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Gauss–Manin Connections for Arrangements, I Eigenvalues

DANIEL C. COHEN[1,★] and PETER ORLIK[2]
[1]*Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A. e-mail: cohen@math.lsu.edu*
[2]*Department of Mathematics, University of Wisconsin, Madison, WI 53706, U.S.A. e-mail: orlik@math.wisc.edu*

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Abstract. We construct a formal connection on the Aomoto complex of an arrangement of hyperplanes, and use it to study the Gauss–Manin connection for the moduli space of the arrangement in the cohomology of a complex rank one local system. We prove that the eigenvalues of the Gauss–Manin connection are integral linear combinations of the weights which define the local system.

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1. Introduction

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, with complement $M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{j=1}^n H_j$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a collection of weights. Associated to $\boldsymbol{\lambda}$, we have a rank one representation $\rho: \pi_1(M) \to \mathbb{C}^*$ given by $\gamma_j \mapsto t_j = \exp(-2\pi i \lambda_j)$ for any meridian loop γ_j about the hyperplane H_j of $\mathcal{A}$, and an associated rank one local system $\mathcal{L}$ on M. The need to calculate the local system cohomology $H^*(M; \mathcal{L})$ arises in various contexts. For instance, such local systems may be used to study the Milnor fiber of the non-isolated hypersurface singularity at the origin obtained by coning the arrangement, see [6, 7]. In mathematical physics, local systems on complements of arrangements arise in the Aomoto–Gelfand theory of multivariable hypergeometric integrals [2, 12, 18] and the representation theory of Lie algebras and quantum groups. These considerations lead to solutions of the Knizhnik–Zamolodchikov differential equation from conformal field theory, see [21, 24]. Here, a central problem is the determination of the Gauss–Manin connection on $H^*(M(\mathcal{A}); \mathcal{L})$ for certain arrangements, and local systems arising from certain weights. In this paper, we study the Gauss–Manin connection on $H^*(M(\mathcal{A}); \mathcal{L})$ for all arrangements, and local systems arising from arbitrary weights.

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The arrangements which arise in the context of the K-Z equations are the discriminantal arrangements of Schechtman and Varchenko [21]. For these arrangements, and certain weights, the Gauss–Manin connection on $H^*(M(\mathcal{A}); \mathcal{L})$ was determined by Aomoto [1] and Kaneko [15]. The monodromy corresponding to this connection is a representation of the fundamental group of the moduli space of all these arrangements, which is a (classical) configuration space, see [24]. Moduli spaces of arbitrary arrangements with a fixed combinatorial type were defined and investigated by Terao [23]. He identified the moduli spaces of certain arrangements, and determined the Gauss–Manin connection for certain weights. A priori, the eigenvalues of this connection are rational functions of the weights. Terao found that the eigenvalues are, in fact, integral linear combinations of the weights. He asked if this is always the case. In Theorem 4.4, we prove that the eigenvalues of the Gauss–Manin connection are indeed integral linear combinations of the weights for all arrangements and all weights.

Fix the combinatorial type of an arrangement $\mathcal{A}$, and let B be a smooth, connected component of the moduli space of arrangements of type $\mathcal{A}$. There is a fiber bundle $\pi\colon \mathsf{M} \to \mathsf{B}$ over B. The fibers of this bundle, $\pi^{-1}(\mathsf{b}) = M(\mathcal{A}_{\mathsf{b}})$, are complements of arrangements $\mathcal{A}_{\mathsf{b}}$ combinatorially equivalent to $\mathcal{A}$, so are diffeomorphic to $M(\mathcal{A})$ (since B is connected). Given weights $\boldsymbol{\lambda}$, we used stratified Morse theory in [5] to construct a complex which computes $H^*(M(\mathcal{A}); \mathcal{L})$. In fact, we constructed a universal complex $(K^\bullet_\Lambda(\mathcal{A}), \Delta^\bullet(\mathbf{x}))$ with the property that the specialization $x_j \mapsto t_j = \exp(-2\pi \mathrm{i} \lambda_j)$ calculates $H^*(M(\mathcal{A}); \mathcal{L})$. Here, $\mathbf{x} = (x_1, \dots, x_n)$ are the coordinate functions on the complex n-torus $(\mathbb{C}^*)^n$, and $\Lambda = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is the coordinate ring. This construction is reviewed in Section 2.

At $\mathsf{b} \in \mathsf{B}$, we have the corresponding universal complex $(K^\bullet_\Lambda(\mathcal{A}_{\mathsf{b}}), \Delta^\bullet(\mathbf{x}))$, its specialization $(K^\bullet(\mathcal{A}_{\mathsf{b}}), \Delta^\bullet(\mathbf{t}))$ and the cohomology of the latter. Loops in B based at b induce automorphisms of all these objects and consequently yield representations of $\pi_1(\mathsf{B}, \mathsf{b})$. In particular, there is a *universal representation* $\pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_\Lambda K^\bullet_\Lambda(\mathcal{A}_{\mathsf{b}})$. Let $\mathbf{y} = (y_1, \dots, y_n)$ be the coordinates of $\mathrm{T}_\mathbf{1}(\mathbb{C}^*)^n = \mathbb{C}^n$, the holomorphic tangent space of $(\mathbb{C}^*)^n$ at the identity element $\mathbf{1} = (1, \dots, 1) \in (\mathbb{C}^*)^n$. The exponential map $\mathrm{T}_\mathbf{1}(\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is induced by $\exp\colon \mathbb{C} \to \mathbb{C}^*$, $y_j \mapsto e^{y_j} = x_j$. We call the formal logarithm associated to the universal representation the *formal connection*.

Since the complex $(K^\bullet(\mathcal{A}_{\mathsf{b}}), \Delta^\bullet(\mathbf{t}))$ computes the cohomology of the local system $\mathcal{L}$ on $M(\mathcal{A}_{\mathsf{b}})$ corresponding to the weights $\boldsymbol{\lambda}$, the representation $\pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_\mathbb{C} H^*(M(\mathcal{A}_{\mathsf{b}}); \mathcal{L})$ is induced by the representation $\pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_\mathbb{C} K^\bullet(\mathcal{A}_{\mathsf{b}})$. We realize the latter as the specialization at $\mathbf{t}$ of the universal representation. Similarly, the specialization $y_j \mapsto \lambda_j$ of formal connection induces the Gauss–Manin connection on the local system cohomology. Given a loop $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, we show in Section 3 that the eigenvalues of the corresponding universal representation matrix are monomials in the x_j with integer exponents. In Section 4, we show that the eigenvalues of the corresponding formal connection matrix are linear forms in the y_j with integer coefficients. It follows that the eigenvalues of the corresponding Gauss–Manin connection matrix in local system cohomology are integral linear combinations of the weights, answering Terao's question affirmatively for all arrangements and all weights.

The formal connection may be viewed as a connection on a combinatorial object, the Aomoto complex. We use the notation and results of [17, 18]. Let $A = A(\mathcal{A})$ be the Orlik–Solomon algebra of $\mathcal{A}$ generated by the one-dimensional classes a_j, $1 \leqslant j \leqslant n$. It is the quotient of the exterior algebra generated by these classes by a homogeneous ideal, hence is a finite-dimensional graded $\mathbb{C}$-algebra. There is an isomorphism of graded algebras $H^*(M; \mathbb{C}) \simeq A(\mathcal{A})$. For weights $\boldsymbol{\lambda}$, the Orlik–Solomon algebra is a cochain complex with differential given by multiplication by $a_{\lambda} = \sum_{j=1}^{n} \lambda_j a_j$. The Aomoto complex $(A^{\bullet}_R(\mathcal{A}), a_{\mathbf{y}}\wedge)$ is a universal complex with the property that the specialization $y_j \mapsto \lambda_j$ calculates $H^*(A^{\bullet}, a_{\lambda}\wedge)$. Here, $R = \mathbb{C}[y_1, \ldots, y_n]$ is the coordinate ring of $\mathbb{C}^n$, the holomorphic tangent space of $(\mathbb{C}^*)^n$ at $\mathbf{1}$. In [5, Thm. 2.13], we showed that the Aomoto complex $(A^{\bullet}_R(\mathcal{A}), a_{\mathbf{y}}\wedge)$ is chain equivalent to the linearization of the universal complex $(K^{\bullet}_{\Lambda}(\mathcal{A}), \Delta^{\bullet}(\mathbf{x}))$.

Call a system of weights $\boldsymbol{\lambda}$ or the corresponding local system $\mathcal{L}$ *combinatorial* if local system cohomology is quasi-isomorphic to Orlik–Solomon algebra cohomology,

$$H^*(M(\mathcal{A}); \mathcal{L}) \simeq H^*(A^{\bullet}(\mathcal{A}), a_{\lambda}\wedge).$$

The set of combinatorial weights is open and dense in $\mathbb{C}^n$. See [10, 20] for sufficient conditions. For combinatorial weights, the Gauss–Manin connection in local system cohomology coincides with that in the cohomology of the Orlik–Solomon algebra. Thus, if the eigenvalues of the former are integer linear combinations of the weights, then so are those of the latter. In Section 5, we show that, in fact, the eigenvalues of the combinatorial Gauss–Manin connection in Orlik–Solomon algebra cohomology are integer linear combinations of the weights for all weights. Since the Aomoto complex $A^{\bullet}_R(\mathcal{A})$ is the linearization of the universal complex $K^{\bullet}_{\Lambda}(\mathcal{A})$, results on the universal representation on $K^{\bullet}_{\Lambda}(\mathcal{A})$ inform on the formal connection on $A^{\bullet}_R(\mathcal{A})$, and its specializations, for arbitrary weights.

Call a system of weights $\boldsymbol{\lambda}$ or the corresponding local system $\mathcal{L}$ *nonresonant* if the Betti numbers of M with coefficients in $\mathcal{L}$ are minimal. The set of nonresonant weights is open and dense in $\mathbb{C}^n$, but does not coincide with the set of combinatorial weights. The cohomology of nonresonant local systems is known. A detailed account, including sufficient conditions, is found in [18]. For nonresonant weights we have

$$H^q(M; \mathcal{L}) = 0 \text{ for } q \neq \ell, \quad \text{and} \quad \dim H^{\ell}(M; \mathcal{L}) = |e(M)|, \tag{1.1}$$

where $e(M)$ is the Euler characteristic, see [10, 20, 25]. If the weights are both combinatorial and nonresonant, the Gauss–Manin connection may be studied effectively by combinatorial means. In particular, explicit bases for the single nonvanishing cohomology group are known, see [11].

Several authors have studied Gauss–Manin connections using such a basis. See Aomoto [1] and Kaneko [15] for discriminantal arrangements, and Kanarek [14] for the connection arising when a single hyperplane in the arrangement is allowed to move. For general position arrangements, the Gauss–Manin connection matrices were computed by Aomoto and Kita [2]. These connection matrices were obtained by Terao [23] for a larger class of arrangements. Like the eigenvalues, the entries of these matrices were known to be rational functions of the weights. Aomoto and Kita, and Terao found that these entries were, in fact, integer linear combinations of the weights, and Terao asked if this is the case in general. Our work here was motivated by these results and this question.

2. Cohomology Complexes

For an arbitrary complex local system $\mathcal{L}$ on the complement of an arrangement $\mathcal{A}$, we used stratified Morse theory in [4] to construct a complex $(K^\bullet(\mathcal{A}), \Delta^\bullet)$, the cohomology of which is naturally isomorphic to $H^*(M; \mathcal{L})$, the cohomology of M with coefficients in $\mathcal{L}$. We now recall this construction in the context of rank one local systems, and record several related complexes and relevant results from [4, 5].

Choose coordinates $\mathbf{u} = (u_1, \ldots, u_\ell)$ on $\mathbb{C}^\ell$, and let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, with complement $M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{j=1}^n H_j$. We assume throughout that $\mathcal{A}$ contains ℓ linearly independent hyperplanes. For each j, let f_j be a linear polynomial with $H_j = \{\mathbf{u} \in \mathbb{C}^\ell \mid f_j(\mathbf{u}) = 0\}$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a system of weights. Associated to $\boldsymbol{\lambda}$, we have

(1) a flat connection on the trivial line bundle over M, with connection form $\nabla = d + \omega_{\boldsymbol{\lambda}} \wedge \colon \Omega^0 \to \Omega^1$, where d is the exterior differential operator with respect to the coordintes $\mathbf{u}$, $\omega_{\boldsymbol{\lambda}} = \sum_{j=1}^n \lambda_j \, d\log(f_j)$, and Ω^q is the sheaf of germs of holomorphic differential forms of degree q on M;
(2) a rank one representation $\rho\colon \pi_1(M) \to \mathbb{C}^*$, given by $\rho(\gamma_j) = t_j$, where $\mathbf{t} = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ is defined by $t_j = \exp(-2\pi \mathrm{i} \lambda_j)$, and γ_j is any meridian loop about the hyperplane H_j of $\mathcal{A}$; and
(3) a rank one local system $\mathcal{L} = \mathcal{L}_{\mathbf{t}} = \mathcal{L}_{\boldsymbol{\lambda}}$ on M associated to the representation ρ (resp., the flat connection ∇).

Note that weights $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ yield identical representations and local systems if $\boldsymbol{\lambda} - \boldsymbol{\lambda}' \in \mathbb{Z}^n$.

Remark 2.1. The arrangement $\mathcal{A}$ determines a Whitney stratification of $\mathbb{C}^\ell$, with codimension zero stratum given by the complement M. To describe the strata of higher codimension, recall that an edge of $\mathcal{A}$ is a nonempty intersection of hyperplanes. Associated to each codimension p edge X, there is a stratum $S_X = X \setminus \bigcup Y$, where the union is over all edges Y of $\mathcal{A}$ which satisfy $Y \subsetneq X$. Note that $S_X = M(\mathcal{A}^X)$ may be realized as the complement of the arrangement $\mathcal{A}^X$ in X, see [17].

Let $\mathcal{F}$ be a complete flag (of affine subspaces) in $\mathbb{C}^\ell$,

$$\mathcal{F}: \quad \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots \subset \mathcal{F}^\ell = \mathbb{C}^\ell, \tag{2.1}$$

transverse to the stratification determined by $\mathcal{A}$, so that $\dim \mathcal{F}^q \cap S_X = q - \operatorname{codim} S_X$ for each stratum, where a negative dimension indicates that $\mathcal{F}^q \cap S_X = \emptyset$. For an

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explicit construction of such a flag, see [4, Section 1]. Let $M^q = \mathcal{F}^q \cap M$ for each q. Let $K^q = H^q(M^q, M^{q-1}; \mathcal{L})$, and denote by Δ^q the boundary homomorphism $H^q(M^q, M^{q-1}; \mathcal{L}) \to H^{q+1}(M^{q+1}, M^q; \mathcal{L})$ of the triple (M^{q+1}, M^q, M^{q-1}). The following compiles several results from [4].

THEOREM 2.2. *Let $\mathcal{L}$ be a complex rank one local system on the complement M of an arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$.*

(1) *For each q, $0 \leqslant q \leqslant \ell$, we have $H^i(M^q, M^{q-1}; \mathcal{L}) = 0$ if $i \neq q$, and $\dim_{\mathbb{C}} H^q(M^q, M^{q-1}; \mathcal{L}) = b_q(\mathcal{A})$ is equal to the qth Betti number of M with trivial local coefficients $\mathbb{C}$.*
(2) *The system of complex vector spaces and linear maps $(K^\bullet, \Delta^\bullet)$,*

$$K^0 \xrightarrow{\Delta^0} K^1 \xrightarrow{\Delta^1} K^2 \longrightarrow \cdots \longrightarrow K^{\ell-1} \xrightarrow{\Delta^{\ell-1}} K^\ell,$$

is a complex $(\Delta^{q+1} \circ \Delta^q = 0)$. The cohomology of this complex is naturally isomorphic to $H^*(M; \mathcal{L})$, the cohomology of M with coefficients in $\mathcal{L}$.

The dimensions of the terms, K^q, of the complex $(K^\bullet, \Delta^\bullet)$ are independent of $\mathbf{t}$ (resp., λ, $\mathcal{L}$). Write $\Delta^\bullet = \Delta^\bullet(\mathbf{t})$ to indicate the dependence of the complex on $\mathbf{t}$, and view these boundary maps as functions of $\mathbf{t}$. Let $\Lambda = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the ring of complex Laurent polynomials in n commuting variables, and for each q, let $K_\Lambda^q = \Lambda \otimes_{\mathbb{C}} K^q$.

THEOREM 2.3 ([5, Thm. 2.9]). *For an arrangement $\mathcal{A}$ of n hyperplanes with complement M, there exists a universal complex $(K_\Lambda^\bullet, \Delta^\bullet(\mathbf{x}))$ with the following properties:*

(1) *The terms are free Λ-modules, whose ranks are given by the Betti numbers of M, $K_\Lambda^q \simeq \Lambda^{b_q(\mathcal{A})}$.*
(2) *The boundary maps, $\Delta^q(\mathbf{x})\colon K_\Lambda^q \to K_\Lambda^{q+1}$ are Λ-linear.*
(3) *For each $\mathbf{t} \in (\mathbb{C}^*)^n$, the specialization $\mathbf{x} \mapsto \mathbf{t}$ yields the complex $(K^\bullet, \Delta^\bullet(\mathbf{t}))$, the cohomology of which is isomorphic to $H^*(M; \mathcal{L}_{\mathbf{t}})$, the cohomology of M with coefficients in the local system associated to $\mathbf{t}$.*

The entries of the boundary maps $\Delta^q(\mathbf{x})$ are elements of the Laurent polynomial ring Λ, the coordinate ring of the complex algebraic n-torus. Via the specialization $\mathbf{x} \mapsto \mathbf{t} \in (\mathbb{C}^*)^n$, we view them as holomorphic functions $(\mathbb{C}^*)^n \to \mathbb{C}$. Similarly, for each q, we view $\Delta^q(\mathbf{x})$ as a holomorphic map $\Delta^q\colon (\mathbb{C}^*)^n \to \mathrm{Mat}(\mathbb{C})$, $\mathbf{t} \mapsto \Delta^q(\mathbf{t})$.

Remark 2.4. If $\mathbf{t} = \mathbf{1}$ is the identity element of $(\mathbb{C}^*)^n$, the associated local system $\mathcal{L}_{\mathbf{1}}$ is trivial. Consequently, the specialization $\mathbf{x} \mapsto \mathbf{1}$ yields a complex $(K^\bullet, \Delta^\bullet(\mathbf{1}))$ whose cohomology gives $H^*(M; \mathbb{C})$. Since $\dim K^q = b_q(\mathcal{A}) = \dim H^q(M; \mathbb{C})$, the boundary maps of this complex are necessarily trivial, $\Delta^q(\mathbf{1}) = 0$ for each q.

There is an analogous universal complex for the cohomology, $H^*(A^\bullet, a_\lambda\wedge)$, of the Orlik–Solomon algebra $A = A(\mathcal{A})$. Let $R = \mathbb{C}[y_1, \dots, y_n]$ be the polynomial ring. The *Aomoto complex* $(A^\bullet_R, a_{\mathbf{y}}\wedge)$ has terms $A^q_R = R \otimes_{\mathbb{C}} A^q \simeq R^{b_q(\mathcal{A})}$, and boundary maps given by $p(\mathbf{y}) \otimes \eta \mapsto \sum y_j p(\mathbf{y}) \otimes a_j \wedge \eta$. For $\boldsymbol{\lambda} \in \mathbb{C}^n$, the specialization $\mathbf{y} \mapsto \lambda$ of the Aomoto complex $(A^\bullet_R, a_{\mathbf{y}}\wedge)$ yields the Orlik–Solomon algebra complex $(A^\bullet, a_\lambda\wedge)$.

A choice of basis for the Orlik–Solomon algebra of $\mathcal{A}$ yields a basis for each term A^q_R of the Aomoto complex. Let $\mu^q(\mathbf{y})$ denote the matrix of $a_{\mathbf{y}} \wedge : A^q_R \to A^{q+1}_R$ with respect to a fixed basis. The following results were established in [5].

THEOREM 2.5. (1) *For each q, the entries of $\mu^q(\mathbf{y})$ are integral linear forms in $y_1, \dots, y_n$.*

(2) *For any arrangement $\mathcal{A}$, the Aomoto complex $(A^\bullet_R, \mu^\bullet(\mathbf{y}))$ is chain equivalent to the linearization of the universal complex $(K^\bullet_\Lambda, \Delta^\bullet(\mathbf{x}))$.*

3. Representations

Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^\ell$ as above, and let B be a smooth, connected component of the moduli space of arrangements with the combinatorial type of $\mathcal{A}$. This moduli space is a locally closed subspace of $(\mathbb{CP}^\ell)^n$. We refer to [18, 23] for the precise definition of this moduli space, and to Section 6 for an example. In this section, we extend the constructions of the previous section to produce representations of the fundamental group of B related to the cohomology of the complement of $\mathcal{A}$ with coefficients in a rank one local system.

Denote the coordinates on $(\mathbb{CP}^\ell)^n$ by $\mathbf{z} = (\mathbf{z}^1, \dots, \mathbf{z}^n)$, where $\mathbf{z}^i = (z^i_0 : \cdots : z^i_\ell)$, and recall that the coordinates on $\mathbb{C}^\ell$ are denoted by $\mathbf{u} = (u_1, \dots, u_\ell)$. There is a fiber bundle $\pi\colon \mathsf{M} \to \mathsf{B}$, see [23, Section 3]. The total space may be described as

$$\mathsf{M} = \{(\mathbf{z}, \mathbf{u}) \in (\mathbb{CP}^\ell)^n \times \mathbb{C}^\ell \mid \mathbf{z} \in \mathsf{B} \text{ and } \mathbf{u} \in \pi^{-1}(\mathbf{z})\},$$

and the projection is given by $\pi(\mathbf{z}, \mathbf{u}) = \mathbf{z}$. For $\mathsf{b} \in \mathsf{B}$, the fiber $\mathsf{M_b} = \pi^{-1}(\mathsf{b})$ is the complement, $\mathsf{M_b} = M(\mathcal{A}_\mathsf{b})$, of the arrangement $\mathcal{A}_\mathsf{b}$ combinatorially equivalent to $\mathcal{A}$. The closure, $\bar{\mathsf{M}}_\mathsf{b}$, of the fiber is homeomorphic to $\mathbb{C}^\ell$, and admits a Whitney stratification determined by the arrangement $\mathcal{A}_\mathsf{b}$ as in Remark 2.1. Let $\mathcal{F}_\mathsf{b}$ be a flag in $\bar{\mathsf{M}}_\mathsf{b}$ that is transverse to $\mathcal{A}_\mathsf{b}$ as in (2.1). Evidently, these flags may be chosen to vary smoothly with b.

Recall that the hyperplanes of $\mathcal{A}$ are defined by linear polynomials $f_j = f_j(\mathbf{u})$. Since B is by assumption connected, for every $\mathsf{b} \in \mathsf{B}$, the arrangement $\mathcal{A}_\mathsf{b}$ is lattice-isotopic to $\mathcal{A}$ in the sense of Randell [19]. Consequently, there are smooth functions $f_j(\mathbf{z}, \mathbf{u})$ on M so that, for each $\mathsf{b} \in \mathsf{B}$, the hyperplanes of $\mathcal{A}_\mathsf{b}$ are defined by $f_j(\mathsf{b}, \mathbf{u})$.

Given $\mathbf{t} \in (\mathbb{C}^*)^n$ (or weights $\boldsymbol{\lambda} \in \mathbb{C}^n$) and $\mathsf{b} \in \mathsf{B}$, denote the corresponding local system on $\mathsf{M_b}$ by $\mathcal{L}(\mathsf{b})$. In this context, the construction of the previous section yields vector bundles $\mathbf{K}^q$ over B for $0 \leqslant q \leqslant \ell$ as follows. For $\mathsf{b} \in \mathsf{B}$, let $\mathsf{M}^q_\mathsf{b} = \mathcal{F}^q_\mathsf{b} \cap \mathsf{M_b}$ and $K^q(\mathsf{b}) = H^q(\mathsf{M}^q_\mathsf{b}, \mathsf{M}^{q-1}_\mathsf{b}; \mathcal{L}(\mathsf{b}))$. Since $\pi\colon \mathsf{M} \to \mathsf{B}$ is locally trivial, the natural projection $\pi^q\colon \mathbf{K}^q \to \mathsf{B}$, where $\mathbf{K}^q = \bigcup_{\mathsf{b} \in \mathsf{B}} K^q(\mathsf{b})$, is a vector bundle. The transition functions of this vector bundle are locally constant.

If $\gamma: I \to \mathsf{B}$ is a path, then the induced bundle $\gamma^*(\mathbf{K}^q)$ is trivial. Consequently there is a canonical linear isomorphism $K^q(\gamma(0)) \to K^q(\gamma(1))$, from the fiber over the initial point of γ to that over the terminal point, which depends only on the homotopy class of the path. Fix a basepoint $\mathsf{b} \in \mathsf{B}$, and write $K^\bullet = K^\bullet(\mathsf{b})$. The operation of parallel translation of fibers over curves in B in the vector bundle $\pi^q: \mathbf{K}^q \to \mathsf{B}$ provides a complex representation of rank $b_q(\mathcal{A})$,

$$\Phi^q: \pi_1(\mathsf{B}, \mathsf{b}) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(K^q). \tag{3.1}$$

To indicate the dependence of the representation Φ^q on $\mathbf{t} \in (\mathbb{C}^*)^n$, write $\Phi^q = \Phi^q(\mathbf{t})$.

THEOREM 3.1. *If* $\mathbf{t} = \mathbf{1}$ *is the identity element of* $(\mathbb{C}^*)^n$, *then the corresponding representation* $\Phi^q(\mathbf{1})$ *is trivial for each* q. *That is, for every* $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, *we have* $\Phi^q(\mathbf{1})(\gamma) = \mathrm{id}: K^q \to K^q$ *for each* q, $0 \leqslant q \leqslant \ell$.

Proof. For the trivial local system $\mathcal{L}(\mathsf{b}) = \mathbb{C}$ associated to $\mathbf{t} = \mathbf{1}$, the long exact cohomology sequence of the pair $(\mathsf{M}_\mathsf{b}^q, \mathsf{M}_\mathsf{b}^{q-1})$ splits into short exact sequences

$$0 \to H^i(\mathsf{M}_\mathsf{b}^q, \mathsf{M}_\mathsf{b}^{q-1}; \mathbb{C}) \to H^i(\mathsf{M}_\mathsf{b}^q; \mathbb{C}) \to H^i(\mathsf{M}_\mathsf{b}^{q-1}; \mathbb{C}) \to 0,$$

see [13, III.3] and [4, Rem. 5.4]. In particular, the qth relative cohomology group K^q is canonically isomorphic to $H^q(\mathsf{M}_\mathsf{b}^q; \mathbb{C}) = H^q(\mathsf{M}_\mathsf{b}; \mathbb{C})$, the qth cohomology of M_b with constant coefficients $\mathbb{C}$, see Remark 2.4. So it suffices to show that the fundamental group of B acts trivially on $H^q(\mathsf{M}_\mathsf{b}; \mathbb{C})$.

Let $i_\mathsf{b}: \mathsf{M}_\mathsf{b} \to \mathsf{M}$ denote the inclusion of the fiber in the total space of the bundle $\pi: \mathsf{M} \to \mathsf{B}$. It is known [22] that the image, $i_\mathsf{b}^* H^*(\mathsf{M}; \mathbb{C}) \subseteq H^*(\mathsf{M}_\mathsf{b}; \mathbb{C})$, of the cohomology of the total space M with (trivial) coefficients in the field $\mathbb{C}$ is invariant under the action of $\pi_1(\mathsf{B}, \mathsf{b})$. Consider the logarithmic forms on M defined by

$$\omega_j(\mathbf{z}, \mathbf{u}) = \frac{d_\mathbf{z} f_j(\mathbf{z}, \mathbf{u}) + d_\mathbf{u} f_j(\mathbf{z}, \mathbf{u})}{f_j(\mathbf{z}, \mathbf{u})}, \quad \text{where} \quad d_\mathbf{y} g = \sum_{i=1}^k \frac{\partial g}{\partial y_i}\, \mathrm{d}y_i$$

denotes the gradient of g with respect to the variables $\mathbf{y} = (y_1, \ldots, y_k)$. Clearly these forms represent nontrivial classes in $H^*(\mathsf{M}; \mathbb{C})$. Furthermore, we have

$$i_\mathsf{b}^* \omega_j(\mathbf{z}, \mathbf{u}) = \omega_j(\mathsf{b}, \mathbf{u}) = \frac{d_\mathbf{z} f_j(\mathsf{b}, \mathbf{u}) + d_\mathbf{u} f_j(\mathsf{b}, \mathbf{u})}{f_j(\mathsf{b}, \mathbf{u})} = \frac{d_\mathbf{u} f_j(\mathsf{b}, \mathbf{u})}{f_j(\mathsf{b}, \mathbf{u})} = d_\mathbf{u} \log(f_j(\mathsf{b}, \mathbf{u})).$$

As is well known, the forms $\omega_j(\mathsf{b}, \mathbf{u})$ generate the cohomology ring of $\mathsf{M}_\mathsf{b} = M(\mathcal{A}_\mathsf{b})$. It follows that M_b is totally nonhomologous to zero in M with respect to $\mathbb{C}$: The inclusion $i_\mathsf{b}: \mathsf{M}_\mathsf{b} \to \mathsf{M}$ induces a surjection $i_\mathsf{b}^*: H^*(\mathsf{M}; \mathbb{C}) \to H^*(\mathsf{M}_\mathsf{b}; \mathbb{C})$ in cohomology with trivial coefficients $\mathbb{C}$. Consequently, the fundamental group $\pi_1(\mathsf{B}, \mathsf{b})$ acts trivially on the $H^q(\mathsf{M}_\mathsf{b}; \mathbb{C})$ for each q, and the representation $\Phi^q(\mathbf{1})$ is trivial. □

Denote the boundary homomorphism of the triple $(\mathsf{M}_\mathsf{b}^{q+1}, \mathsf{M}_\mathsf{b}^q, \mathsf{M}_\mathsf{b}^{q-1})$ in cohomology with local coefficients $\mathcal{L}(\mathsf{b})$ determined by $\mathbf{t}$ by $\Delta^q(\mathbf{t}) = \Delta_\mathsf{b}^q(\mathbf{t}): K^q \to K^{q+1}$.

COROLLARY 3.2. *For each* $\mathbf{t} \in (\mathbb{C}^*)^n$ *and each* $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, *the automorphisms* $\Phi^q(\mathbf{t})(\gamma): K^q \to K^q$, $0 \leqslant q \leqslant \ell$, *comprise a chain automorphism* $\Phi^\bullet(\mathbf{t})(\gamma)$ *of the complex* $(K^\bullet, \Delta^\bullet(\mathbf{t}))$.

Proof. By Theorem 3.1, the result holds at $\mathbf{t} = \mathbf{1}$. Therefore it holds for $\mathbf{t}$ close to $\mathbf{1}$. The result follows. □

We abbreviate the above result by writing $\Phi^\bullet(\mathbf{t}): \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_{\mathbb{C}}(K^\bullet)$. For $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, the automorphism $\Phi^q(\gamma) = \Phi^q(\mathbf{t})(\gamma)$ may be viewed as a holomorphic function of $\mathbf{t}$:

$$\Phi^q(\gamma): (\mathbb{C}^*)^n \to \mathrm{Aut}_{\mathbb{C}}(K^q), \quad \mathbf{t} \mapsto \Phi^q(\mathbf{t})(\gamma).$$

Recall that $\Lambda = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $(K_\Lambda^\bullet, \Delta^\bullet(\mathbf{x}))$ be the universal complex of the arrangement $\mathcal{A}_\mathsf{b}$ from Theorem 2.3. By the continuity of the functions $\Phi^q(\gamma)$, we have the following extension of this result.

THEOREM 3.3. *For each* $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, *there is a chain map* $\Phi^\bullet(\mathbf{x})(\gamma): K_\Lambda^\bullet \to K_\Lambda^\bullet$ *so that the specialization* $\mathbf{x} \mapsto \mathbf{t}$ *yields the chain automorphism* $\Phi^\bullet(\mathbf{t})(\gamma)$ *of the complex* $(K^\bullet, \Delta^\bullet(\mathbf{t}))$. *This provides a representation* $\Phi^\bullet(\mathbf{x}): \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{End}_\Lambda(K_\Lambda^\bullet)$ *which specializes to the representation* $\Phi^\bullet(\mathbf{t}): \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_{\mathbb{C}}(K^\bullet)$.

Call $\Phi^\bullet(\mathbf{x})$: $\pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{End}_\Lambda(K_\Lambda^\bullet)$ the *universal representation*.

THEOREM 3.4. *For each* q *and each* $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, *the eigenvalues of* $\Phi^q(\mathbf{x})(\gamma)$ *are monomials functions of the form* $r(\mathbf{x}) = x_1^{m_1} \cdots x_n^{m_n}$, *where* $m_j \in \mathbb{Z}$.

Proof. Given q and $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, let $r(\mathbf{x})$ be an eigenvalue of $\Phi^q(\mathbf{x})(\gamma)$. Then $r(\mathbf{t})$ is an eigenvalue of $\Phi^q(\mathbf{t})(\gamma) \in \mathrm{Aut}_{\mathbb{C}}(K^q) \simeq \mathrm{GL}(b_q(\mathcal{A}), \mathbb{C})$ for every $\mathbf{t} \in (\mathbb{C}^*)^n$. It follows that the function $r: (\mathbb{C}^*)^n \to \mathbb{C}$, $\mathbf{t} \mapsto r(\mathbf{t})$ is single-valued and has no poles. Thus, $r(\mathbf{x})$ is a Laurent polynomial in $x_1, \ldots, x_n$. Write $r(\mathbf{x}) = x_1^{m_1} \cdots x_n^{m_n} \cdot p(\mathbf{x})$, where $p(\mathbf{x})$ is a polynomial. Since $\Phi^q(\mathbf{t})(\gamma)$ is an automorphism for every $\mathbf{t} \in (\mathbb{C}^*)^n$, we have $p(\mathbf{t}) \neq 0$ for all $\mathbf{t}$. Thus, $p(\mathbf{x}) = c \in \mathbb{C}^*$ is a nonzero constant, and $r(\mathbf{x}) = c \cdot x_1^{m_1} \cdots x_n^{m_n}$ is a unit in Λ. Using Theorem 3.1, we have $c = 1$. □

Thus for every $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, the maps $\Phi^q(\mathbf{x})(\gamma)$ are automorphisms, so we write $\Phi^\bullet(\mathbf{x}): \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_\Lambda(K_\Lambda^\bullet)$.

COROLLARY 3.5. *For each* $\mathbf{t} \in (\mathbb{C}^*)^n$, *the eigenvalues of the automorphism* $\Phi^q(\mathbf{t})(\gamma)$ *are evaluations* $r(\mathbf{t})$ *of the monomial functions* $r(\mathbf{x})$.

Given $\mathbf{t} \in (\mathbb{C}^*)^n$ with associated local system $\mathcal{L}(\mathsf{b})$ on M_b, there are also vector bundles $\mathbf{H}^q \to \mathsf{B}$ over the moduli space, defined by $\mathbf{H}^q = \bigcup_{\mathsf{b} \in \mathsf{B}} H^q(\mathsf{M}_\mathsf{b}; \mathcal{L}(\mathsf{b}))$ for each q. As above, parallel translation of the fibers in this bundle over curves in the base gives rise to a representation

$$\Psi^q = \Psi^q(\mathbf{t}): \pi_1(\mathsf{B}, \mathsf{b}) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(H^q(\mathsf{M}_\mathsf{b}; \mathcal{L}(\mathsf{b}))).$$

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By Theorem 2.2.2, the cohomology of the complex $(K^{\bullet}, \Delta^{\bullet}(\mathbf{t}))$ is naturally isomorphic to $H^{*}(\mathsf{M}_{\mathsf{b}}; \mathcal{L}(\mathsf{b}))$. Furthermore, parallel translation induces the representation $\Phi^{\bullet}(\mathbf{t}): \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_{\mathbb{C}}(K^{\bullet})$ of (3.1) on this complex. Thus by functoriality, we have the following theorem:

THEOREM 3.6. *The cohomology representation* $\Psi^{q}(\mathbf{t}) = \Phi^{q}(\mathbf{t})^{*}$ *is induced by* $\Phi^{\bullet}(\mathbf{t})$. *In other words, for each* $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, *the automorphism* $\Psi^{\bullet}(\mathbf{t})(\gamma)$ *in cohomology is induced by the automorphism* $\Phi^{\bullet}(\mathbf{t})(\gamma)$ *of the complex* $(K^{\bullet}, \Delta^{\bullet}(\mathbf{t}))$.

COROLLARY 3.7. *For each* $\mathbf{t} \in (\mathbb{C}^{*})^{n}$, *the eigenvalues of the automorphism* $\Psi^{q}(\mathbf{t})(\gamma)$ *are evaluations of monomial functions.*

4. Connections

The vector bundles $\mathbf{K}^{q} \to \mathsf{B}$ and $\mathbf{H}^{q} \to \mathsf{B}$ over the moduli space constructed above support Gauss–Manin connections corresponding to the representations $\Phi^{q}(\mathbf{t})$ and $\Psi^{q}(\mathbf{t})$ of the fundamental group of B. We now study these connections in light of the results of the previous section.

Over a manifold such as B, there is a well known equivalence between local systems and complex vector bundles equipped with flat connections, see [9, 16]. Let $\mathbf{V} \to \mathsf{B}$ be such a bundle, with connection ∇. The latter is a $\mathbb{C}$-linear map $\nabla: \Omega^{0}(\mathbf{V}) \to \Omega^{1}(\mathbf{V})$, where $\Omega^{p}(\mathbf{V})$ denotes the complex p-forms on B with values in $\mathbf{V}$, which satisfies $\nabla(f\sigma) = \sigma \mathrm{d}f + f\nabla(\sigma)$ for a function f and $\sigma \in \Omega^{0}(\mathbf{V})$. The connection extends to a map $\nabla: \Omega^{p}(\mathbf{V}) \to \Omega^{p+1}(\mathbf{V})$ for $p \geqslant 0$, and is flat if the curvature $\nabla \circ \nabla$ vanishes. Call two connections ∇ and ∇' on $\mathbf{V}$ isomorphic if ∇' is obtained from ∇ by a gauge transformation, $\nabla' = g \circ \nabla \circ g^{-1}$ for some $g: \mathsf{B} \to \mathrm{Hom}(\mathbf{V}, \mathbf{V})$.

The aforementioned equivalence is given by $(\mathbf{V}, \nabla) \mapsto \mathbf{V}^{\nabla}$, where $\mathbf{V}^{\nabla}$ is the local system, or locally constant sheaf, of horizontal sections $\{\sigma \in \Omega^{0}(\mathbf{V}) \mid \nabla(\sigma) = 0\}$. There is also a well known equivalence between local systems on B and finite-dimensional complex representations of the fundamental group of B. Note that isomorphic connections give rise to the same representation. Under these equivalences, the local systems induced by the representations $\Phi^{q}(\mathbf{t})$ and $\Psi^{q}(\mathbf{t})$ correspond to flat connections on the vector bundles $\mathbf{K}^{q} \to \mathsf{B}$ and $\mathbf{H}^{q} \to \mathsf{B}$, called Gauss–Manin connections.

For weights $\boldsymbol{\lambda}$ that are both nonresonant and combinatorial, it follows from (1.1) that the only nonvanishing cohomology vector bundle is $\mathbf{H}^{\ell} \to \mathsf{B}$. In [23], Terao shows that this vector bundle is trivial, and that the corresponding Gauss–Manin connection has logarithmic poles along the irreducible components of the codimension one divisor $\mathsf{D} = \bar{\mathsf{B}} \backslash \mathsf{B}$, where $\bar{\mathsf{B}}$ denotes the closure of B in $(\mathbb{CP}^{\ell})^{n}$.

For general weights, the vector bundles $\mathbf{K}^{q} \to \mathsf{B}$ and $\mathbf{H}^{q} \to \mathsf{B}$ need not be trivial. However, the restriction of any one of these vector bundles to a circle is trivial, since any map from the circle to the relevant classifying space is null-homotopic. Thus we make a local study of these bundles, their Gauss–Manin connections, and the corresponding local systems and fundamental group representations as follows.

Let $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, and choose a representative path $\tilde{g}: I \to \mathsf{B}$. Then, of course, $\tilde{g}(0) = \tilde{g}(1) = \mathsf{b}$, so $\tilde{g}$ defines a map $g: \mathbb{S}^1 \to \mathsf{B}$. If ϕ denotes one of the representations $\Phi^q(\mathbf{t})$ or $\Psi^q(\mathbf{t})$, there is an induced representation of $\pi_1(\mathbb{S}^1, 1) = \langle \zeta \rangle = \mathbb{Z}$, given by $\zeta \mapsto X$, where $X = \phi(\gamma)$. Denote the matrix of X by the same symbol.

Now let $\mathbf{V} \to \mathsf{B}$ denote one of the vector bundles $\mathbf{K}^q \to \mathsf{B}$ or $\mathbf{H}^q \to \mathsf{B}$, and let $g^*(\mathbf{V}) \to \mathbb{S}^1$ be the induced vector bundle over the circle. Pulling back the relevant Gauss–Manin connection ∇, we have a corresponding connection $g^*(\nabla)$ on the bundle $g^*(\mathbf{V}) \to \mathbb{S}^1$, which, as noted above, is necessarily a trivial vector bundle. Specifying the flat connection $g^*(\nabla)$ on this trivial bundle amounts to choosing a logarithm, Y, of the matrix X arising from the above representation. In summary:

PROPOSITION 4.1. *The connection matrix Y satisfies $X = \exp(-2\pi \mathrm{i} Y)$.*

The representations $\Phi^\bullet(\mathbf{t})$ and $\Psi^\bullet(\mathbf{t})$ are induced by the universal representation $\Phi^\bullet(\mathbf{x}): \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{Aut}_\Lambda(K^\bullet_\Lambda)$, see Theorems 3.3 and 3.6. We now define a corresponding formal connection. Recall from Theorem 2.5.2 that the Aomoto complex $(A^\bullet_R, \mu^\bullet) = (A^\bullet_R(\mathsf{b}), \mu^\bullet_{\mathsf{b}})$ is chain equivalent to the linearization (at $\mathbf{1} \in (\mathbb{C}^*)^n$) of the universal complex $(K^\bullet_\Lambda, \Delta^\bullet(\mathbf{x}))$. Choosing bases appropriately, we can assume that the Aomoto complex is equal to this linearization, see the proof of [5, Thm. 2.13].

For each $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$ and each q, let $\Omega^q(\mathbf{y})(\gamma)$ denote the linear term in the power series expansion of $\Phi^q(\mathbf{x})(\gamma)$ in $\mathbf{y}$, where $\mathbf{x} = \exp(\mathbf{y})$, that is $x_j = \exp(y_j)$ for $1 \leqslant j \leqslant n$. This defines a map (in fact, a representation)

$$\Omega^q(\mathbf{y}): \pi_1(\mathsf{B}, \mathsf{b}) \longrightarrow \mathrm{End}_R(A^q_R). \tag{4.1}$$

By construction, the entries of the matrix of $\Omega^q(\mathbf{y})(\gamma)$ are linear forms in $y_1, \dots, y_n$, with integer coefficients, see Theorem 2.5.1. We call the collection $\Omega^\bullet(\mathbf{y})$ the *formal connection*.

THEOREM 4.2. *The eigenvalues of the formal connection are integral linear forms in the variables $y_1, \dots, y_n$. In other words, for each $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$ and each q, the eigenvalues of the formal connection matrix $\Omega^q(\mathbf{y})(\gamma)$ are integral linear forms in $y_1, \dots, y_n$.*

Proof. Recall from Theorem 3.4 that the eigenvalues of $\Phi^q(\mathbf{x})(\gamma)$ are monomials of the form $x_1^{m_1} \cdots x_n^{m_n}$. Since $\Omega^q(\mathbf{y})(\gamma)$ is the linear term in the power series expansion of $\Phi^q(\exp(\mathbf{y}))(\gamma)$ in $\mathbf{y}$, the result follows. □

THEOREM 4.3. *Let $\boldsymbol{\lambda}$ be a system of weights, and let $\mathbf{t} = \exp(-2\pi \mathrm{i} \boldsymbol{\lambda})$. For each $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, the evaluation $\Omega^q(\boldsymbol{\lambda})(\gamma)$ of the formal connection matrix $\Omega^q(\mathbf{y})(\gamma)$ at $\boldsymbol{\lambda}$ is a Gauss–Manin connection matrix corresponding to the automorphism $\Phi^q(\mathbf{t})(\gamma)$.*

Proof. Given $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, the endomorphism $\Omega^q(\mathbf{y})(\gamma)$ of A^q_R is the linearization of the automorphism $\Phi^q(\mathbf{x})(\gamma)$ of K^q_Λ. Recall from Theorem 3.1 that $\Phi^q(\mathbf{1})(\gamma) = \mathrm{id}$. It follows that $\Omega^q(\mathbf{y})(\gamma)$ may be realized as a logarithmic derivative of $\Phi^q(\mathbf{x})(\gamma)$ at $\mathbf{t} = \mathbf{1}$. This being the case, we have $\Phi^q(\mathbf{x})(\gamma) = \exp(\Omega^q(\mathbf{y})(\gamma))$, where $\mathbf{x} = \exp(\mathbf{y})$. Since $\mathbf{t} = \exp(-2\pi \mathrm{i} \boldsymbol{\lambda})$, the specialization $\mathbf{x} \mapsto \mathbf{t}$ yields $\Phi^q(\mathbf{t})(\gamma) = \exp(\Omega^q(-2\pi \mathrm{i} \boldsymbol{\lambda})(\gamma))$. Thus a

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Gauss–Manin connection matrix $Y(\gamma)$ satisfies $-2\pi\mathrm{i}\,Y(\gamma) = \Omega^q(-2\pi\mathrm{i}\boldsymbol{\lambda})(\gamma)$, see Proposition 4.1. Now the entries of $\Omega^q(\mathbf{y})(\gamma)$ are linear forms in $y_1, \dots, y_n$. Consequently, we have $\Omega^q(-2\pi\mathrm{i}\boldsymbol{\lambda})(\gamma) = -2\pi\mathrm{i}\Omega^q(\boldsymbol{\lambda})(\gamma)$. Therefore, the specialization $\mathbf{y} \mapsto \boldsymbol{\lambda}$ yields the Gauss–Manin connection matrix $Y(\gamma) = \Omega^q(\boldsymbol{\lambda})(\gamma)$. □

These results lead to the following theorem, the main result of this paper, which provides an affirmative answer to the question of Terao stated in the Introduction.

THEOREM 4.4. *For any arrangement $\mathcal{A}$ and any system of weights $\boldsymbol{\lambda}$, if $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, then the eigenvalues of a corresponding Gauss–Manin connection matrix in local system cohomology are evaluations of linear forms with integer coefficients, and are thus integral linear combinations of the weights.*

Proof. Let $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, and consider the corresponding induced bundles over the circle as discussed above. By Theorem 4.3, the Gauss–Manin connection on the vector bundle $\mathbf{K}^q \to \mathbb{S}^1$ is given by the matrix $\Omega^q(\boldsymbol{\lambda})(\gamma)$ for each q. By Theorem 4.2, the eigenvalues of the connection matrix $\Omega^q(\boldsymbol{\lambda})(\gamma)$ are evaluations at $\boldsymbol{\lambda}$ of linear forms with integer coefficients for each q. Passage to cohomology yields a connection matrix $\bar{\Omega}^q(\boldsymbol{\lambda})(\gamma)$, corresponding to the cohomology representation $\Psi^q(\mathbf{t})(\gamma)$, whose eigenvalues satisfy the same condition. □

Remark 4.5. As noted above, for weights $\boldsymbol{\lambda}$ that are both nonresonant and combinatorial, the only nonvanishing cohomology vector bundle $\mathbf{H}^\ell \to \mathsf{B}$, corresponding to the representation $\Psi^\ell(\mathbf{t})$, is trivial. A trivialization is given by the β**nbc** basis for the local system cohomology group $H^\ell(\mathsf{M}_\mathsf{b}; \mathcal{L}(\mathsf{b}))$, see [11, 23]. In this context, Terao [23] shows that the Gauss–Manin connection is determined by a connection 1-form $\sum d\log \mathsf{D}_j \otimes \nabla_j$, where $\nabla_j \in \mathrm{End}_\mathbb{C} H^\ell(\mathsf{M}_\mathsf{b}; \mathcal{L}(\mathsf{b}))$, each $d\log \mathsf{D}_j$ denotes a 1-form on $\bar{\mathsf{B}}$ with a simple logarithmic pole along the irreducible component D_j of the divisor $\mathsf{D} = \bar{\mathsf{B}} \setminus \mathsf{B}$, and the sum is over all such irreducible components. In cases where the codimension of $\bar{\mathsf{B}}$ in $(\mathbb{CP}^\ell)^n$ is small, the endomorphisms ∇_j have been explicitly determined, by Aomoto and Kita [2] in the codimension zero case, and by Terao [23] in the codimension one case. See [18] for an exposition of these results.

If γ_j is a simple loop in B linking the component D_j, then the endomorphisms ∇_j and $\Omega_j = \bar{\Omega}^\ell(\boldsymbol{\lambda})(\gamma_j)$ give rise to conjugate automorphisms $\exp(-2\pi\mathrm{i}\nabla_j)$ and $\Psi^\ell(\mathbf{t})(\gamma_j) = \exp(-2\pi\mathrm{i}\Omega_j)$ of $H^\ell(\mathsf{M}_\mathsf{b}; \mathcal{L}(\mathsf{b}))$. It follows that the connections on the trivial vector bundle over the circle corresponding to ∇_j and Ω_j are isomorphic. By Theorem 4.4, the eigenvalues of the latter connection matrix are integral linear combinations of the weights. □

5. Combinatorial Connections

In this section, we investigate the combinatorial implications of the formal connection defined on the Aomoto complex. Recall that the Aomoto complex is a universal

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complex, $(A^\bullet_R(\mathcal{A}), a_{\mathbf{y}}\wedge)$, with the property that the specialization $y_j \mapsto \lambda_j$ calculates the Orlik–Solomon algebra cohomology $H^*(A^\bullet(\mathcal{A}), a_\lambda\wedge)$.

Let $\mathbf{A}^q \to \mathsf{B}$ be the vector bundle over the moduli space whose fiber at b is $A^q(\mathcal{A}_{\mathsf{b}})$, the qth graded component of the Orlik–Solomon algebra of the arrangement $\mathcal{A}_{\mathsf{b}}$. Given weights $\boldsymbol{\lambda}$, the cohomology of the complex $(A^\bullet(\mathcal{A}_{\mathsf{b}}), a_\lambda\wedge)$ gives rise to an additional vector bundle $\mathbf{H}^q(A) \to \mathsf{B}$ whose fiber at b is the qth cohomology group of the Orlik–Solomon algebra, $H^q(A^\bullet(\mathcal{A}_{\mathsf{b}}), a_\lambda\wedge)$. Like their topological counterparts studied in the previous sections, these combinatorial vector bundles admit *combinatorial connections*.

Fix a basepoint $\mathsf{b} \in \mathsf{B}$, and denote the Aomoto complex of $\mathcal{A}_{\mathsf{b}}$ by simply $(A^\bullet_R, a_{\mathbf{y}}\wedge)$. As before, let $\mu^\bullet(\mathbf{y})$ denote the boundary map with respect to a given basis. Recall from (4.1) that the formal connection is comprised of maps $\mathbf{\Omega}^\bullet(\mathbf{y})\colon \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{End}_R(A^q_R)$.

PROPOSITION 5.1. *For each $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, the endomorphisms $\mathbf{\Omega}^q(\mathbf{y})(\gamma)\colon A^q_R \to A^q_R$, $0 \leqslant q \leqslant \ell$, comprise a chain map $\mathbf{\Omega}^\bullet(\mathbf{y})(\gamma)$ of the Aomoto complex $(A^\bullet_R(\mathsf{b}), \mu^\bullet(\mathbf{y}))$.*

Proof. For $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, we have an automorphism $\Phi^\bullet(\mathbf{x})(\gamma)$ of the universal complex $(K^\bullet_\Lambda, \Delta^\bullet(\mathbf{x}))$ by Theorems 3.3 and 3.4. Write $\Phi^q = \Phi^q(\mathbf{x})(\gamma)$ and $\Delta^q = \Delta^q(\mathbf{x})$, and consider these maps as matrices with entries in Λ. Then, for each q, $\Delta^q \cdot \Phi^{q+1} = \Phi^q \cdot \Delta^q$.

Now make the substitution $\mathbf{x} = \exp(\mathbf{y})$, and denote power series expansions in $\mathbf{y}$ by $\Delta^q = \sum_{k \geqslant 0} \Delta^q_k$ and $\Phi^q = \sum_{k \geqslant 0} \Phi^q_k$. In this notation, $\mathbf{\Omega}^q(\mathbf{y})(\gamma) = \Phi^q_1$. Comparing terms of degree two in the above equality, we obtain

$$\Delta^q_0 \cdot \Phi^{q+1}_2 + \Delta^q_1 \cdot \Phi^{q+1}_1 + \Delta^q_2 \cdot \Phi^{q+1}_0 = \Phi^q_0 \cdot \Delta^q_2 + \Phi^q_1 \cdot \Delta^q_1 + \Phi^q_2 \cdot \Delta^q_0. \tag{5.1}$$

By Remark 2.4, we have $\Delta^q_0 = 0$. By Theorem 2.5.2, the linearization of Δ^q is equal to the boundary map of the Aomoto complex, $\Delta^q_1 = \mu^q(\mathbf{y})$. Also, Theorem 3.1 implies that $\Phi^q_0 = \mathrm{id}$ and $\Phi^{q+1}_0 = \mathrm{id}$. These facts, together with (5.1), imply that $\mu^q(\mathbf{y}) \cdot \Phi^{q+1}_1 = \Phi^q_1 \cdot \mu^q(\mathbf{y})$. In other words, $\Phi^\bullet_1 = \mathbf{\Omega}^\bullet(\mathbf{y})(\gamma)$ is a chain map of the Aomoto complex. □

So we write $\mathbf{\Omega}^\bullet(\mathbf{y})\colon \pi_1(\mathsf{B}, \mathsf{b}) \to \mathrm{End}_R(A^\bullet_R)$. By Theorem 4.2, the eigenvalues of the formal connection $\mathbf{\Omega}^\bullet(\mathbf{y})$ on the Aomoto complex $A^\bullet_R$ are integral linear forms in $\mathbf{y}$. Using this fact and the above Proposition, we obtain the following combinatorial analogue of Theorem 4.4.

THEOREM 5.2. *For any arrangement $\mathcal{A}$ and any system of weights $\boldsymbol{\lambda}$, the eigenvalues of the combinatorial connection in Orlik–Solomon algebra cohomology are evaluations of linear forms with integer coefficients, and are thus integral linear combinations of the weights.*

Proof. For $\gamma \in \pi_1(\mathsf{B}, \mathsf{b})$, the formal connection $\mathbf{\Omega}^\bullet(\mathbf{y})(\gamma)$ is a chain map on the Aomoto complex, which induces upon specialization the Gauss–Manin connection in Orlik–Solomon algebra cohomology. The result follows. □

6. An Example

We conclude by illustrating the results of the previous sections with an explicit example. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ with hyperplanes

$$H_1 = \{u_1 + u_2 = 0\}, \qquad H_2 = \{2u_1 + u_2 = 0\},$$
$$H_3 = \{3u_1 + u_2 = 0\}, \qquad H_4 = \{1 + 5u_1 + u_2 = 0\}.$$

6.1. UNIVERSAL COMPLEXES

We first record the universal complex $K^\bullet_\Lambda$ and the Aomoto complex $A^\bullet_R$ of $\mathcal{A}$. The universal complex is equivalent to the cochain complex of the maximal Abelian cover of the complement $M = M(\mathcal{A})$. For any $\mathbf{t} \in (\mathbb{C}^*)^4$, the specializations at $\mathbf{t}$ of the two complexes are quasi-isomorphic. The latter complex may be obtained by applying the Fox calculus to a presentation of the fundamental group of the complement, see for instance [7]. A presentation of this group is

$$\pi_1(M) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid [\gamma_3\ \gamma_1, \gamma_2],\ [\gamma_1\ \gamma_2, \gamma_3],\ [\gamma_i, \gamma_4] \text{ for } i = 1,2,3\rangle, \tag{6.1}$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Using this presentation, we obtain $K^\bullet_\Lambda\colon \Lambda \xrightarrow{\Delta^0} \Lambda^4 \xrightarrow{\Delta^1} \Lambda^5$, where, in matrix form, $\Delta^0 = \Delta^0(\mathbf{x}) = \begin{bmatrix} x_1 - 1 & x_2 - 1 & x_3 - 1 & x_4 - 1 \end{bmatrix}$ and

$$\Delta^1 = \Delta^1(\mathbf{x}) = \begin{bmatrix} x_3 - x_2x_3 & 1 - x_3 & 1 - x_4 & 0 & 0 \\ x_1x_3 - 1 & x_1 - x_1x_3 & 0 & 1 - x_4 & 0 \\ 1 - x_2 & x_1x_2 - 1 & 0 & 0 & 1 - x_4 \\ 0 & 0 & x_1 - 1 & x_2 - 1 & x_3 - 1 \end{bmatrix}.$$

By Theorem 2.5(2), the Aomoto complex $A^\bullet_R$ is the linearization of the complex $K^\bullet_\Lambda$. Fixing the **nbc**-basis [18, Section 5.2] for the Orlik–Solomon algebra of $\mathcal{A}$ yields a corresponding basis for $A^\bullet_R$. With respect to this basis, the Aomoto complex is given by $A^\bullet_R\colon\ R \xrightarrow{\mu^0} R^4 \xrightarrow{\mu^1} R^5$, where $\mu^0 = \mu^0(\mathbf{y}) = [y_1\ y_2\ y_3\ y_4]$ and

$$\mu^1 = \mu^1(\mathbf{y}) = \begin{bmatrix} -y_2 & -y_3 & -y_4 & 0 & 0 \\ y_1 + y_3 & -y_3 & 0 & -y_4 & 0 \\ -y_2 & y_1 + y_2 & 0 & 0 & -y_4 \\ 0 & 0 & y_1 & y_2 & y_3 \end{bmatrix}.$$

6.2. THE MODULI SPACE AND RELATED BUNDLES

The moduli space of the arrangement $\mathcal{A}$ was studied in detail by Terao [23], see also [18, Ex. 10.4.2]. This moduli space may be described as

$$\mathsf{B} = \mathsf{B}(\mathcal{A}) = \left\{ \begin{pmatrix} z_0^1 & z_0^2 & z_0^3 & z_0^4 & 1 \\ z_1^1 & z_1^2 & z_1^3 & z_1^4 & 0 \\ z_2^1 & z_2^2 & z_2^3 & z_2^4 & 0 \end{pmatrix} \middle| \begin{matrix} D_{i,j,k} = 0 \text{ if } \{i,j,k\} = \{1,2,3\} \\ D_{i,j,k} \neq 0 \text{ if } \{i,j,k\} \neq \{1,2,3\} \end{matrix} \right\}.$$

Here, $(z_0^i : z_1^i : z_2^i) \in \mathbb{CP}^2$ for $1 \leqslant i \leqslant 4$, and $D_{i,j,k}$ denotes the determinant of the submatrix of the above matrix with columns i, j, and k, for $1 \leqslant i < j < k \leqslant 5$. This moduli space is smooth, see [18, Prop. 9.3.3]. Recall the fiber bundle $\pi\colon \mathsf{M} \to \mathsf{B}$ of [23, Section 3], with fiber $\pi^{-1}(\mathsf{b}) = \mathsf{M}_{\mathsf{b}} = M(\mathcal{A}_{\mathsf{b}})$, the complement of the arrangement $\mathcal{A}_{\mathsf{b}}$ combinatorially equivalent to $\mathcal{A}$. The total space of this bundle is given by

$$\mathsf{M} = \{(\mathsf{b}, \mathbf{u}) \in \mathsf{B} \times \mathbb{C}^2 \mid \mathbf{u} \in \mathsf{M}_{\mathsf{b}}\}.$$

For brevity, in 6.3–6.5 below, we calculate various representation and connection matrices for a single element $\alpha \in \pi_1(\mathsf{B}, \mathsf{b}_0)$, where $\mathsf{b}_0 \in \mathsf{B}$ is the basepoint (corresponding to $\mathcal{A}$) given below. View $\mathbb{S}^1$ as the set of complex numbers of length one, and define $g : \mathbb{S}^1 \to \mathsf{B}(\mathcal{A})$, $s \mapsto g(s)$, by the following formula.

$$\mathsf{b}_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 5 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad g(s) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ \frac{3-s}{2} & \frac{3+s}{2} & 3 & 5 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Note that g is a loop based at b_0 about the divisor defined by $D_{1,2,5} = 0$ in $\bar{\mathsf{B}} \backslash \mathsf{B}$, so represents an element α of the fundamental group $\pi_1(\mathsf{B}, \mathsf{b}_0)$. We will determine the action of α on the universal complex $K^\bullet_\Lambda$.

For this, consider the induced bundle $g^*(\mathsf{M})$, with total space

$$E = \{(s, (\mathsf{b}, \mathbf{u})) \in \mathbb{S}^1 \times (\mathsf{B} \times \mathbb{C}^2) \mid g(s) = \mathsf{b} \text{ and } \mathbf{u} \in \mathsf{M}_{\mathsf{b}}\},$$

and projection $\pi'(s, (\mathsf{b}, \mathbf{u})) = s$. A similar bundle over $\mathbb{S}^1$ arises in the context of configuration spaces. We refer to [3] as a general reference on configuration spaces and braid groups. Let $F_n(\mathbb{C}) = \{\mathbf{v} \in \mathbb{C}^n \mid v_i \neq v_j \text{ if } i \neq j\}$ be the configuration space of n ordered points in $\mathbb{C}$, the complement of the braid arrangement. There is a well known bundle $p\colon F_{n+1}(\mathbb{C}) \to F_n(\mathbb{C})$, which admits a section. Writing $F_{n+1}(\mathbb{C}) = \{(\mathbf{v}, w) \in F_n(\mathbb{C}) \times \mathbb{C} \mid w \in \mathbb{C}\backslash\{v_j\}\}$, the projection is $p(\mathbf{v}, w) = \mathbf{v}$. The fiber of this bundle is $p^{-1}(\mathbf{v}) = \mathbb{C}\backslash\{v_j\}$, the complement of n points in $\mathbb{C}$.

Define $g_1\colon \mathbb{S}^1 \to F_4(\mathbb{C})$ by

$$g_1(s) = \left(\frac{3-s}{2}, \frac{3+s}{2}, 3, 4\right).$$

This loop represents the standard generator $A_{1,2}$ of the pure braid group $P_4 = \pi_1(F_4(\mathbb{C}), \mathbf{v}_0)$, the fundamental group of the configuration space $F_4(\mathbb{C})$, where $\mathbf{v}_0 = (1, 2, 3, 4)$. Let $g_1^*(F_5(\mathbb{C}))$ be the pullback of the bundle $p\colon F_5(\mathbb{C}) \to F_4(\mathbb{C})$ along the map $g_1\colon \mathbb{S}^1 \to F_4(\mathbb{C})$. The bundle $g_1^*(F_5(\mathbb{C}))$ has total space

$$E_1 = \{(s, (\mathbf{v}, w)) \in \mathbb{S}^1 \times (F_4(\mathbb{C}) \times \mathbb{C}) \mid g_1(s) = \mathbf{v} \text{ and } w \in \mathbb{C}\backslash\{v_j\}\},$$

and projection $p'(s, (\mathbf{v}, w)) = s$. The two induced bundles $g^*(\mathsf{M})$ and $g_1^*(F_5(\mathbb{C}))$ are related as follows. If $\mathbf{v} = g_1(s)$ and $w \in \mathbb{C} \setminus \{v_j\}$, it is readily checked that the point $\mathbf{u} = (-1, w)$ is in $\mathsf{M}_{g(s)}$, the fiber of $g^*(\mathsf{M})$ over $s \in \mathbb{S}^1$. This defines a map $h: E_1 \to E$, $(s, (\mathbf{v}, w)) \mapsto (s, (g(s), \mathbf{u}))$, where $\mathbf{u} = (-1, w)$. Checking that $\pi' \circ h = p'$, we see that $h: g_1^*(F_5(\mathbb{C})) \to g^*(\mathsf{M})$ is a map of bundles.

6.3. UNIVERSAL REPRESENTATIONS AND FORMAL CONNECTIONS

The fiber bundles $\xi_1 = g_1^*(F_5(\mathbb{C}))$ and $\xi = g^*(\mathsf{M})$ admit compatible sections, induced by the section of the configuration space bundle $p : F_5(\mathbb{C}) \to F_4(\mathbb{C})$ and the bundle map h defined above. Consequently, upon passage to fundamental groups, we obtain the following commutative diagram with split short exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathbb{C}\backslash\{v_j\}) & \longrightarrow & \pi_1(E_1) & \longrightarrow & \pi_1(\mathbb{S}^1) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} & & \| & & \\ 1 & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(E) & \longrightarrow & \pi_1(\mathbb{S}^1) & \longrightarrow & 1 \end{array}$$

Via the bundle map h, the fiber $\pi_1(\mathbb{C}\backslash\{v_j\})$ of $g_1^*(\xi_1)$ may be realized as the intersection of the line $\{u_1 = -1\}$ with the fiber of $g^*(\xi)$ in $\mathbb{C}^2$. Thus, the map $h_*: \pi_1(\mathbb{C}\backslash\{v_j\}) \to \pi_1(M)$ is the natural projection of the free group on four generators, $\pi_1(\mathbb{C}\backslash\{v_j\}) = \mathbb{F}_4 = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle$, onto the group $\pi_1(M)$ with presentation (6.1). Let ζ denote the standard generator of $\pi_1(\mathbb{S}^1, 1)$, mapping to $A_{1,2} \in P_4 = \pi_1(F_4(\mathbb{C}), \mathbf{v}_0)$ and to $\alpha \in \pi_1(\mathsf{B}, \mathsf{b}_0)$ under the homomorphisms induced by the maps g_1 and g. The action of ζ on the free group $\mathbb{F}_4$ coincides with that of $A_{1,2}$ on $\mathbb{F}_4$, and is well known. It is given by the Artin representation:

$$\zeta(\gamma_i) = A_{1,2}(\gamma_i) = \begin{cases} \gamma_1\gamma_2\gamma_i\gamma_2^{-1}\gamma_1^{-1}, & \text{if } i = 1 \text{ or } i = 2, \\ \gamma_i, & \text{otherwise.} \end{cases}$$

By virtue of the commutativity of the above diagram, this action descends to an action of $\alpha \in \pi_1(\mathsf{B}, \mathsf{b}_0)$ on $\pi_1(M)$ defined by the same formula. The resulting action of α on the universal complex $K_\Lambda^\bullet$ – the universal representation – may be determined using the Fox calculus, see for instance [8] for similar computations. The action on K_Λ^0 is trivial since α acts on $\pi_1(M)$ by conjugation. The action on K_Λ^1 is familiar. It is obtained by applying the Gassner representation to the pure braid $A_{1,2}$. We suppress the calculation of the action of α on K_Λ^2, and record only the result below.

Denote the universal representation and formal connection matrices corresponding to $\alpha \in \pi_1(\mathsf{B}, \mathsf{b}_0)$ by $\Phi^q = \Phi^q(\mathbf{x})(\alpha)$ and $\Omega^q = \Omega^q(\mathbf{y})(\alpha)$ respectively. These matrices provide chain maps of the universal and Aomoto complexes:

$$\begin{array}{ccccc} \Lambda & \xrightarrow{\Delta^0} & \Lambda^4 & \xrightarrow{\Delta^1} & \Lambda^5 \\ \downarrow{\scriptstyle \Phi^0} & & \downarrow{\scriptstyle \Phi^1} & & \downarrow{\scriptstyle \Phi^2} \\ \Lambda & \xrightarrow{\Delta^0} & \Lambda^4 & \xrightarrow{\Delta^1} & \Lambda^5 \end{array}, \qquad \begin{array}{ccccc} R & \xrightarrow{\mu^0} & R^4 & \xrightarrow{\mu^1} & R^5 \\ \downarrow{\scriptstyle \Omega^0} & & \downarrow{\scriptstyle \Omega^1} & & \downarrow{\scriptstyle \Omega^2} \\ R & \xrightarrow{\mu^0} & R^4 & \xrightarrow{\mu^1} & R^5 \end{array}$$

and are given by $\Phi^0 = 1$, $\Omega^0 = 0$,

$$\Phi^1 = \begin{bmatrix} 1-x_1+x_1x_2 & 1-x_2 & 0 & 0 \\ x_1-x_1^2 & x_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \Omega^1 = \begin{bmatrix} y_2 & -y_2 & 0 & 0 \\ -y_1 & y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Phi^2 = \begin{bmatrix} x_1x_2 & 0 & 0 & 0 & 0 \\ x_2-1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-x_1+x_1x_2 & 1-x_2 & 0 \\ 0 & 0 & x_1-x_1^2 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \Omega^2 = \begin{bmatrix} y_1+y_2 & 0 & 0 & 0 & 0 \\ y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & -y_2 & 0 \\ 0 & 0 & -y_1 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6.4. NONRESONANT LOCAL SYSTEMS

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be a system of weights in $\mathbb{C}^4$, and $\mathbf{t} = (t_1, t_2, t_3, t_4)$ the corresponding point in $(\mathbb{C}^*)^4$. The induced local system $\mathcal{L}$ on M is nonresonant, and $H^2(M; \mathcal{L}) \simeq \mathbb{C}^2$, provided the rank of the matrix $\Delta^1(\mathbf{t})$ is equal to three. If this is the case, then $\mathbf{t} \neq \mathbf{1}$ and rank $\Delta^0(\mathbf{t}) = 1$.

Let $\Xi(\mathbf{x}) : \Lambda^5 \to \Lambda^2$ and $\Upsilon(\mathbf{y}) : R^5 \to R^2$ be the linear maps with matrices

$$\Xi = \Xi(\mathbf{x}) = \begin{bmatrix} x_4-1 & 0 \\ 0 & x_4-1 \\ x_3-x_2x_3 & 1-x_3 \\ x_1x_3-1 & x_1-x_1x_3 \\ 1-x_2 & x_1x_2-1 \end{bmatrix} \quad \text{and} \quad \Upsilon = \Upsilon(\mathbf{y}) = \begin{bmatrix} y_4 & 0 \\ 0 & y_4 \\ -y_2 & -y_3 \\ y_1+y_3 & -y_3 \\ -y_2 & y_1+y_2 \end{bmatrix}.$$

Note that Υ is the linearization of Ξ. It is readily checked that $\Xi \circ \Delta^1 = 0$, $\Upsilon \circ \mu^1 = 0$, and that rank $\Xi(\mathbf{t}) = 2$ if $\mathbf{t} \in (\mathbb{C}^*)^4$ induces a nonresonant local system $\mathcal{L}$ on M. Consequently, the projection $\mathbb{C}^5 \simeq K^2 \twoheadrightarrow H^2(M; \mathcal{L}) \simeq \mathbb{C}^2$ may be realized as the specialization at $\mathbf{t}$ of the map Ξ.

Via $\Xi\colon K^2_\Lambda \to \Lambda^2$ and $\Upsilon\colon A^2_R \to R^2$, the chain maps $\Phi^\bullet\colon K^\bullet_\Lambda \to K^\bullet_\Lambda$ and $\Omega^\bullet\colon A^\bullet_R \to A^\bullet_R$ induce maps $\bar{\Phi}(x)\colon \Lambda^2 \to \Lambda^2$ and $\bar{\Omega}(\mathbf{y})\colon R^2 \to R^2$, given by

$$\bar{\Phi} = \bar{\Phi}(\mathbf{x}) = \begin{bmatrix} x_1x_2 & 0 \\ x_2-1 & 1 \end{bmatrix} \quad \text{and} \quad \bar{\Omega} = \bar{\Omega}(\mathbf{y}) = \begin{bmatrix} y_1+y_2 & 0 \\ y_2 & 0 \end{bmatrix}.$$

Specializing at $\mathbf{t} \in (\mathbb{C}^*)^4$ and $\boldsymbol{\lambda} \in \mathbb{C}^4$ yields the representation matrix $\Psi^2(\mathbf{t})(\alpha) = \bar{\Phi}(\mathbf{t})$ and the corresponding Gauss–Manin connection matrix $\bar{\Omega}^2(\boldsymbol{\lambda})(\alpha) = \bar{\Omega}(\boldsymbol{\lambda})$ in the cohomology of the nonresonant local system $\mathcal{L}$. These matrices are

$$\Psi^2(\mathbf{t})(\alpha) = \begin{bmatrix} t_1t_2 & 0 \\ t_2-1 & 1 \end{bmatrix} \quad \text{and} \quad \bar{\Omega}^2(\boldsymbol{\lambda})(\alpha) = \begin{bmatrix} \lambda_1+\lambda_2 & 0 \\ \lambda_2 & 0 \end{bmatrix}.$$

Up to a transpose, the latter recovers Terao's calculation of the connection matrix corresponding to the divisor $D_{1,2,5}$, denoted by Ω_4 in [18, Ex. 10.4.2].

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6.5. RESONANT LOCAL SYSTEMS

Now let $\mathcal{L}$ be a nontrivial resonant local system on M. Such a local system corresponds to a point $\mathbf{1} \neq \mathbf{t} \in (\mathbb{C}^*)^4$ satisfying $t_1t_2t_3 = 1$ and $t_4 = 1$. For each such $\mathbf{t}$, we have $H^1(M; \mathcal{L}) \simeq \mathbb{C}$ and $H^2(M; \mathcal{L}) \simeq \mathbb{C}^3$. Representation and Gauss–Manin connection matrices corresponding to the loop $\alpha \in \pi_1(\mathsf{B}, \mathsf{b})$ in resonant local system cohomology may be obtained by methods analogous to those used in the nonresonant case above.

Define $\Xi\colon \Lambda^5 \to \Lambda^3$ and $\Upsilon\colon R^5 \to R^3$ by

$$\Xi = \begin{bmatrix} x_1x_2 - 1 & 0 & 0 \\ x_2 - 1 & 0 & 0 \\ 0 & x_2 - 1 & 0 \\ 0 & 1 - x_1 & x_3 - 1 \\ 0 & 0 & 1 - x_2 \end{bmatrix} \quad \text{and} \quad \Upsilon = \begin{bmatrix} y_1 + y_2 & 0 & 0 \\ y_2 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & -y_1 & y_3 \\ 0 & 0 & -y_2 \end{bmatrix}.$$

As before, $\Xi \circ \Delta^1 = 0$, and $\Upsilon \circ \mu^1 = 0$, and Υ is the linearization of Ξ. For each $\mathbf{t} \in (\mathbb{C}^*)^4$ satisfying $t_1t_2t_3 = 1$ and $t_4 = 1$, we have rank $\Xi(\mathbf{t}) = 3$. So the projection $\mathbb{C}^5 \simeq K^2 \twoheadrightarrow H^2(M; \mathcal{L}) \simeq \mathbb{C}^3$ may be realized as the specialization $\Xi(\mathbf{t})$.

Via $\Xi\colon K^2_\Lambda \to \Lambda^3$ and $\Upsilon\colon A^2_R \to R^3$, the chain maps $\Phi^\bullet\colon K^\bullet_\Lambda \to K^\bullet_\Lambda$ and $\Omega^\bullet\colon A^\bullet_R \to A^\bullet_R$ induce $\bar{\Phi}\colon \Lambda^3 \to \Lambda^3$ and $\bar{\Omega}\colon R^3 \to R^3$, given by

$$\bar{\Phi} = \begin{bmatrix} x_1x_2 & 0 & 0 \\ 0 & x_1x_2 & 1 - x_3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \bar{\Omega} = \begin{bmatrix} y_1 + y_2 & 0 & 0 \\ 0 & y_1 + y_2 & -y_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Specializing yields the representation matrix $\Psi^2(\mathbf{t})(\alpha)$ and the corresponding Gauss–Manin connection matrix $\bar{\Omega}^2(\boldsymbol{\lambda})(\alpha)$ in the second cohomology of the resonant local system $\mathcal{L}$.

Using the universal complex $K^\bullet_\Lambda$ and the conditions satisfied by a point $\mathbf{t} \in (\mathbb{C}^*)^4$ inducing the resonant local system $\mathcal{L}$, one can show that the representation matrix $\Psi^1(\mathbf{t})(\alpha)$ in first cohomology is given by $\Psi^1(\mathbf{t})(\alpha) = [t_1t_2]$. The corresponding Gauss–Manin connection matrix is, of course, $\bar{\Omega}^1(\boldsymbol{\lambda})(\alpha) = [\lambda_1 + \lambda_2]$.

Remark 6.6. For this arrangement, every local system $\mathcal{L}$ is combinatorial. Given $\mathcal{L}$, there are weights $\boldsymbol{\lambda} \in \mathbb{C}^4$ for which the local system cohomology $H^*(M; \mathcal{L})$ is quasi-isomorphic to Orlik–Solomon algebra cohomology $H^*(A^\bullet, a_{\boldsymbol{\lambda}}\wedge)$. Thus, for such weights, the combinatorial connection matrices in the cohomology of the Orlik–Solomon algebra coincide with the Gauss–Manin connection matrices in local system cohomology computed above.

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