ON COMMUTATIVE NON-SELF-ADJOINT OPERATOR ALGEBRAS

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1. Introduction. A proof is given here of a theorem of Sarason [**9**, Theorem 2], the proof being valid in an arbitrary (non-separable) complex Hilbert space. Sarason's proof uses a theorem and lemma of Wermer which may both fail when the separability hypothesis is omitted [**3**]. By using a special case of Sarason's theorem and another result of Sarason [**10**, Lemma 1] a simplified and shortened proof is given of a result of Scroggs [**11**, Corollary 1].

In general, terminology and notation are similar to those in Halmos's book [**5**]. In addition, throughout this paper, $L(H)$ denotes the algebra of bounded linear operators on the complex Hilbert space H, $\mathscr{A}$ denotes a commutative, identity containing, weakly closed algebra of normal operators in $L(H)$ and $\mathscr{A}^w$ denotes the Von Neumann algebra generated by $\mathscr{A}$.

2. Sarason's Theorem. This is the following result [**9**].

THEOREM 1. *If the operator B, in $L(H)$, leaves invariant every closed invariant subspace of $\mathscr{A}$, then B belongs to $\mathscr{A}$.*

Before proving this theorem we require some preliminary definitions and results.

DEFINITION. A Boolean algebra of projections, $\mathscr{B}$, on a Hilbert space H is *complete* if and and only if, for each subset $\{E_\alpha\} \subseteq \mathscr{B}$,

(a) H admits the orthogonal direct sum decomposition $H = M \oplus N$, where $M = \text{clm}\,\{E_\alpha H\}$, $N = \bigcap_\alpha (I - E_\alpha)H$

(b) the projection E_0 with range M belongs to $\mathscr{B}$ (see [**1**]).

Let $\mathscr{P}$ denote the set of projections in $\mathscr{A}$. Then $\mathscr{P}$ forms a complete Boolean algebra of projections. This follows from [**4**, p. 2201] and the fact that, since $\mathscr{A}$ is closed in the weak operator topology, it is also closed in the strong operator topology.

LEMMA 1. *$\mathscr{P}$ can be regarded as a self-adjoint spectral measure $E(\cdot)$ over its Stone representation space Ω, and every element of $\mathscr{A}^w$ can be expressed in the form $\int_\Omega f(\lambda)E(d\lambda)$, where $f \in C(\Omega)$.*

Proof. Certainly $\mathscr{P}$ is isomorphic with the Boolean algebra of all open and closed subsets of Ω (a compact, Hausdorff, extremally disconnected space). Call this isomorphism $E'(\cdot)$. Now the set τ of finite linear combinations of open and closed sets of Ω is norm dense in $C(\Omega)$. Define a map ϕ from τ to $\mathscr{A}^w$ by

$$\phi\Big(\sum_{i=1}^{n} \lambda_i \chi_{\Omega i}\Big) = \sum_{i=1}^{n} \lambda_i E'(\Omega_i)$$

($\lambda_i \in \mathbb{C}$; Ω_i open and closed in Ω for $i = 1, \ldots, n$; $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$), where, for any set S, χ_S is the characteristic function of S. Then ϕ is a continuous algebra homomorphism. Since

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τ is norm dense in $C(\Omega)$, ϕ can be extended to the whole of $C(\Omega)$ and, since the uniform closure of finite linear combinations of elements of $\mathscr{P}$ is equal to $\mathscr{A}^w$ [**7**, p. 18, Lemma 1], the image of $C(\Omega)$ under ϕ is $\mathscr{A}^w$. Hence there exists a uniquely determined spectral measure $E(\cdot)$ defined on the Borel subsets of Ω and such that [**4**, p. 2186]

$$\phi(f) = \int_\Omega f(\lambda)E(d\lambda) \quad (f \in C(\Omega)).$$

Also, if δ is any open and closed subset of Ω, then

$$\phi(\chi_\delta) = \int_\Omega \chi_\delta E(d\lambda) = E(\delta) = E'(\delta).$$

Hence the values of the spectral measure $E(\cdot)$ on the open and closed subsets of Ω generate $\mathscr{P}$; whence the result.

For m a natural number, we let H_m denote the orthogonal direct sum of m copies of H and, for $A \in L(H)$, we let A_m denote the direct sum of A with itself m times. $\mathscr{A}_m = \{A_m : A \in \mathscr{A}\}$ and $E_m(\cdot)$ is the direct sum of $E(\cdot)$ with itself m times.

DEFINITION. The *cyclic subspace* $M(x)$ corresponding to x in H is given by

$$M(x) = \text{clm}\,\{E(\delta)x : \delta \in \Sigma\},$$

where Σ denotes the σ-algebra of Borel subsets of Ω.

LEMMA 2. *Suppose that there exist vectors $x \in H_m$, $y \in H$ such that $\langle E(\cdot)y, y\rangle = \langle E_m(\cdot)x, x\rangle$. Let N be the smallest closed reducing subspace for $\mathscr{A}$ containing the vector y and Y the smallest closed reducing subspace of $\mathscr{A}_m$ containing the vector x. Then, for any A in $\mathscr{A}^w$, $A_m \,|\, Y$ and $A \,|\, N$ are unitarily equivalent via an isometry V of N onto Y such that $A_m \,|\, Y = VAV^{-1}$, where V is independent of the choice of A.* (For convenience we write A instead of $A \,|\, N$.)

Proof. First notice that $N = M(y)$ and $Y = \text{clm}\,\{E_m(\delta)x : \delta \in \Sigma\}$. Let

$$\langle E(\cdot)y, y\rangle = \langle E_m(\cdot)x, x\rangle = \mu.$$

Then there exist isometric isomorphisms U_1, U_2 taking $L_2(\mu)$ onto $M(y)$ and $L_2(\mu)$ onto $\text{clm}\,\{E_m(\delta)x : \delta \in \Sigma\}$, respectively, such that

$$U_1^{-1}E_m(\delta)U_1 f = \chi_\delta f,$$
$$U_2^{-1}E_m(\delta)U_2 f = \chi_\delta f,$$

where $f \in L_2(\mu)$, $\delta \in \Sigma$ [**5**, p. 95]. Therefore $E_m(\delta) = VE(\delta)V^{-1}$, where $V = U_1 U_2^{-1}$, is an isometric isomorphism of N onto Y. $V \,|\, N$ is unitary. Thus, if $A \in \mathscr{A}^w$, there is an $f \in C(\Omega)$ such that $A = \int_\Omega f(\lambda)E(d\lambda)$, by Lemma 1, and this implies that

$$\begin{aligned}\langle Az_1, z_2\rangle &= \int_\Omega f(\lambda)\, d\langle E(\lambda)z_1, z_2\rangle \\ &= \int_\Omega f(\lambda)\, d\langle V^{-1}E_m(\lambda)Vz_1, z_2\rangle \\ &= \int_\Omega f(\lambda)\, d\langle E_m(\lambda)Vz_1, Vz_2\rangle \\ &= \langle A_m Vz_1, Vz_2\rangle = \langle V^{-1}A_m Vz_1, z_2\rangle,\end{aligned}$$

for all $z_1, z_2 \in H$. Therefore $A_m = VAV^{-1}$.

LEMMA 3. *Suppose that B in $L(H)$ leaves invariant every closed invariant subspace of $\mathscr{A}$. Then $B \in \mathscr{A}^w$.*

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Proof. By hypothesis, B commutes with every projection that commutes with $\mathscr{A}^w$. Let $S \in (\mathscr{A}^w)'$ and suppose that $S = S^*$. Then, if $T \in \mathscr{A}^w$, Fuglede's theorem tells us that T commutes with all the spectral projections of S and hence all these lie in $(\mathscr{A}^w)'$. Therefore, by the spectral theorem, $BS = SB$. Since any operator $S \in (\mathscr{A}^w)'$ can be expressed as a linear combination of self-adjoint operators in $(\mathscr{A}^w)'$, namely $S = \frac{1}{2}(S+S^*)+i\{(1/2i)(S-S^*)\}$, it follows that B commutes with every operator in $(\mathscr{A}^w)'$. Therefore $B \in (\mathscr{A}^w)'' = \mathscr{A}^w$. This completes the proof.

Let $\mathscr{R}$ be a von Neumann algebra; then x is said to be a separating vector for $\mathscr{R}$ if and only if $A \in \mathscr{R}$, $Ax = 0$ implies that $A = 0$. If E is a projection in $\mathscr{R}$, then E is said to be countably decomposable in $\mathscr{R}$ if and only if every orthogonal family $\{E_\alpha\} \subseteq \mathscr{R}$ of nonzero subprojections of E is at most countable. $\mathscr{R}$ is said to be countably decomposable if and only if I is countably decomposable in $\mathscr{R}$.

Now, since $\mathscr{P}$ is complete, we can define carrier projections in $\mathscr{P}$ thus:

$$C(x) = \bigwedge\{E \colon E \in \mathscr{P},\ Ex = x\}$$

is the carrier projection of x.

A subset $\mathscr{D}$ of $\mathscr{P}$ is said to be an ideal if and only if (1) E, $F \in \mathscr{D}$ implies that $E \vee F \in \mathscr{D}$, (2) $G \leqq H$, $H \in \mathscr{D}$ implies that $G \in \mathscr{D}$. A σ-ideal is an ideal closed under countable unions, and an ideal is dense if and only if every element of $\mathscr{P}$ is a union of elements of $\mathscr{D}$. Now let $\mathscr{C}$ be the set of countably decomposable elements in $\mathscr{P}$. Then $\mathscr{C}$ is a dense σ-ideal and a projection in $\mathscr{P}$ belongs to $\mathscr{C}$ if and only if it is the carrier projection of a vector in H [**4**, p. 2266].

We are now in a position to prove the main lemma in this section.

LEMMA 4. *Suppose that B in $L(H)$ leaves invariant every closed invariant subspace of $\mathscr{A}$. Then B_m leaves invariant every closed invariant subspace of $\mathscr{A}_m$ ($m = 1, 2, 3, \ldots$).*

Proof. Let $x = (x_1, x_2, \ldots, x_m) \in H_m$ and let Y be the smallest closed reducing subspace of $\mathscr{A}_m$ containing x. Consider the projection $\bigvee_{i=1}^{m} C(x_i)$ with range M, say. Then $\bigvee_{i=1}^{m} C(x_i)$ is countably decomposable, since $\mathscr{C}$ is a σ-ideal. Also, M is invariant under $E(\cdot)$ and hence under $\mathscr{A}^w$. Therefore $\mathscr{A}^w \mid M$ is a countably decomposable commutative von Neumann algebra over H. Hence $\mathscr{A}^w \mid M$ has a separating vector $\tilde{x}$ in H [7, p. 30]. Let $\tilde{E}(\cdot) = E(\cdot) \mid M$. Then $\tilde{E}(\partial)\tilde{x} = 0 \Rightarrow \tilde{E}(\partial) = 0$ $(\partial \in \Sigma)$. Hence

$$\begin{aligned}\langle \tilde{E}(\partial)\tilde{x}, \tilde{x}\rangle = 0 &\Rightarrow \tilde{E}(\partial)\tilde{x} = 0\\ &\Rightarrow \tilde{E}(\partial) = 0\\ &\Rightarrow \langle E(\partial)x_i, x_i\rangle = 0 \quad (i = 1, \ldots, m)\\ &\Rightarrow \langle E_m(\partial)x, x\rangle = 0.\end{aligned}$$

Hence the measure $\langle E_m(\cdot)x, x\rangle$ is absolutely continuous with respect to the measure $\langle E(\cdot)\tilde{x}, \tilde{x}\rangle$. So, by [**5**, p. 95, p. 104], there exists a vector y in $M(\tilde{x})$ such that $\langle E(\cdot)y, y\rangle = \langle E_m(\cdot)x, x\rangle$. Let N be the smallest closed reducing subspace for $\mathscr{A}$ containing the vector y. Then, by Lemma 2, for any A in $\mathscr{A}^w$, $A_m \mid Y$ and $A \mid N$ are unitarily equivalent via an isometry V of N onto Y such that $A_m \mid Y = VAV^{-1}$. Since, by Lemma 3, B is in $\mathscr{A}^w$, then $B_m \mid Y = VBV^{-1}$. Hence V maps closed invariant subspaces of B onto closed invariant subspaces of $B_m \mid Y$.

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Let L be the smallest closed subspace of H_m invariant under $\mathscr{A}_m$ and containing x. Then $V^{-1}L$ is invariant under $\mathscr{A}$ and hence also under B. Hence L is invariant under B_m. If now Y_0 is an arbitrary closed subspace of H_m containing x and invariant under $\mathscr{A}_m$, then $B_m x \in L \subseteq Y_0$. Therefore $B_m Y_0 \subseteq Y_0$.

Proof of Theorem 1. Let $x_1, \ldots, x_m, y_1, \ldots, y_m$ be unit vectors in H and let $\varepsilon > 0$ be given. Define U to be the set of all operators T in $L(H)$ such that

$$|\langle Tx_j, y_j\rangle - \langle Bx_j, y_j\rangle| < \varepsilon \quad (j = 1, \ldots, m).$$

Then U is a neighbourhood of B and the family of all such U is a base of neighbourhoods of B in the weak operator topology. It remains only to prove that U contains an element of $\mathscr{A}$.

Put

$$x = (x_1, \ldots, x_m) \in H_m, \quad \mathscr{A}x = \{Ax \colon A \in \mathscr{A}\}, \quad \mathscr{A}_m x = \{A_m x \colon A \in \mathscr{A}\}.$$

Consider the closed linear subspace $\operatorname{clm} \mathscr{A}_m x$. This subspace is invariant under $\mathscr{A}_m$ and so it is invariant under B_m. Therefore $B_m x \in \operatorname{clm} \mathscr{A}_m x$. Hence there exists an element A in $\mathscr{A}$ such that $\|A_m x - B_m x\| < \varepsilon$. Hence

$$\|Ax_i - Bx_i\| < \varepsilon \quad (i = 1, \ldots, m)$$

and so

$$|\langle Ax_i, y_i\rangle - \langle Bx_i, y_i\rangle| \leqq \|Ax_i - Bx_i\| \, \|y_i\| < \varepsilon \quad (i = 1, \ldots, m).$$

Therefore A is in U and so B is in $\mathscr{A}$.

REMARK: It was noted in [**6**] that, in the enunciation of Theorem 1, the word " commutative " is unnecessary. This follows from the fact that, if $\mathscr{U}$ is a linear space of normal operators, then $\mathscr{U}$ is commutative. For, if $A, B \in \mathscr{U}$, then

$$2(B^*A - AB^*)$$
$$= (A+B)^*(A+B) - (A+B)(A+B)^* + i\{(A+iB)^*(A+iB) - (A+iB)(A+iB)^*\} = 0.$$

Hence, by Fuglede's theorem, $AB = BA$.

3. A Result of Scroggs.

DEFINITION. A normal operator is said to have *property* (P) if and only if every closed invariant subspace of the operator is also reducing for the operator.

Scroggs proved the following result [**11**].

THEOREM 2. *If T is a normal operator and if* $\operatorname{int} \sigma(T) \neq \emptyset$, *then property* (P) *fails for* T.

A direct proof of this result is given here, based on a lemma of Sarason [**10**, Lemma 1]. We require a preliminary lemma, the proof of which is given for completeness.

LEMMA 5. *Let H be a Hilbert space and let T be a bounded normal operator on H. Then there is a closed separable reducing subspace K for T such that $\sigma(T/K) = \sigma(T)$.*

Proof. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a countable dense subset of the complex plane, containing a dense subset of $\sigma(T)$. Let λ_k be a particular element of the sequence. With λ_k as centre construct a sequence of open discs, say $\{S_j\}_{j=1}^{\infty}$, so that, if r_j is the radius of S_j, $\lim_{j\to\infty} r_j = 0$. For each disc,

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choose a vector x_{kj} belonging to the range of the projection $E(S_j)$, where $E(\cdot)$ is the resolution of the identity for the bounded normal operator T. If $S_j \cap \sigma(T) = \emptyset$, then $x_{kj} = 0$; otherwise $x_{kj} \neq 0$. In this way we obtain an infinite sequence of vectors $\{x_{kj}\}_{j=1}^{\infty}$ associated with λ_k. Repeating this process for each λ_k $(k = 1, 2, \ldots)$, we obtain a doubly indexed sequence of vectors $\{x_{ij}\}_{i,j=1}^{\infty}$. The cycle generated by each x_{ij}, i.e., the subspace spanned by the $E(M)x_{ij}$ for each Borel set M, is a separable subspace of H [**3**, Corollary 2.4]. Hence the countable union of such cycles is separable. Let K be the subspace spanned by these cycles. Then K is separable. Since each of these cycles is reducing for T, it follows from the linearity and continuity of T that K reduces T. Finally, we show that $\sigma(T|K) = \sigma(T)$. It suffices to show that $\sigma(T) \subseteq \sigma(T|K)$. Suppose that $\lambda_k \in \sigma(T)$. Then, given any neighbourhood $N(\lambda_k)$ of λ_k, some $S_i(\lambda_k)$ has the following properties:

$$(1)\ S_i(\lambda_k) \subseteq N(\lambda_k), \qquad (2)\ S_i(\lambda_k) \cap \sigma(T) \neq \emptyset.$$

By definition, there is an $x_{ki} \neq 0$ such that $E(S_i)x_{ki} = x_{ki}$. But this means that $S_i(\lambda_k)$ and hence $N(\lambda_k)$ contains a point of the set $\sigma(T|K)$. If this point is not λ_k, then this shows that λ_k is a limit point of $\sigma(T|K)$ and hence $\lambda_k \in \sigma(T|K)$, since this set is closed. Hence $\sigma(T) \subseteq \sigma(T|K)$, since $\sigma(T)$ has a dense subset consisting of points λ_k, and the proof is complete.

In the proof of Theorem 2 we shall use a special case of Theorem 1, namely the following.

T has property (P) if and only if T^* is in the closed subalgebra of $L(H)$, generated by I and T, in the weak operator topology.

In the following proof, $E(\cdot)$ will be the resolution of the identity for T and for x in H, $M(x)$ will denote the closed linear subspace generated by the $E(M)x$, for each Borel subset M of the complex plane.

Proof of Theorem 2. By Lemma 5 there exists a separable closed subspace Y of H which is reducing for T and such that $\sigma(T|Y) = \sigma(T)$. Let $\tilde{x}$ be a separating vector for the spectral measure $E(\cdot)|Y$ [**3**, Theorem 2.7]. Now $\sigma(T|M(\tilde{x}))$ is the support of $E(\cdot)|M(\tilde{x})$ and this is the same as the support of $E(\cdot)|Y$; so $\sigma(T|M(\tilde{x})) = \sigma(T)$.

Define $\mu(\cdot) = \langle E(\cdot)\tilde{x}, \tilde{x}\rangle$. Then $\mu(\cdot)$ is a positive measure with compact support; $\operatorname{supp}\mu = \sigma(T|M(\tilde{x})) = \sigma(T)$.

We now suppose that T has property (P) and obtain a contradiction. From the special case of Theorem 1 we see that T^* belongs to the weak closure of polynomials in T, i.e., there exists a net of polynomials $\{p_\alpha\}$ such that

$$\lim_\alpha \langle p_\alpha(T)x, y\rangle = \langle T^*x, y\rangle \quad \text{for all } x, y \text{ in } H.$$

Put $S = T|M(\tilde{x})$. Then S is a normal operator and $S^* = T^*|M(\tilde{x})$; hence

$$\lim_\alpha \langle p_\alpha(S)x, y\rangle = \langle S^*x, y\rangle \quad \text{for all } x, y \text{ in } M(\tilde{x}).$$

Now, by [**4**, p. 95], there exists an isometric isomorphism U of $L_2(\mu)$ onto $M(\tilde{x})$ with the property that $U^{-1}E(M)Uf = \chi_M f$, for all Borel sets M and all f in $L_2(\mu)$. We have

$$\begin{aligned}\langle p_\alpha(S)x, y\rangle &= \langle p_\alpha(S)Uf, Ug\rangle \quad \text{for some } f, g \text{ in } L_2(\mu)\\ &= \int_{\sigma(\tau)} p_\alpha(\lambda)\, d\langle E(\lambda)Uf, Ug\rangle.\end{aligned}$$

Now

$$\langle E(M)Uf, Ug\rangle = \langle UU^{-1}E(M)Uf, Ug\rangle = \langle U^{-1}E(M)Uf, g\rangle = \langle \chi_M f, g\rangle = \int_M f\bar{g}\, d\mu,$$

for all Borel sets M. Hence,

$$\langle p_\alpha(S)x, y\rangle = \int_{\sigma(\tau)} p_\alpha(\lambda)f(\lambda)\overline{g(\lambda)}\, d\mu(\lambda).$$

So

$$\lim_\alpha \int p_\alpha f\bar{g}\, d\mu = \int \bar{z} f\bar{g}\, d\mu, \quad \text{for all } f,\ g \text{ in } L_2(\mu).$$

Hence

$$\lim_\alpha \int p_\alpha h\, d\mu = \int \bar{z} h\, d\mu, \quad \text{for all } h \text{ in } L_2(\mu),$$

and therefore $\bar{z}$ is in the weak-star closure of polynomials in $L^\infty(\mu)$, thus contradicting [**9**, Lemma 1]. For completeness we show how the contradiction arises.

Since $\text{int}(\text{supp}\,\mu) = \text{int}(\sigma(T) \neq \emptyset$ we consider the set M of functions holomorphic in $G = \text{int}(\sigma(T))$. We show that M is closed in the weak-star topology of $L^\infty(\mu)$. We need only show that M is weak-star sequentially closed [**2**, p. 124]. By considering a sequence $\{f_n\}$ converging weak-star in $L^\infty(\mu)$ to f we see that $\{f_n\}$ is bounded in $L^\infty(\mu)$ [**2**, p. 123]. Hence it is uniformly bounded in G. By Montel's theorem [**8**, p. 272], $\{f_n\}$ has a subsequence which converges uniformly on compact subsets of G to the function g, say, where g is holomorphic in G. Hence $f = g$ a.e. (μ) in G. Thus $f \in M$. Therefore M is closed in the weak-star topology of $L^\infty(\mu)$ and this of course shows that $\bar{z}$ does not belong to the weak-star closure of polynomials in $L^\infty(\mu)$, since $\bar{z} \notin M$.

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