrals converge in both $\mathcal{S}_{\text{flag}}(\mathbb{H}^n)$ and $\mathcal{S}'_{\text{flag}}(\mathbb{H}^n)$.

We characterize the flag Lipschitz space $\text{Lip}^\alpha_{\text{flag}}$ by the following theorem.

Theorem 1.6 *$f\in\text{Lip}^\alpha_{\text{flag}}$ with $\alpha=(\alpha_1,\alpha_2),\alpha_1,\alpha_2>0$ if and only if $f\in(\mathcal{S}_{\text{flag}})'(\mathbb{H}^n)$ and*

$$\sup_{s,t>0,(z,u)\in\mathbb{H}^n}s^{-\alpha_1}t^{-\alpha_2}|\psi_{s,t}*f(z,u)|\le C<\infty.$$

Moreover,

$$\|f\|_{\text{Lip}^\alpha_{\text{flag}}}\sim\sup_{s,t>0,(z,u)\in\mathbb{H}^n}s^{-\alpha_1}t^{-\alpha_2}|\psi_{s,t}*f(z,u)|.$$

The main result of this paper is the following theorem.

Theorem 1.7 *The flag singular integral operator T is bounded on $\text{Lip}^\alpha_{\text{flag}}$ with $\alpha=(\alpha_1,\alpha_2),\alpha_1,\alpha_2>0$. Moreover, $\|Tf\|_{\text{Lip}^\alpha_{\text{flag}}}\le C\|f\|_{\text{Lip}^\alpha_{\text{flag}}}$.*

As a consequence, the Marcinkiewicz multipliers are bounded on the flag Lischitz space $\text{Lip}^\alpha_{\text{flag}}$.

2 Proof of Theorem 1.5

Observe that for $F_1,F_2\in\mathcal{S}_\infty(\mathbb{H}^n\times\mathbb{R})$, then $\pi(F_1*F_2)=(\pi F_1)*(\pi F_2)$. Suppose that $f\in\mathcal{S}_{\text{flag}}(\mathbb{H}^n)$ and $f(x,y)=\pi(F)$ with $F\in\mathcal{S}_\infty(\mathbb{H}^n\times\mathbb{R})$. Set $\Psi_{s,t}(z,u,r)=$

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$\psi_s^{(1)}(z,u)\psi_t^{(2)}(r)$ then $\psi_{s,t} = \pi(\Psi_{s,t})$. We rewrite the Calderón reproducing formula in (1.6) as

$$(\pi F)(z,u) = \int_0^\infty \int_0^\infty \pi(\Psi_{s,t}) * (\pi\Psi_{s,t}) * (\pi F)(z,u) \frac{ds}{s}\frac{dt}{t}.$$

To show Theorem 1.5, it suffices to prove that for $f \in \mathcal{S}_{\text{flag}}(\mathbb{H}^n)$,

$$\Big\| \pi\Big(F - \int_{N^{-1}}^N \int_{N^{-1}}^N \Psi_{s,t} * \Psi_{s,t} * F \frac{ds}{s}\frac{dt}{t}\Big) \Big\|_{\mathcal{S}_{\text{flag}}} \longrightarrow 0$$

as N tends to ∞.

By the definition of the norm of $\mathcal{S}_{\text{flag}}(\mathbb{H}^n)$, one only needs to show that

$$\Big\| F - \int_{N^{-1}}^N \int_{N^{-1}}^N \Psi_{s,t} * \Psi_{s,t} * F(z,u,r) \frac{ds}{s}\frac{dt}{t} \Big\|_{\mathcal{S}_\infty} \longrightarrow 0$$

as N tends to ∞.

To do this, observe that

$$\begin{aligned}
&\Psi_{s,t} *_{\mathbb{H}^n\times\mathbb{R}} \Psi_{s,t} *_{\mathbb{H}^n\times\mathbb{R}} F\big((z,u),r\big) \\
&\quad = (\psi_s^{(1)}\psi_t^{(2)}) *_{\mathbb{H}^n\times\mathbb{R}} (\psi_s^{(1)}\psi_t^{(2)}) *_{\mathbb{H}^n\times\mathbb{R}} F\big((z,u),r\big) \\
&\quad = \big[(\psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)})(\psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)})\big] *_{\mathbb{H}^n\times\mathbb{R}} F\big((z,u),r\big)
\end{aligned}$$

yields

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \Psi_{s,t} *_{\mathbb{H}^n\times\mathbb{R}} \Psi_{s,t} *_{\mathbb{H}^n\times\mathbb{R}} F((z,u),r) \frac{ds}{s}\frac{dt}{t} \\
&\quad = \int_0^\infty \psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)} *_{\mathbb{H}^n} \Big\{ \int_0^\infty \psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)} *_{\mathbb{R}} F\big((z,u),r\big) \frac{dt}{t} \Big\} \frac{ds}{s} \\
&\quad = \int_0^\infty \psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)} *_{\mathbb{H}^n} F\big((z,u),r\big) \frac{ds}{s} \\
&\quad = F\big((z,u),r\big).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\Big| F - \int_{N^{-1}}^N \int_{N^{-1}}^N \Psi_{s,t} * \Psi_{s,t} * F(z,u,r) \frac{ds}{s}\frac{dt}{t} \Big| \\
&\quad \le \int_0^\infty \int_0^{N^{-1}} |\Psi_{s,t} * \Psi_{s,t} * F(z,u,r)| \frac{ds}{s}\frac{dt}{t} \\
&\qquad + \int_0^\infty \int_N^\infty |\Psi_{s,t} * \Psi_{s,t} * F(z,u,r)| \frac{ds}{s}\frac{dt}{t} \\
&\qquad + \int_0^{N^{-1}} \int_0^\infty |\Psi_{s,t} * \Psi_{s,t} * F(z,u,r)| \frac{ds}{s}\frac{dt}{t} \\
&\qquad + \int_N^\infty \int_0^\infty |\Psi_{s,t} * \Psi_{s,t} * F(z,u,r)| \frac{ds}{s}\frac{dt}{t} \\
&\quad = I + II + III + IV.
\end{aligned}$$

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We claim that for $N \geq 1$ and any given an positive integer L, there exists a constant C_L independent of N such that

$$I \leq C_L \big(1 + |(z,u)|^2 + |r|\big)^{-L} N^{-1} \|F\|_{\mathcal{S}_\infty}.$$

To show the claim, we write

$$\begin{aligned}
&\Psi_{s,t} *_{\mathbb{R}\times\mathbb{H}^n} \Psi_{s,t} *_{\mathbb{H}^n\times\mathbb{R}} F\big((z,u),r\big) \\
&\quad = \big[(\psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)})(\psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)})\big] *_{\mathbb{H}^n\times\mathbb{R}} F\big((z,u),r\big) \\
&\quad = \int_{\mathbb{R}\times\mathbb{H}^n} \psi_s^1(z',u')\psi_t^2(r') F\big((z,u)\circ(z',u')^{-1}, r-r'\big)\, dz'du'dr',
\end{aligned}$$

where $\psi_s^1 = \psi_s^{(1)} *_{\mathbb{H}^n} \psi_s^{(1)}$ and $\psi_t^2 = \psi_t^{(2)} *_{\mathbb{R}} \psi_t^{(2)}$.

Considering the case where $0 < t \leq 1$ first, the cancellation conditions on ψ_s^1 and ψ_t^2 imply that

$$\begin{aligned}
(2.1)\ &\int_{\mathbb{R}\times\mathbb{H}^n} \psi_s^1(z',u')\psi_t^2(r')F((z,u)\circ(z',u')^{-1}, r-r')dz'du'dr' \\
&\quad = \int_{\mathbb{R}\times\mathbb{H}^n} \psi_s^1(z',u')\psi_t^2(r')[F((z,u)\circ(z',u')^{-1}, r-r') - F((z,u), r-r') \\
&\qquad - F((z,u)\circ(z',u')^{-1}, r) + F((z,u),r)]dz'du'dr'.
\end{aligned}$$

Noting that $|(z,u)|^2 = |z|^2 + |u|$ and applying the smoothness conditions on F, we obtain the following estimates. For any given positive integer L there exists a constant C such that

(a)
$$\begin{aligned}
&|F((z,u)\circ(z',u')^{-1}, r-r') - F((z,u), r-r') - F((z,u)\circ(z',u')^{-1}, r) \\
&\qquad\qquad + F((z,u),r)| \\
&\leq C\frac{|(z',u')|}{(1+|(z,u)|)^{L+2n+2}} \frac{|r'|}{(1+|r|)^{L+1}} \|F\|_{\mathcal{S}_\infty},
\end{aligned}$$

for $|(z',u')| \leq \frac{1}{2\gamma}(1+|(z,u)|)$ and $|r'| \leq \frac{1}{2}(1+|r|)$;

(b)
$$\begin{aligned}
&|F((z,u)\circ(z',u')^{-1}, r-r') - F((z,u), r-r') - F((z,u)\circ(z',u')^{-1}, r) \\
&\qquad\qquad + F((z,u),r)| \\
&\leq C\frac{|(z',u')|}{(1+|(z,u)|)^{L+2n+2}} \left[\frac{1}{(1+|r-r'|)^{L+1}} + \frac{1}{(1+|r|)^{L+1}}\right] \|F\|_{\mathcal{S}_\infty}
\end{aligned}$$

for $|(z',u')| \leq \frac{1}{2\gamma}(1+|(z,u)|)$ and $|r'| \geq \frac{1}{2}(1+|r|)$;

(c)
$$\begin{aligned}
&|F((z,u)\circ(z',u')^{-1}, r-r') - F((z,u), r-r') - F((z,u)\circ(z',u')^{-1}, r) \\
&\qquad\qquad + F((z,u),r)| \\
&\leq C\Big(\frac{1}{(1+|(z,u)\circ(z',u')^{-1}|)^{L+2n+2}} + \frac{1}{(1+|(z,u)|)^{L+2n+2}}\Big) \frac{|r'|}{(1+|r|)^{L+1}} \|F\|_{\mathcal{S}_\infty}
\end{aligned}$$

for $|(z',u')| \geq \frac{1}{2\gamma}(1+|(z,u)|)$ and $|r'| \leq \frac{1}{2}(1+|r|)$;

(d)
$$|F((z,u)\circ(z',u')^{-1}, r-r') - F((z,u), r-r') - F((z,u)\circ(z',u')^{-1}, r) + F((z,u),r)|$$
$$\leq C\Big(\frac{1}{(1+|(z,u)\circ(z',u')^{-1}|)^{L+2n+2}} + \frac{1}{(1+|(z,u)|)^{L+2n+2}}\Big) \times \Big(\frac{1}{(1+|r-r'|)^{L+1}} + \frac{1}{(1+|r|)^{L+1}}\Big)\|F\|_{\mathcal{S}_\infty}$$

for $|(z',u')| \geq \frac{1}{2\gamma}(1+|(z,u)|)$ and $|r'| \geq \frac{1}{2}(1+|r|)$.

Plugging the above estimates together with the size conditions

$$|\psi_s^1(z',u')| \leq Cs^{-2n-2}\frac{1}{\big(1+|\frac{z'}{s}|^2+|\frac{u'}{s^2}|\big)^{n+2+L}},$$
$$|\psi_t^2(r')| \leq Ct^{-1}\frac{1}{\big(1+|\frac{r'}{t}|\big)^{2+L}}$$

into (2.1) yields that whenever $0 < t \leq 1$,

$$\Big|\int_{\mathbb{H}^n\times\mathbb{R}} \psi_s^1(z',u')\psi_t^2(r')F((z,u)\circ(z',u')^{-1}, r-r')dz'du'dr'\Big| \leq Cst\frac{1}{(1+|(z,u)|^2+|r|)^L}\|F\|_{\mathcal{S}_\infty}.$$

We now consider the case with $1 \leq t < \infty$. For this case, applying the cancellation conditions on ψ_s^1 and $F((z,u),r)$ with respect to the variable r, we write

$$\text{(2.2)}\quad \int_{\mathbb{R}\times\mathbb{H}^n} \psi_s^1(z',u')\psi_t^2(r')F((z,u)\circ(z',u')^{-1}, r-r')dz'du'dr'$$
$$= \int_{\mathbb{R}\times\mathbb{H}^n} \psi_s^1(z',u')\Big[\psi_t^2(r') - \sum_{j\leq L} c_j(\psi_t^2)^{(j)}(r)(r'-r)^j\Big] \times\Big[F\big((z,u)\circ(z',u')^{-1}, r-r'\big) - F\big((z,u), r-r'\big)\Big]dz'du'dr',$$

where $\sum_{j\leq L} c_j(\psi_t^2)^{(j)}(r)(r'-r)^j$ is the Taylor series of ψ_t^2 at r with the degree L. Inserting estimates

(a)
$$|\psi_t^2(r') - \sum_{j\leq L} c_j(\psi_t^2)^{(j)}(r)(r'-r)^j| \leq C\Big(\frac{|r-r'|}{t+|r|}\Big)^L\frac{t^L}{(t+|r|)^{1+L}}$$

for $|r-r'| \leq \frac{1}{2}(t+|r|)$;

(b)
$$|\psi_t^2(r') - \sum_{j\leq L} c_j(\psi_t^2)^{(j)}(r)(r'-r)^j| \leq C\Big[\frac{t^L}{(t+|r'|)^{1+L}} + \sum_{j\leq L} c_j\frac{|r'-r|^j}{t^j}\frac{t^L}{(t+|r|)^{1+L}}\Big]$$

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for $|r-r'|\geq\frac{1}{2}(t+|r|)$, together with

(c)
$$|F((z,u)\circ(z',u')^{-1},r-r')-F((z,u),r-r')|\leq$$
$$C\frac{|(z',u')|}{(1+|(z,u)|)^{L+2n+2}}\frac{1}{(1+|r-r'|)^{L+2}}\|F\|_{\mathcal{S}_\infty},$$
for $|(z',u')|\leq\frac{1}{2\gamma}(1+|(z,u)|)$;

(d)
$$|F((z,u)\circ(z',u')^{-1},r-r')-F((z,u),r-r')|\leq$$
$$C\Big(\frac{1}{(1+|(z,u)\circ(z',u')^{-1}|)^{L+2n+2}}+\frac{1}{(1+|(z,u)|)^{L+2n+2}}\Big)\frac{1}{(1+|r-r'|)^{L+1}}\|F\|_{\mathcal{S}_\infty}$$
for $|(z',u')|\geq\frac{1}{2\gamma}(1+|(z,u)|)$ into (2.2) gives that if $t\geq1$,
$$\Big|\int_{\mathbb{H}^n\times\mathbb{R}}\psi_s^1(z',u')\psi_t^2(r')F((z,u)\circ(z',u')^{-1},r-r')dz'du'dr'\Big|\leq C\frac{s}{t}\frac{1}{(1+|(z,u)|^2+|r|)^L}\|F\|_{\mathcal{S}_\infty}.$$

These estimates imply the proof of the claim. Note that $(\partial_z^\alpha\partial_u^\beta\partial_r^\gamma F)$ satisfy similar regularity and cancellation conditions as F does, and
$$\partial_z^\alpha\partial_u^\beta\partial_r^\gamma\check{\Psi}_{s,t}*_{\mathbb{H}^n\times\mathbb{R}}\Psi_{s,t}*_{\mathbb{H}^n\times\mathbb{R}}F\big((z,u),r\big)=\check{\Psi}_{s,t}*_{\mathbb{H}^n\times\mathbb{R}}\Psi_{s,t}*_{\mathbb{H}^n\times\mathbb{R}}(\partial_z^\alpha\partial_u^\beta\partial_r^\gamma F)\big((z,u),r\big).$$

These facts yield that $\|I\|_{\mathcal{S}_\infty}\leq CN^{-1}$, and hence $\|I\|_{\mathcal{S}_\infty}\to0$ as N tends to ∞. The proofs for II, III, and IV are similar, and thus the proof of Theorem 1.5 is concluded.

3 Proof of Theorem 1.6

We first show that if $f\in\mathrm{Lip}_{\mathrm{flag}}^\alpha$ with $\alpha=(\alpha_1,\alpha_2)$, $0<\alpha_1,\alpha_2<1$, then $f\in\mathcal{S}'_{\mathrm{flag}}$. To see this, for each $g(z,u)\in\mathcal{S}_{\mathrm{flag}}$ where $g=\pi G$ with $G\in\mathcal{S}_\infty$ and $\|G\|_{\mathcal{S}_\infty}=1$, we observe that
$$\begin{aligned}\langle g,f\rangle&=\int_{\mathbb{H}^n}\int_{\mathbb{R}}G((z,u-v),v)f(z,u)dvdzdu\\&=\int_{\mathbb{H}^n}\int_{\mathbb{R}}G((z,u),v)f(z,u+v)dvdzdu.\end{aligned}$$
Applying the cancellation conditions on $G((z,u),v)$, we have
$$\langle g,f\rangle=\int_{\mathbb{H}^n}\int_{\mathbb{R}}G\big((z,u),v\big)\big[f(z,u+v)-f(0,v)-f(z,u)+f(0,0)\big]dvdzdu.$$
The size conditions on G and the fact that $f\in\mathrm{Lip}_{\mathrm{flag}}^\alpha$ imply that
$$|\langle g,f\rangle|\leq C\|f\|_{\mathrm{Lip}_{\mathrm{flag}}^\alpha},$$

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which shows that $f \in (\mathcal{S}_{\mathrm{flag}})'$.

Observe that

$$\psi_{s,t} * f(z,u) = \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}(z',u')\psi_t^{(2)}(v) f\big((z,u)\circ(z',u'+v)^{-1}\big)\,dv dz' du'.$$

Applying the cancellation conditions on $\psi_s^{(1)}$ and $\psi_t^{(2)}$, we have

$$\begin{aligned}\psi_{s,t} * f(z,u) &= \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}(z',u')\psi_t^{(2)}(v)[f((z,u)\circ(z',u'+v)^{-1})\\ &\quad - f((z,u)\circ(z',u')^{-1}) - f((z,u)\circ(0,v)^{-1}) + f(z,u)]dv dz' du'\\ &= \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}(z',u')\psi_t^{(2)}(v)\Delta_v^2\Delta_{(z',u')}^1(f)(z,u) dv dz' du'.\end{aligned}$$

The size conditions on $\psi_s^{(1)}$ and $\psi_t^{(2)}$ and the fact that $f \in \mathrm{Lip}_{\mathrm{flag}}^{\alpha}$ yield that

$$|\psi_{s,t} * f(z,u)| \le C s^{\alpha_1} t^{\alpha_2}\|f\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}},$$

which implies that $s^{-\alpha_1}t^{-\alpha_2}|\psi_{s,t} * f(z,u)| \le C\|f\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}}$ for any $s,t>0$, and $(z,u)\in \mathbb{H}^n$.

When $\alpha = (\alpha_1,\alpha_2)$ with $\alpha_1 = 1$, $0<\alpha_2<1$, noting first that $\psi_s^{(1)}$ is a radial function and then applying the cancellation conditions on $\psi_t^{(2)}$, we have

$$\begin{aligned}&\psi_{s,t} * f(z,u)\\ &\quad= \frac{1}{2}\int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}(z',u')\psi_t^{(2)}(v)\big[f\big((z,u)\circ(z',u'-v)\big)\\ &\qquad\qquad + f((z,u)\circ(z',u'+v)^{-1}\big)\big] dz' du' dv\\ &\quad= \frac{1}{2}\int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}(z',u')\psi_t^{(2)}(v)\big[f((z,u)\circ(z',u'-v))\\ &\qquad + f((z,u)\circ(z',u'+v)^{-1}) - 2f((z,u)\circ(0,v)^{-1})\big] dz' du' dv\\ &\quad= \frac{1}{2}\int\int \psi_s^{(1)}(z',u')\psi_t^{(2)}(v)\big\{ [f((z,u)\circ(z',u'-v))\\ &\qquad + f((z,u)\circ(z',u'+v)^{-1}) - 2f((z,u)\circ(0,v)^{-1})]\\ &\qquad - [f((z,u)\circ(z',u')) + f((z,u)\circ(z',u')^{-1}) - 2f(z,u)]\big\} dz' du' dv\\ &\quad= \frac{1}{2}\int\int \psi_s^{(1)}(z',u')\psi_t^{(2)}(v)\Delta_v^2\Delta_{(z',u')}^{1,Z}(f)(z,u) dz' du' dv.\end{aligned}$$

Applying the same proof gives the desired estimate for this case. All other cases $\alpha = (\alpha_1,\alpha_2)$ where $0<\alpha_1<1$, $\alpha_2 = 1$ or $\alpha_1=\alpha_2=1$ can be handled similarly. For the case where $\alpha = (\alpha_1,\alpha_2)$ with $\alpha_1,\alpha_2>1$, set $\partial^{m_1+m_2}\widetilde{\psi}(z,u) = \psi$ with $s^{-m_1}t^{-m_2}\widetilde{\psi}_{s,t}$ satisfying the similar smoothness, size, and cancellation conditions as $\psi_{s,t}$. Thus, repeating the same proof gives

$$\begin{aligned}|\psi_{s,t} * f| &= |s^{m_1}t^{m_2}(s^{-m_1}t^{-m_2}\widetilde{\psi}_{s,t}) * \partial^{m_1+m_2} f| \le C s^{m_1}t^{m_2}s^{r_1}t^{r_2}\|f\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}}\\ &= C s^{\alpha_1}t^{\alpha_2}\|f\|_{\mathrm{Lip}_{\mathrm{flag}}^{\alpha}}.\end{aligned}$$

Therefore, this case can be also handled similarly.

We now prove the converse implication of Theorem 1.6. Suppose that $f \in \mathcal{S}'_{\text{flag}}(\mathbb{H}^n)$ and $|\psi_{s,t} * f(z,u)| \le Cs^{\alpha_1}t^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$. We first show that f is a continuous function on $\mathbb{H}^n$. To do this, by Theorem 1.5,

$$f(z,u) = \int_0^\infty \int_0^\infty \psi_{s,t} * \psi_{s,t} * f(z,u) \frac{ds}{s}\frac{dt}{t},$$

where the integral converges in $\mathcal{S}'_{\text{flag}}(\mathbb{H}^n)$. We split the above integral into the following four cases:

(a) $s, t < 1$;
(b) $s \ge 1, t < 1$;
(c) $s < 1, t \ge 1$;
(d) $s, t \ge 1$.

Then we write $f = f_1 + f_2 + f_3 + f_4$ in $\mathcal{S}'_{\text{flag}}(\mathbb{H}^n)$ according to the above four cases of s and t. As for f_1, observe that the fact $|\psi_{s,t} * \psi_{s,t} * f(z,u)| \le Cs^{\alpha_1}t^{\alpha_2}$ implies that the integral

$$\int_0^1 \int_0^1 \psi_{s,t} * \psi_{s,t} * f(z,u) \frac{ds}{s}\frac{dt}{t}$$

converges uniformly and hence, f_1 is a continuous function on $\mathbb{H}^n$.

To see that f_2 is continuous, for $g \in \mathcal{S}_{\text{flag}}$,

$$\langle f_2, g\rangle = \int_0^1 \int_1^\infty \langle \psi_{s,t} * \psi_{s,t} * f(z,u), g(z,u)\rangle \frac{ds}{s}\frac{dt}{t}.$$

Note that by the cancellation condition on g, for any fixed positive integer N, we have

$$\begin{aligned}
&\Big\langle \psi_{s,t} * \psi_{s,t} * f(z,u), g(z,u)\Big\rangle \\
&\quad= \Big\langle \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}((z,u)\circ(z',u'+v)^{-1})\psi_t^{(2)}(v)\psi_{s,t} \\
&\qquad\quad * f(z',u')dvdz'du', g(z,u)\Big\rangle \\
&\quad= \Big\langle \int_{\mathbb{H}^n}\int_{\mathbb{R}} \Big[\psi_s^{(1)}((z,u)\circ(z',u'+v)^{-1}) \\
&\qquad\quad - \sum_{|\alpha|+2\beta\le 2N} c_{\alpha,\beta}\partial_z^\alpha\partial_u^\beta\psi_s^{(1)}((z',u'+v)^{-1})(z,u)^{\alpha,\beta}\Big] \\
&\qquad\quad \times \psi_t^{(2)}(v)\psi_{s,t} * f(z',u')dvdz'du', g(z,u)\Big\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
f_2(z,u) = \int_0^1\int_1^\infty \int_{\mathbb{H}^n}\int_{\mathbb{R}} &\Big[\psi_s^{(1)}((z,u)\circ(z',u'+v)^{-1}) \\
&- \sum_{|\alpha|+2\beta\le 2N} c_{\alpha,\beta}\partial_z^\alpha\partial_u^\beta\psi_s^{(1)}((z',u'+v)^{-1})(z,u)^{\alpha,\beta}\Big]\psi_t^{(2)}(v)\psi_{s,t} \\
&* f(z',u')dvdz'du'\frac{ds}{s}\frac{dt}{t}
\end{aligned}$$

in the sense of $\mathcal{S}'_{\text{flag}}(\mathbb{H}^n)$.

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And hence,

$$\Big| \int_{\mathbb{H}^n} \int_{\mathbb{R}} \Big[\psi_s^{(1)}((z,u)\circ(z',u'+v)^{-1}) - \sum_{|\alpha|+2\beta\le 2N} c_{\alpha,\beta}\partial_z^{\alpha}\partial_u^{\beta}\psi_s^{(1)}((z',u'+v)^{-1})(z,u)^{\alpha,\beta}\Big] \times \psi_t^{(2)}(v)\psi_{s,t} * f(z',u')dvdz'du' \Big| \le Cs^{-2N+\alpha_1}t^{\alpha_2}|(z,u)|^{2N}.$$

The above estimate implies that for any given $R > 0$, the integral for f_2 converges uniformly for $|(z,u)|^{2N} \le R$, and thus f_2 is a continuous function on any compact subset in $\mathbb{H}^n$.

Note that $(z,u)\circ(z',u'+v)^{-1} = (z,-u')\circ(z',-u+v)^{-1}$. We write

$$\begin{aligned}
\langle \psi_{s,t} * \psi_{s,t} * f(z,u), g(z,u)\rangle
&= \Big\langle \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}((z,-u')\circ(z',-u+v)^{-1})\psi_t^{(2)}(v)\psi_{s,t} * f(z',u')dvdz'du', g(z,u)\Big\rangle\\
&= \Big\langle \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}((z,-u')\circ(z',v)^{-1})\psi_t^{(2)}(u+v)\psi_{s,t} * f(z',u')dvdz'du', g(z,u)\Big\rangle\\
&= \Big\langle \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}((z,-u')\circ(z',v)^{-1})\Big[\psi_t^{(2)}(u+v) - \sum_{\gamma\le 2N} c_\gamma \frac{d^\gamma}{dv^\gamma}\psi_t^{(2)}(v)v^\gamma\Big] \times \psi_{s,t} * f(z',u')dvdz'du', g(z,u)\Big\rangle.
\end{aligned}$$

Similarly,

$$\Big| \int_{\mathbb{H}^n}\int_{\mathbb{R}} \psi_s^{(1)}((z,-u')\circ(z',v)^{-1}) \times \Big[\psi_t^{(2)}(u+v) - \sum_{\gamma\le 2N} c_\gamma \frac{d^\gamma}{dv^\gamma}\psi_t^{(2)}(v)v^\gamma\Big]\psi_{s,t} * f(z',u')dvdz'du' \Big| \le Cs^{\alpha_1}t^{-2N+\alpha_2}|u|^{2N},$$

and hence f_3 is a continuous function on any compact subset in $\mathbb{H}^n$.

Taking the geometric means of these two estimates shows that f_4 is a continuous function.

Now we show that $f \in \mathrm{Lip}^{\alpha}_{\mathrm{flag}}$. First, if $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1, \alpha_2 < 1$, then

$$\begin{aligned}
&\Big|\Delta^2_w \Delta^1_{(u,v)}(f)(z,r)\Big| \\
&\quad = \Big| f\big((z,r)\circ(u,v+w)^{-1}\big) - f\big((z,r)\circ(u,v)^{-1}\big) \\
&\qquad - f\big((z,r)\circ(0,w)^{-1}\big) + f(z,r)\Big| \\
&\quad = \Big| \int_0^\infty \int_0^\infty \int_{\mathbb{H}^n} \Big[\psi_{s,t}\big((z,r)\circ(u,v+w)^{-1}\circ(z',r')^{-1}\big) \\
&\qquad - \psi_{s,t}\big((z,r)\circ(u,v)^{-1}\circ(z',r')\big) \\
&\qquad - \psi_{s,t}\big((z,r)\circ(0,w)^{-1}\circ(z',r')^{-1}\big) + \psi_{s,t}\big((z,r)\circ(z',r')\big)\Big]\psi_{s,t} \\
&\qquad * f(z',r')dz'dr'\frac{ds}{s}\frac{dt}{t}\Big|.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\psi_{s,t}\big((z,r)\circ(u,v+w)^{-1}\circ(z',r')^{-1}\big) - \psi_{s,t}\big((z,r)\circ(u,v)^{-1}\circ(z',r')\big) \\
&\qquad - \psi_{s,t}\big((z,r)\circ(0,w)^{-1}\circ(z',r')^{-1}\big) + \psi_{s,t}\big((z,r)\circ(z',r')\big) \\
&\quad = \int_{\mathbb{R}} \Big[\psi^{(1)}_s\big((z,r)\circ(u,v)^{-1}\circ(z',v'+r')^{-1}\big) \\
&\qquad - \psi^{(1)}_s\big((z,r)\circ(z',v'+r')^{-1}\big)\Big]\Big[\psi^{(2)}_t(v'-w) - \psi^{(2)}_t(v')\Big]dv'.
\end{aligned}$$

We have

$$\begin{aligned}
A &= \int_{\mathbb{H}^n}\Big[\psi_{s,t}\big((z,r)\circ(u,v+w)^{-1}\circ(z',r')^{-1}\big) - \psi_{s,t}\big((z,r)\circ(u,v)^{-1}\circ(z',r')\big) \\
&\quad - \psi_{s,t}\big((z,r)\circ(0,w)^{-1}\circ(z',r')^{-1}\big) \\
&\quad + \psi_{s,t}\big((z,r)\circ(z',r')\big)\Big]\psi_{s,t} * f(z',r')dz'dr' \\
&= \int_{\mathbb{H}^n}\int_{\mathbb{R}}\Big[\psi^{(1)}_s\big((z,r)\circ(u,v)^{-1}\circ(z',v'+r')^{-1}\big) - \psi^{(1)}_s\big((z,r)\circ(z',v'+r')^{-1}\big)\Big] \\
&\quad \times\Big[\psi^{(2)}_t(v'-w) - \psi^{(2)}_t(v')\Big]\psi_{s,t} * f(z',r')dv'dz'dr'.
\end{aligned}$$

We now choose t_0 and s_0 such that $t_0 \le |(u,v)| < 2t_0$ and $s_0 \le |w| < 2s_0$, and we split

$$\begin{aligned}
\int_0^\infty\int_0^\infty A &= \int_0^{s_0}\int_0^{t_0} A + \int_{s_0}^\infty\int_0^{t_0} A + \int_0^{s_0}\int_{t_0}^\infty A + \int_{s_0}^\infty\int_{t_0}^\infty A \\
&=: A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

To deal with A_1, applying the size conditions on both $\psi^{(1)}_s$ and $\psi^{(2)}_t$ yields

$$|A| \le C|\psi_{s,t} * f(z',r')| \le Cs^{\alpha_1}t^{\alpha_2}, \text{ for all } s,t > 0,$$

and hence A_1 is dominated by

$$C\int_0^{s_0}\int_0^{t_0} s^{\alpha_1}t^{\alpha_2}\frac{ds}{s}\frac{dt}{t} \le Cs_0^{\alpha_1}t_0^{\alpha_2} \le C|(u,v)|^{\alpha_1}|w|^{\alpha_2}.$$

To estimate A_2, applying the smoothness condition on $\psi_s^{(1)}$ and the size condition on $\psi_t^{(k)}$ implies

$$|A| \lesssim \frac{|(u,v)|}{s}|\psi_{s,t} * f(z',r')| \lesssim s^{\alpha_1-1}t^{\alpha_2}|(u,v)|.$$

This implies that A_2 is bounded by

$$C\int_{s_0}^{\infty}\int_0^{t_0} s^{\alpha_1-1}t^{\alpha_2}\frac{ds}{s}\frac{dt}{t}|(u,v)| \le Cs_0^{\alpha_1-1}t_0^{\alpha_2}|(u,v)| \le C|(u,v)|^{\alpha_1}|w|^{\alpha_2}.$$

The estimate for A_3 is similar to the estimate for A_2. Finally, to handle A_4, applying the smoothness conditions on both $\psi_s^{(1)}$ and $\psi_t^{(k)}$ we obtain that

$$|A| \lesssim \frac{|(u,v)|}{s}\cdot\frac{|w|}{t}|\psi_{s,t} * f(z',r')| \lesssim s^{\alpha_1-1}t^{\alpha_2-1}|(u,v)||w|.$$

Hence, this implies that A_4 is dominated by

$$C\int_{s_0}^{\infty}\int_{t_0}^{\infty} s^{\alpha_1-1}t^{\alpha_2-1}\frac{ds}{s}\frac{dt}{t}|(u,v)||w| \le Cs_0^{\alpha_1-1}t_0^{\alpha_2-1}|(u,v)||w| \le C|(u,v)|^{\alpha_1}|w|^{\alpha_2}.$$

When $\alpha = (\alpha_1,\alpha_2)$ with $\alpha_1 = \alpha_2 = 1$, observe that

$$\begin{aligned}
&\Delta_w^{2,Z}\Delta_{(u,v)}^{1,Z}(f)(z,r)\\
&\quad= \Big[f((z,r)\circ(u,v+w)) + f((z,r)\circ(u,v-w)^{-1}) - 2f((z,r)\circ(0,w))\Big]\\
&\qquad+\Big[f\big((z,r)\circ(u,v-w)\big)\\
&\qquad+ f((z,r)\circ(u,v+w)^{-1}) - 2f((z,r)\circ(0,w)^{-1})\Big]\\
&\qquad-2\Big[f((z,r)\circ(u,v)) + f((z,r)\circ(u,v)^{-1}) - 2f(z,r)\Big]\\
&\quad= \int_0^\infty\int_0^\infty\int_{\mathbb{H}^n}\int_{\mathbb{R}}\Big[\psi_s^{(1)}\big((z,r)\circ(u,v)^{-1}\circ(z',v'+r')^{-1}\big)\\
&\qquad+\psi_s^{(1)}\big((z,r)\circ(u,v)\circ(z',v'+r')^{-1}\big)\\
&\qquad-2\psi_s^{(1)}\big((z,r)\circ(z',v'+r')^{-1}\big)\Big]\\
&\qquad\times\Big[\psi_t^{(2)}(v'-w)+\psi_t^{(2)}(v'+w)-2\psi_t^{(2)}(v')\Big]\psi_{j,k} * f(\omega)dzd\omega.
\end{aligned}$$

Repeating a similar calculation gives the desired result for this case. The other two cases, where $\alpha_1 = 1, 0<\alpha_2<1$ and $0<\alpha_1<1, \alpha_2 = 1$, can be handled similarly. Lastly, when $\alpha_1 = m_1 + r_1, \alpha_2 = m_2 + r_2$, note that

$$\begin{aligned}
&\partial^{m_1+m_2}f\big((z,r)\circ(u,v+w)^{-1}\big) - \partial^{m_1+m_2}f\big((z,r)\circ(u,v)^{-1}\big)\\
&\qquad-\partial^{m_1+m_2}f\big((z,r)\circ(0,w)^{-1}\big) + \partial^{m_1+m_2}f(z,r)\\
&\quad= \int_{\mathbb{H}^n}\int_{\mathbb{R}}\Big[\partial^{m_1}\psi_s^{(1)}\big((z,r)\circ(u,v)^{-1}\circ(z',v'+r')^{-1}\big)\\
&\qquad-\partial^{m_1}\psi_s^{(1)}((z,r)\circ(z',v'+r')^{-1})\Big]\\
&\qquad\times\Big[\partial^{m_2}\psi_t^{(2)}(v'-w)-\partial^{m_2}\psi_t^{(2)}(v')\Big]\psi_{s,t} * f(z',r')dv'dz'dr'.
\end{aligned}$$

Again observe that the properties of $\partial^{m_1}\psi_s^{(1)}$ and $\partial^{m_2}\psi_t^{(2)}$ are similar to $s^{-m_1}\psi_s^{(1)}$ and $t^{-m_2}\psi_t^{(1)}$, respectively, and hence the estimate for this case is the same as the proof for the case where $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1, \alpha_2 \le 1$. We leave the details to the reader. The proof of Theorem 1.6 is concluded. ∎

4 Proof of Theorem 1.7

We first show that if $f \in \mathrm{Lip}^\alpha_{\mathrm{flag}}$ with $\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 > 0$, then there exists a sequence $\{f_n\}$ such that $f_n \in L^2 \cap \mathrm{Lip}^\alpha_{\mathrm{flag}}$ and f_n converges to f in the distribution sense. Moreover, $\|f_n\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}} \le C\|f\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}$, where the constant C is independent of f_n and f. To do this, note that, by Theorem 1.5, for each $f \in \mathrm{Lip}^\alpha_{\mathrm{flag}}$,

$$f(z,u) = \int_0^\infty \int_0^\infty \psi_{s,t} * \psi_{s,t} * f(z,u) \frac{ds}{s}\frac{dt}{t},$$

where the integral converges in $\mathcal{S}'_{\mathrm{flag}}(\mathbb{H}^n)$. Set

$$f_n = \int_{n^{-1}}^n \int_{n^{-1}}^n \psi_{s,t} * \psi_{s,t} * f(z,u) \frac{ds}{s}\frac{dt}{t}.$$

Obviously, $f_n \in L^2$ and converges to f in the distribution sense. To see that $f_n \in \mathrm{Lip}^\alpha_{\mathrm{com}}$, by Theorem 1.6,

$$\|f_n\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}} \le C \sup_{\substack{s,t>0\\(z,u)\in\mathbb{H}^n}} s^{-\alpha_1}t^{-\alpha_2}|\psi_{s,t} * f_n(z,u)|.$$

Note that

$$\psi_{s,t} * f_n(x) = \int_{n^{-1}}^n \int_{n^{-1}}^n \psi_{s,t} * \psi_{s',t'} * \psi_{s',t'} * f(z,u) \frac{ds'}{s'}\frac{dt'}{t'}.$$

By an estimate given in [14], that there exists a constant $C = C(M)$ depending only on M such that if $(s \vee s')^2 \le t \vee t'$, then

$$\big|\psi_{s,t} * \psi_{s',t'}(z,u)\big| \le C\Big(\frac{s}{s'} \wedge \frac{s'}{s}\Big)^{2M}\Big(\frac{t}{t'} \wedge \frac{t'}{t}\Big)^{M} \frac{(s \vee s')^{2M}}{(s \vee s' + |z|)^{2n+2M}} \frac{(t \vee t')^M}{(t \vee t' + |u|)^{1+M}},$$

and if $(s \vee s')^2 \ge t \vee t'$, then

$$\big|\psi_{s,t}\psi_{s',t'}(z,u)\big| \le C\Big(\frac{s}{s'} \wedge \frac{s'}{s}\Big)^{M}\Big(\frac{t}{t'} \wedge \frac{t'}{t}\Big)^{M} \frac{(s \vee s')^{M}}{(s \vee s' + |z|)^{2n+M}} \frac{(s \vee s')^M}{(s \vee s' + \sqrt{|u|})^{2+2M}}.$$

These estimates imply that if $M > \alpha_1 \vee \alpha_2$,

$$\begin{aligned} s^{-\alpha_1}t^{-\alpha_2}&|\psi_{s,t} * f_n(z,u)| \\ &\le Cs^{-\alpha_1}t^{-\alpha_2}\int_0^\infty\int_0^\infty \Big(\frac{s}{s'} \wedge \frac{s'}{s}\Big)^{M}\Big(\frac{t}{t'} \wedge \frac{t'}{t}\Big)^{M}(s')^{\alpha_1}(t')^{\alpha_2}\frac{ds'}{s'}\frac{dt'}{t'}\|f\|_{\mathrm{Lip}_{\mathrm{flag}}} \\ &\le C\|f\|_{\mathrm{Lip}_{\mathrm{flag}}}, \end{aligned}$$

which implies that $\|f_n\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}} \le C\|f\|_{\mathrm{Lip}^\alpha_{\mathrm{flag}}}$.

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Now we claim that if $f \in L^2$ and $T = K * f$ is a flag singular integral operator on $\mathbb{H}^n$ with a flag kernel K as given in [23], then

$$\|T(f)\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \lesssim \|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.$$

Indeed, by Theorem 1.6,

$$\|T(f)\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \lesssim \sup_{\substack{s,t>0\\(z,u)\in\mathbb{H}^n}} s^{-\alpha_1} t^{-\alpha_2} |\psi_{s,t} * T(f)(z,u)|.$$

Observe that, by a result in [23], T is bounded on $L^2(\mathbb{H}^n)$, and hence

$$\psi_{s,t} * T(f)(z,u) = \int_0^\infty \int_0^\infty \psi_{s,t} * K * \psi_{s',t'} * \psi_{s',t'} * f(z,u) \frac{ds'}{s'} \frac{dt'}{t'}.$$

Again, By a result in [23], $K(z,u) = \int_{\mathbb{R}} K^\sharp(z, u-v, v)dv$, where $K^\sharp(z,u,v)$, $(z,u) \in \mathbb{H}^n$, $v \in \mathbb{R}$, is a product singular integral kernel on $\mathbb{H}^n \times \mathbb{R}$. Note that

$$\psi_{s,t} * K * \psi_{s',t'}(z,u) = \int_{\mathbb{R}} \Psi_{s,t} *_{\mathbb{H}^n\times\mathbb{R}} K^\sharp *_{\mathbb{H}^n\times\mathbb{R}} \Psi_{s',t'}(z, u-v, v)dv,$$

where $\pi\Psi_{s,t} = \psi_{s,t}$.

Applying the classical almost orthogonal estimates with $\Psi_{s,t}, K^\sharp$ and $\Psi_{s',t'}$ on $\mathbb{H}^n \times \mathbb{R}$, we have that for any positive integer M,

$$|\Psi_{s,t} *_{\mathbb{H}^n\times\mathbb{R}} K^\sharp *_{\mathbb{H}^n\times\mathbb{R}} \Psi_{s',t'}\big((z,u),v\big)| \lesssim \\ C\Big(\frac{s}{s'} \wedge \frac{s'}{s}\Big)^{2M} \Big(\frac{t}{t'} \wedge \frac{t'}{t}\Big)^{M} \frac{(s\vee s')^{4M}}{((s\vee s')^2 + |z|^2 + |u|)^{n+1+2M}} \frac{(t\vee t')^{2M}}{(t\vee t' + |v|)^{1+2M}}.$$

Thus, we obtain that there exists a constant $C = C(M)$ depending only on M such that if $(s\vee s')^2 \le t \vee t'$, then

$$\big|\psi_{s,t} *_{\mathbb{H}^n} *K *_{\mathbb{H}^n} \psi_{s',t'}(z,u)\big| \le \\ C\Big(\frac{s}{s'} \wedge \frac{s'}{s}\Big)^{2M} \Big(\frac{t}{t'} \wedge \frac{t'}{t}\Big)^{M} \frac{(s\vee s')^{2M}}{(s\vee s' + |z|)^{2n+2M}} \frac{(t\vee t')^{M}}{(t\vee t' + |u|)^{1+M}},$$

and if $(s\vee s')^2 \ge t\vee t'$, then

$$\big|\psi_{s,t} *_{\mathbb{H}^n} *K *_{\mathbb{H}^n} \psi_{s',t'}(z,u)\big| \le \\ C\Big(\frac{s}{s'} \wedge \frac{s'}{s}\Big)^{M} \Big(\frac{t}{t'} \wedge \frac{t'}{t}\Big)^{M} \frac{(s\vee s')^{M}}{(s\vee s' + |z|)^{2n+M}} \frac{(s\vee s')^{M}}{(s\vee s' + \sqrt{|u|})^{2+2M}}.$$

These estimates imply that if $M > \alpha_1 \vee \alpha_2$,

$$s^{-\alpha_1} t^{-\alpha_2} |\psi_{s,t} * T(f)(z,u)| \\ \le C s^{-\alpha_1} t^{-\alpha_2} \int_0^\infty \int_0^\infty \Big(\frac{s}{s'} \wedge \frac{s'}{s}\Big)^{M} \Big(\frac{t}{t'} \wedge \frac{t'}{t}\Big)^{M} (s')^{\alpha_1} (t')^{\alpha_2} \frac{ds'}{s'} \frac{dt'}{t'} \|f\|_{\mathrm{Lip}_{\mathrm{flag}}} \\ \le C\|f\|_{\mathrm{Lip}_{\mathrm{flag}}},$$

which yields the proof of the claim.

We now extend T to $\mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ as follows. First, if $f \in \mathrm{Lip}^{\alpha}_{\mathrm{flag}}$, then, as mentioned above, there exists a sequence $\{f_n\}_{n\in\mathbb{Z}} \in L^2 \cap \mathrm{Lip}^{\alpha}_{\mathrm{flag}}$ such that f_n converges to f in the distribution sense and $\|f_n\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \le C\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}$. It follows from the claim that

$$\|T(f_n) - T(f_m)\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \le C\|f_n - f_m\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}},$$

and hence $T(f_n)$ converges in the distribution sense. We now define

$$T(f) = \lim_{n\to\infty} T(f_n)$$

in the distribution sense. We obtain, by Theorem 1.6 and the above claim,

$$\begin{aligned}\|T(f)\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} &\lesssim \sup_{\substack{s,t>0\\(z,u)\in\mathbb{H}^n}} s^{-\alpha_1}t^{-\alpha_2}|\psi_{s,t} * T(f)(z,u)| \\ &\lesssim \sup_{\substack{s,t>0\\(z,u)\in\mathbb{H}^n}} s^{-\alpha_1}t^{-\alpha_2}|\lim_{n\to\infty}\psi_{s,t} * T(f_n)(z,u)| \\ &\lesssim \liminf_{n\to\infty} \sup_{\substack{s,t>0\\(z,u)\in\mathbb{H}^n}} s^{-\alpha_1}t^{-\alpha_2}|\psi_{s,t} * T(f_n)(z,u)| \\ &\lesssim \liminf_{n\to\infty}\|f_n\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}} \lesssim \|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{flag}}}.\end{aligned}$$

The proof of Theorem 1.7 is concluded. ∎

References

[1] L. Carleson, *A counterexample for measures bounded on H^p for the bidisc.* Mittag-Leffler Report, 7, 1974.

[2] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces.* Geom. Funct. Anal. **9**(1999), no. 3, 428–517. http://dx.doi.org/10.1007/s000390050094

[3] S-Y. A. Chang, *Carleson measures on the bi-disc.* Ann. of Math. **109**(1979), 613–620. http://dx.doi.org/10.2307/1971229

[4] S-Y. A. Chang and R. Fefferman, *Some recent developments in Fourier analysis and H^p theory on product domains.* Bull. Amer. Math. Soc. **12**(1985), 1–43. http://dx.doi.org/10.1090/S0273-0979-1985-15291-7

[5] ———, *The Calderón–Zygmund decomposition on product domains.* Amer. J. Math. **104**(1982), 455–468. http://dx.doi.org/10.2307/2374150

[6] ———, *A continuous version of duality of H^1 with BMO on the bidisc.* Ann. of Math. **112**(1980), 179–201. http://dx.doi.org/10.2307/1971324

[7] C. Fefferman and E. M. Stein, *H^p spaces of several variables.* Acta Math. **129**(1972),137–193. http://dx.doi.org/10.1007/BF02392215

[8] R. Fefferman, *Multi-parameter Fourier analysis.* Beijing Lectures in Harmonic Analysis, Ann. of Math. Stud., 112, Princeton University Press, Princeton, NJ, 1986, pp. 47–130.

[9] ———, *Harmonic analysis on product spaces.* Ann. of Math. **126**(1987), 109–130. http://dx.doi.org/10.2307/1971346

[10] ———, *Multiparameter Calderón–Zygmund theory.* In: Harmonic analysis and partial differential equations (Chicago, IL, 1996), Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999, pp. 207–221.

[11] ———, *Singular integrals on product spaces.* Adv. Math. **45**(1982), 117–143. http://dx.doi.org/10.1016/S0001-8708(82)80001-7

[12] D. Geller and A. Mayeli, *Continuous wavelets and frames on stratified Lie groups. I.* J. Fourier Anal. Appl. **12**(2006), 543–579. http://dx.doi.org/10.1007/s00041-006-6002-4

[13] R. Gundy and E. M. Stein, *H^p theory for the polydisk.* Proc. Nat. Acad. Sci. **76**(1979), no. 3, 1026–1029. http://dx.doi.org/10.1073/pnas.76.3.1026

[14] Y. Han, G. Lu, and E. Sawyer, *Flag Hardy spaces and Marcinkiewicz multipliers on the Heisenberg group.* Anal. PDE **7**(2014), no. 7, 1465–1534. http://dx.doi.org/10.2140/apde.2014.7.1465

[15] E. Harboure, O. Salinas, and B. Viviani, *Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces.* Trans. Amer. Math. Soc. **349**(1997), no. 1, 235–255. http://dx.doi.org/10.1090/S0002-9947-97-01644-9

[16] S. Jansono, M. Taibleson, and G. Weiss, *Elementary characterizations of the Morrey-Campanato spaces.* In: Harmonic analysis (Cortona, 1982), Lecture Notes in Math., 992, Springer, Berlin, 1983, pp. 101–114.

[17] J. L. Journé, *Calderón–Zygmund operators on product spaces.* Rev. Mat. Iberoamericana **1**(1985), 55–91. http://dx.doi.org/10.4171/RMI/15

[18] ———, *A covering lemma for product spaces.* Proc. Amer. Math. Soc. **96**(1986), 593–598. http://dx.doi.org/10.1090/S0002-9939-1986-0826486-9

[19] ———, *Two problems of Calderón–Zygmund theory on product spaces.* Ann. Inst. Fourier(Grenoble), **38**(1988), 111–132. http://dx.doi.org/10.5802/aif.1125

[20] S. G. Krantz, *Geometric Lipschitz spaces and applications to complex function theory and nilpotent groups.* J. Funct. Anal. **34**(1979), no. 3, 456–471. http://dx.doi.org/10.1016/0022-1236(79)90087-9

[21] ———, *Lipschitz spaces on stratified groups.* Trans. Amer. Math. Soc. **269**(1982), no. 1, 39–66. http://dx.doi.org/10.1090/S0002-9947-1982-0637028-6

[22] W. R. Madych and N. M. Rivière, *Multiplies of the Hölder classes.* J. Funct. Anal. **21**(1976), no. 4, 369–379. http://dx.doi.org/10.1016/0022-1236(76)90032-X

[23] D. Müller, F. Ricci, and E. M. Stein, *Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups. I.* Invent. Math. **119**(1995), 119–233. http://dx.doi.org/10.1007/BF01245180

[24] ———, *Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups. II.* Math. Z. **221**(1996), 267–291. http://dx.doi.org/10.1007/BF02622116

[25] A. Nagel, F. Ricci, and E. M. Stein, *Singular integrals with flag kernels and analysis on quadratic CR manifolds.* J. Func. Anal. **181**(2001), 29–118. http://dx.doi.org/10.1006/jfan.2000.3714

[26] A. Nagel, F. Ricci, E. M. Stein, and S. Wainger, *Singular integrals with flag kernels on homogeneous groups. I.* Rev. Mat. Iberoam. **28**(2012), 631–722. http://dx.doi.org/10.4171/RMI/688

[27] ———, *Algebras of singular integral operators with kernels controlled by multiple norms.* arxiv:1511.05702

[28] D. H. Phong and E. M. Stein, *Some further classes of pseudo-differential and singular integral operators arising in boundary valve problems. I. composition of operators.* Amer. J. Math. **104**(1982), 141–172. http://dx.doi.org/10.2307/2374071

[29] J. Pipher, *Journé's covering lemma and its extension to higher dimensions.* Duke Math. J. **53**(1986), 683–690. http://dx.doi.org/10.1215/S0012-7094-86-05337-8

[30] E. M. Stein, *Singular integral and differentiability properties of functions.* Princeton Univ. Press **30**(1970).

[31] ———, *Singular integrals and estimates for the Cauchy-Riemann equations.* Bull. Amer. Math. Soc. **79**(1973), no. 2, 440–445. http://dx.doi.org/10.1090/S0002-9904-1973-13205-7

School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China
e-mail: 20051017@m.scnu.edu.cn

Department of Mathematics, Auburn University, Auburn, AL 36849, USA
e-mail: hanyong@auburn.edu

Department of Mathematics, Macquarie University, Sydney NSW 2109, Australia
e-mail: ji.li@mq.edu.au

Department of Mathematics, Shantou University, Shantou, Guangdong 515041, China
e-mail: cqtan@stu.edu.cn