PERMANENT PRESERVERS ON THE SPACE OF DOUBLY STOCHASTIC MATRICES

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1. Introduction. Let M_n be the linear space of n-square matrices with real elements. For a matrix $X = (x_{ij}) \in M_n$ the permanent is defined by

$$\operatorname{per} X = \sum_{\sigma} \prod_{i=1}^{n} x_{i\sigma(i)},$$

where σ runs over all permutations of $1, 2, \ldots, n$. In **(2)** Marcus and May determine the nature of all linear transformations T of M_n into itself such that $\operatorname{per} T(X) = \operatorname{per} X$ for all $X \in M_n$. For such a permanent preserver T, and for $n \geqslant 3$, there exist permutation matrices P, Q, and diagonal matrices D, L in M_n, such that $\operatorname{per} DL = 1$ and either

$$T(X) = DPXQL \text{ for all } X \in M_n,$$

or

$$T(X) = DPX'QL \text{ for all } X \in M_n.$$

Here X' denotes the transpose of X. In the case $n = 2$, a different type of transformation is also possible.

In the present paper we consider those linear mappings which preserve the permanents of doubly stochastic matrices. A matrix is doubly stochastic (d.s.) if its elements are non-negative real numbers and its row and column sums are all 1. The set of d.s. matrices in M_n forms a convex polyhedron Ω_n in which the vertices are permutation matrices. By a *permanent preserver* on Ω_n we mean a mapping T of Ω_n into itself such that, for $A, B \in \Omega_n$ and for real numbers α, β, $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$, $\alpha + \beta = 1$,

$$T(\alpha A + \beta B) = \alpha T(A) + \beta T(B). \tag{1.1}$$

$$\operatorname{per} T(A) = \operatorname{per} A. \tag{1.2}$$

We shall show that for such T there exist fixed permutation matrices P, Q such that either

$$T(A) = PAQ \text{ for all } A \in \Omega_n,$$

or

$$T(A) = PA'Q \text{ for all } A \in \Omega_n.$$

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2. Results. Let T be a mapping of Ω_n into itself satisfying (1.1) and (1.2). In (**3**, Lemma 1) it is shown that for $A \in \Omega_n$, per $A = 1$ if and only if A is a permutation matrix. It follows that T maps permutation matrices into permutation matrices. Suppose that $T(I) = P_I$, where I is the identity matrix. Define the mapping ϕ on Ω_n by

$$\phi(A) = P_I' T(A).$$

Then ϕ has properties (1.1), (1.2), and

$$\phi(I) = I. \tag{2.1}$$

If $A \in \Omega_n$, then A is in the convex hull of at most $(n-1)^2 + 1$ permutation matrices (**1**); i.e.,

$$A = \sum_{j=1}^{k} \theta_j P_j, \qquad k \leqslant (n-1)^2 + 1, \tag{2.2}$$

where $\theta_j > 0$, $j = 1, \ldots, k$, and $\sum^k{}_{j=1}\theta_j = 1$. Then

$$\phi(A) = \sum_{j=1}^{k} \theta_j \phi(P_j). \tag{2.3}$$

It is thus sufficient to discuss the action of ϕ on permutation matrices.

We shall say that two permutation matrices P_1 and P_2 *coincide* in the position (i, j) if the elements of P_1 and P_2 in this position are both 1; and we shall denote by $c[P_1, P_2]$ the number of positions in which P_1 and P_2 coincide.

Lemma 1. *If* $c[P_1, P_2] = \alpha$ *and*

$$A = \theta P_1 + (1-\theta) P_2, \qquad 0 < \theta < 1,$$

then there exist integers

$$e_j > 0, j = 1, \ldots, r, \sum_{j=1}^{r} e_j = n - \alpha,$$

such that

$$\text{per}\, A = \prod_{j=1}^{r} [\theta^{e_j} + (1-\theta)^{e_j}].$$

Proof. There exists a permutation matrix P such that

$$P'(P_1'P_2)P = I_\alpha \dotplus \sum_{j=1}^{r}{}^{\cdot}\, R_{e_j}, \quad \sum_{j=1}^{r} e_j = n - \alpha,$$

where I_α is the identity in Ω_α and

$$R_t = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & 0 & & & 0 \end{pmatrix} \tag{2.4}$$

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is t-square. (All elements not shown as 1's are zero.) Here $\dot{+}$ and $\sum^{\cdot}$ indicate direct sums. Note that $P'(P_1'P_1)P = I$. Hence we have

$$\begin{aligned}\operatorname{per} A &= \operatorname{per}\,[P'(P_1'A)P] \\ &= \operatorname{per}\,[\theta I + (1-\theta)P'P_1'P_2P] \\ &= \operatorname{per}\left[I_\alpha \dot{+} \sum_{j=1}^{r}{}^{\cdot}\ (\theta I_{e_j} + (1-\theta)R_{e_j})\right] \\ &= \prod_{j=1}^{r} \operatorname{per}\,[\theta I_{e_j} + (1-\theta)R_{e_j}] \\ &= \prod_{j=1}^{r} [\theta^{e_j} + (1-\theta)^{e_j}].\end{aligned}$$

LEMMA 2. *For two permutation matrices* P_1, P_2,

$$c[\phi(P_1), \phi(P_2)] = c[P_1, P_2].$$

Proof. Let $c[P_1, P_2] = \alpha$ and $c[\phi(P_1), \phi(P_2)] = \beta$. For $A(\theta) = \theta P_1 + (1-\theta)P_2$ $0 < \theta < 1$, $\phi(A(\theta)) = \theta\phi(P_1) + (1-\theta)\phi(P_1)$. By Lemma 1,

$$\operatorname{per} A(\theta) = \prod_{j=1}^{r} [\theta^{e_j} + (1-\theta)^{e_j}],$$

where

$$\sum_{j=1}^{r} e_j = n - \alpha;$$

and

$$\operatorname{per} \phi(A(\theta)) = \prod_{j=1}^{s} [\theta^{f_j} + (1-\theta)^{f_j}],$$

where

$$\sum_{j=1}^{s} f_j = n - \beta.$$

Since ϕ preserves permanents,

$$\prod_{j=1}^{r} [\theta^{e_j} + (1-\theta)^{e_j}] = \prod_{j=1}^{s} [\theta^{f_j} + (1-\theta)^{f_j}] \tag{2.5}$$

for all θ, $0 < \theta < 1$, and hence for all real θ. We may assume that $e_1 \leqslant \ldots \leqslant e_r$, $f_1 \leqslant \ldots \leqslant f_s$, and $e_r \leqslant f_s$. Now the polynomial $\theta^{f_s} + (1-\theta)^{f_s}$ in θ has at least one root which is not a root of $\theta^t + (1-\theta)^t$ for any t, $0 < t < f_s$. Since this root must occur in both sides of (2.5), $e_r = f_s$. By induction, $e_j = f_j$ for $j = 1, \ldots, s$, and $r = s$. Hence $\alpha = \beta$.

Let $\mathfrak{A}_i^{(t)}$ be the set of those permutation matrices in Ω_t which have 1 in position (i, i). Since we shall be dealing mainly with the case $t = n$, we shall write $\mathfrak{A}_i$ for $\mathfrak{A}_i^{(n)}$.

LEMMA 3. *There exists a permutation* σ *of* $1, 2, \ldots, n$ *such that* $P \in \mathfrak{A}_i$ *implies* $\phi(P) \in \mathfrak{A}_{\sigma(i)}$, $i = 1, \ldots, n$.

Proof. If $n = 2$, σ is the identity. For $n \geqq 3$, set $R = I_1 \dot{+} R_{n-1}$ (see (2.4)). Since $c[I, R] = 1$, $c[I, \phi(R)] = 1$ by Lemma 2, $\phi(R) \in \mathfrak{A}_{\sigma_1}$ for a unique positive integer $\sigma_1 \leqslant n$. Similarly, for any $P \in \mathfrak{A}_1$, $\phi(P) \in \mathfrak{A}_\tau$ for at least one τ. We shall show that $\phi(P) \in \mathfrak{A}_{\sigma_1}$.

Case (i): $c[I, P] = 1$. Let P_{ij} be the permutation matrix which coincides with I except in rows i and j, $i \neq j$. By Lemma 2, $c[\phi(R), \phi(P_{1j})] = 0$; hence $\phi(P_{1j}) \notin \mathfrak{A}_{\sigma_1}$. Similarly, $c[\phi(P), \phi(P_{1j})] = 0$; hence $\phi(P_{1j}) \notin \mathfrak{A}_\tau$. Now $c[I, \phi(P_{1j})] = n - 2$; thus, if $\tau \neq \sigma_1$, $\phi(P_{1j}) = P_{\tau\sigma_1}$, valid for $j = 2, \ldots, n$. This contradicts Lemma 2; hence $\tau = \sigma_1$.

Case (ii): $c[I, P] = k > 1$. If $k \geqslant n - 1$, $P = I$, which is in all $\mathfrak{A}_i$. If $k < n - 1$, we can choose a matrix Q such that $c[R, Q] = c[P, Q] = 0$, while $c[I, Q] = n - k$. In fact there is no loss in generality in assuming that $P = I_k \dot{+} P_1$ for some permutation matrix $P_1 \in \Omega_{n-k}$; in which case $Q = R_k' \dot{+} I_{n-k}$ will do. By Lemma 2, $c[\phi(R), \phi(Q)] = c[\phi(P), \phi(Q)] = 0$, while $c[I, \phi(P)] = k$ and $c[I, \phi(Q)] = n - k$. This forces $\phi(P)$ into $\mathfrak{A}_{\sigma_1}$.

We have shown that for any $P \in \mathfrak{A}_1$, $\phi(P) \in \mathfrak{A}_{\sigma_1}$, for some particular integer σ_1, $1 \leqslant \sigma_1 \leqslant n$. Similarly, we can find for each i, $1 \leqslant i \leqslant n$, an integer σ_i, such that $P \in \mathfrak{A}_i$ implies $\phi(P) \in \mathfrak{A}_{\sigma_1}$. Clearly $\sigma_i \neq \sigma_j$ if $i \neq j$; thus $\sigma(i) = \sigma_i$ gives the desired permutation.

Let P_σ be the permutation matrix whose $\sigma(i)$th column is the ith column of I; σ is the permutation given by Lemma 3. Define the linear transformation ψ on Ω_n:

$$\psi(A) = P_\sigma \phi(A) P_\sigma', \qquad A \in \Omega_n. \tag{2.6}$$

Denote by $\mathfrak{G}_t$ *the set of permanent preservers on* Ω_t *which map* $\mathfrak{A}_i^{(t)}$ *into* $\mathfrak{A}_i^{(t)}$, $i = 1, 2, \ldots, t$. It follows at once that $\psi \in \mathfrak{G}_n$. Let E be the identity mapping on Ω_n, and let F be the transpose mapping on Ω_n; that is, $F(A) = A'$ for all $A \in \Omega_n$.

LEMMA 4. *If* $G \in \mathfrak{G}_n$ *then* $G = E$ *or* $G = F$.

Proof. The proof is by induction on n. For $n = 1, 2$ it is immediate that $G = E$. For $n = 3$ it is easily checked that G must be E or F.

For $n \geqslant 4$, each permutation matrix $P \in \mathfrak{A}_1$ can be written: $P = I_1 \dot{+} \tilde{P}$, where $\tilde{P}$ is a permutation matrix in Ω_{n-1}. Thus G induces a linear mapping $\tilde{G}$ on Ω_{n-1} defined by: $G(P) = I_1 \dot{+} \tilde{G}(\tilde{P})$ and (2.3); moreover, $\tilde{G} \in \mathfrak{G}_{n-1}$. By the induction hypothesis $\tilde{G}$ is the identity or transpose mapping, and hence $G = E$ or F on $\mathfrak{A}_1$. Similarly $G = E$ or F on $\mathfrak{A}_2$. To see that $G = E$ or F uniformly on $\mathfrak{A}_1 \cup \mathfrak{A}_2$, consider the matrices:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & 0 & & \dots & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since $c[P_1, P_2] = n - 3$ while $c[P_1, P_2'] = 0$, it follows from Lemma 2 that $G(P_2) = P_2$ if $G(P_1) = P_1$ and $G(P_2) = P_2'$, if $G(P_1) = P_1'$. Similarly $G = E$ or F *uniformly* on all $\mathfrak{A}_i$, $i = 1, \dots, n$.

There remains to show that, for $P \notin \mathfrak{A}_i$, $i = 1, \dots, n$, $G(P) = P$ or P' according as $G = E$ or F on the $\mathfrak{A}_i$. We shall discuss the case where $G = E$ on $\mathfrak{A}_i$; the argument for transposition is the same. Let $P_{\alpha\beta}$ be the matrix obtained from I by permuting columns α and β. For each α, $1 \leqslant \alpha \leqslant n$, $\exists \gamma \neq \alpha \ni PP_{\alpha\gamma} \in \mathfrak{A}_\alpha$. Then $G(PP_{\alpha\gamma}) = PP_{\alpha\gamma}$. Since $c[G(P), G(PP_{\alpha\gamma})] = n-2$ and $G(P) \notin \mathfrak{A}_\alpha$, $G(P) = PP_{\alpha\gamma}P_{\alpha\delta}$ for some δ. When $n \geqslant 4$, this cannot hold for all α unless $\delta = \gamma$. Thus $G(P) = P$, and the proof of the lemma is complete.

Since $T(A) = P_I P_\sigma' \psi(A) P_\sigma$, we have immediately our main result:

THEOREM: *Let T be a linear mapping of Ω_n into Ω_n such that* per $T(A)$ = per A *or all $A \in \Omega_n$. Then there exist permutation matrices P and Q such that either*

$$T(A) = PAQ, \text{ all } A \in \Omega_n,$$

or

$$T(A) = PA'Q, \text{ all } A \in \Omega_n.$$

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