The Second Integral Theorem of Mean Value: a geometrical proof.

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The theorem in question is the following:—if $\phi(x)$ is a function that either always increases or else always decreases as x increases from a to b, then

$$\int_a^b \phi(x)\psi(x)dx = \phi(a)\int_a^\xi \psi(x)dx + \phi(b)\int_\xi^b \psi(x)dx \quad - \quad - \quad (1)$$

where $a < \xi < b$.

The usual proof by means of Abel's inequalities leaves nothing to be desired in respect of simplicity and rigour from the analytical standpoint. A theorem, however, may become a little clearer when it can be geometrically interpreted, and I therefore venture to submit the following discussion of the Mean-Value Theorem in the hope that it may be of some interest.

I take the form

$$\int_a^b \phi(x)\psi(x)dx = \phi(a)\int_a^\xi \psi(x)dx \quad - \qquad (2)$$

where $\phi(x)$ is a positive decreasing function; from this form the theorem (1) may be readily deduced. For if $\phi(x)$ is a decreasing function, positive or negative, then $\phi(x) - \phi(b)$ is a positive decreasing function; if $\phi(x)$ is an increasing function, then $\phi(b) - \phi(x)$ is a positive decreasing function. The substitution of $\phi(x) - \phi(b)$ or of $\phi(b) - \phi(x)$ in place of $\phi(x)$ in (2) will give (1).

Let $\quad F(x) = \phi(a)\int_a^x \psi(x)dx, \quad f(x) = \int_a^x \phi(x)\,\psi(x)dx;$

then, $\quad F'(x) = \phi(a)\,\psi(x), \qquad f'(x) = \phi(x)\,\psi(x),$

and therefore $F'(x)$ is numerically greater than $f'(x)$.

Hence so long as $\psi(x)$ is positive $F(x)$ increases more rapidly than $f(x)$, while so long as $\psi(x)$ is negative $F(x)$ decreases more rapidly than $f(x)$.

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1. Suppose $\psi(x)$ to be either always positive or else always negative.

When $x=a$, $F(a)=0=f(a)$. Therefore when $\psi(x)$ is positive, $F(x)$ and $f(x)$ are both positive increasing functions, and the graph of $F(x)$ will lie above that of $f(x)$ (Fig. 1). On the other hand, when $\psi(x)$ is negative, $F(x)$ and $f(x)$ are both negative decreasing functions, and the graph of $F(x)$ will lie below that of $f(x)$ (Fig. 2).

If therefore b is any value of x greater than a, the ordinate $f(b)=\mathrm{MP}$ will be equal to one and to only one ordinate $\mathrm{NR}=F(\xi)$;

that is $$\int_a^b \phi(x)\,\psi(x)dx=\phi(a)\int_a^\xi \psi(x)dx$$

where $a<\xi=\mathrm{ON}<b$.

2. Next let $\psi(x)$ be sometimes positive and sometimes negative and divide the interval (a, x) into sub-intervals

$$(a,\ a_1),\quad (a_1,\ a_2),\quad (a_2,\ a_3)\ .\ .\ .$$

Suppose that in the first of these intervals $\psi(x)$ is positive, in the second negative, in the third positive, and so on. Both $F(x)$ and $f(x)$ will turn when x is equal to $a_1,\ a_2,\ a_3\ .\ .\ .$ since their derivatives change sign as x passes through these values.

We will show that so long as $F(x)$ remains positive $f(x)$ will also be positive, and that if x_1 is the value of x for which $F(x)$ first vanishes, $f(x_1)$ is positive (not zero). It follows from these results that the graph of $F(x)$, which at first lies above that of $f(x)$, must, if it ever meets the x axis, cross below that of $f(x)$ at a point whose ordinate is positive; it is easy to see by drawing graphs that equation (2) will not be true unless the graphs cross in the way stated.

Let there be n sub-intervals and denote by $u_1,\ u_2,\ u_3\ldots\ldots$ the integral of $\psi(x)$ from a to a_1, from a_1 to a_2, from a_2 to $a_3\ldots\ldots$. Let x' be any value of x in the n^{th} interval $(a_{n-1},\ a_n)$. Then

$$\begin{aligned} F(x')&=\phi(a)\left[\int_a^{a_1}\psi(x)dx+\int_{a_1}^{a_2}\psi(x)dx+\ldots\ldots+\int_{a_{n-1}}^{x'}\psi(x)dx\right]\\ &=\phi(a)[u_1+u_2+\ldots\ldots+u_n]. \end{aligned}$$

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Again, since $\psi(x)$ does not change sign in any of the sub-intervals, we can apply the First Theorem of Mean Value to each of the n integrals into which $f(x')$ may be decomposed. For example

$$\int_{a_r}^{a_{r+1}} \phi(x)\,\psi(x)dx = \phi_{r+1}\int_{a_r}^{a_{r+1}} \psi(x)dx = \phi_{r+1}u_{r+1}$$

where ϕ_{r+1} lies between $\phi(a_r)$ and $\phi(a_{r+1})$.

Hence $f(x') = \phi_1 u_1 + \phi_2 u_2 + \ldots\ldots + \phi_n u_n$

where ϕ_1, $\phi_2 \ldots\ldots$ are values of $\phi(x)$ in the first, second..... intervals.

If now $F(x')$ be positive for every value of x in the interval (a, x') then the sum

$$s_r = u_1 + u_2 + \ldots\ldots u_r$$

is positive, where r is any integer from 1 to n. But we may write

$$u_1 = s_1, \; u_2 = s_2 - s_1, \; u_3 = s_3 - s_2 \ldots\ldots$$

and then $f(x')$ becomes, after rearrangement,

$$f(x') = (\phi_1 - \phi_2)s_1 + (\phi_2 - \phi_3)s_2 + \ldots\ldots + (\phi_{n-1} - \phi_n)s_{n-1} + \phi_n s_n.$$

Since $\phi(x)$ is a decreasing function each of the differences $\phi_1 - \phi_2$, $\phi_2 - \phi_3, \ldots\ldots$is positive, and therefore, since each of the sums s_1, $s_2 \ldots\ldots$ is also positive, $f(x')$ must be positive.

Next let $F(x)$ first vanish when $x = x_1$ and let x_1 be in the interval (a_{n-1}, a_n). To find $f(x_1)$ we have only to replace s_n by zero; it is then obvious that $f(x_1)$ is positive since each of the sums $s_1, s_2, \ldots\ldots s_{n-1}$ is positive.

Hence before $F(x)$ vanishes there is a point at which the graph of $F(x)$ passes below that of $f(x)$. There is nothing in the proof to determine whether there are more points than one; if there are more points than one, let C_1 be the first of them.

Similar reasoning shows that if $\psi(x)$ is negative in the first interval (a, a_1) the graph of $F(x)$ is at first below that of $f(x)$, but must pass above it before $F(x)$ first vanishes.

If we now transfer the origin of coordinates to C_1, it is clear, by the same reasoning as before, that the graph of $F(x)$ must cross that of $f(x)$ before it again crosses the new x-axis. If C_2 is the first of these points of crossing, transfer the origin to C_2 and proceed as before.

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Hence the most general form of the graphs of $F(x)$ and $f(x)$ will be as in Figure 3, and it is thus evident that if $b = OM$, there is always an ordinate $F(\xi)$ or NR equal to MP or $f(b)$ where $\xi = ON < b$. In the general case there may obviously be several values of ξ; but there is always one such that $a < \xi < b$.

It may be observed that whether s_r be positive or negative the equation

$$f(x') = (\phi_1 - \phi_2)s_1 + (\phi_2 - \phi_3)s_2 + \ldots\ldots + \phi_n s_n$$

may be written in the form

$$f(x') = \phi_1 s_m$$

where s_m is a quantity lying between the greatest and the least of the quantities $s_1, s_2, \ldots\ldots s_n$, and is therefore of the form

$$\int_a^{\xi'} \psi(x)dx.$$

But the form thus obtained for $f(x')$ is not that required by the theorem. The First Mean Value Theorem is only used to deduce the form of the graphs. Of course, as is well known, the theorem may be deduced by integrating by parts and then using the First Mean Value Theorem.