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INEQUALITIES IN DISCRETE SUBGROUPS OF $PSL(2, \mathbf{R})$

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1. Introduction. Conditions for a subgroup, F, of $PSL(2, \mathbf{R})$ to be discrete have been investigated by a number of authors. Jørgensen's inequality [**5**] gives an elegant necessary condition for discreteness for subgroups of $PSL(2, \mathbf{C})$. Purzitsky, Rosenberger, Matelski, Knapp, and Van Vleck, among others [**12**, **13**, **14**, **9**, **16**, **17**, **18**, **19**, **20**, **7**, **21**] studied two generator discrete subgroups of $PSL(2, \mathbf{R})$ in a long series of papers. The complete classification of two generator subgroups was surprisingly complicated and elusive. The most complete result appears in [**20**].

In this paper we use the results of [**20**] to prove that a non-elementary subgroup F of $PSL(2, \mathbf{R})$ is discrete if and only if every non-elementary subgroup, G, generated by two hyperbolics is discrete (Theorem 5.2) and that F contains no elliptics if and only if each such G is free (Theorem 5.1). Thus, we produce necessary and sufficient conditions for a non-elementary subgroup F of $PSL(2, \mathbf{R})$ to be a discrete group without elliptic elements (Theorem 6.1) or a discrete group containing only hyperbolic elements (Theorem 7.1). The conditions are that for each pair of hyperbolic elements one of three inequalities in their multipliers and their cross ratio are satisfied. Unlike many results in this area, the inequalities are independent of any normalization, and thus may be more useful. For example, these results show us how to construct three hyperbolic transformations A, B and D for which B and D have the same multiplier where A and B could never lie in a compact surface group, but A and D could.

If F is a discrete group and $A \in F$ is a hyperbolic transformation, we are interested in understanding when the group generated by F and an n-th root of A is discrete because of its connection with conformal automorphisms of Riemann surfaces. (See [**4**].) Our main results enable us to obtain answers for some special cases.

Notation and normalization, which we eventually eliminate, are established in Sections 2 and 3. Preliminary lemmas appear in Section 3. Section 4 summarizes Rosenberger's most recent results. General results about discreteness appear in Section 5; the main results appear in Sections 6 and 7 and adjoining roots is discussed in Section 8.

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In addition to providing inequalities which give necessary and sufficient conditions for discreteness, this work provides a conceptual basis for constructing examples. In a subsequent paper we show that the inequalities imply Jørgensen's inequality in the hyperbolic case and use the conceptual framework to construct counterexamples to results claimed in **[9]**.

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2. Preliminaries. If A and B are matrices in $SL(2, \mathbf{R})$, the group of 2×2 real matrices with determinant equal to 1, and I is the identity matrix, then

$$PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\langle I, -I \rangle.$$

Here $\langle x_1, \ldots, x_n \rangle$ denotes the group generated by $x_1, \ldots, x_n$. An element $Q \in PSL(2, \mathbf{R})$ is a pair $Q = \{A, -A\}$ where $A \in SL(2, \mathbf{R})$, and we also write A for the element of $PSL(2, \mathbf{R})$ when no confusion is apt to arise.

A subgroup G of $PSL(2, \mathbf{R})$ is *elementary* if the commutator of any two elements of infinite order has trace equal to 2 and is non-elementary otherwise. A Fuchsian group is a non-elementary discrete subgroup of $PSL(2, \mathbf{R})$.

We let $[A, B] = ABA^{-1}B^{-1}$ denote the group commutator of A and B and $\operatorname{tr} A$ the trace of A.

3. Notation, normalization and definitions. While our final results are independent of any normalization, we begin with some normalizations. If g and h are hyperbolic transformations then, after a conjugation, we may assume that g fixes 0 and ∞ with ∞ the attracting fixed point. We let ξ_1 and ξ_2 be the fixed points of h and assume that $\xi_1 < \xi_2$. If the axes of g and h intersect, but are not parallel, then $\xi_1 < 0$ and $\xi_2 > 0$. If the axes are disjoint, then either

(1) $\quad \xi_1 < \xi_2 < 0 \quad$ or

(2) $\quad 0 < \xi_1 < \xi_2$.

We further make the normalization that ξ_2 is the attracting fixed point of h. Thus, there are real numbers R and K with $R > 1$ and $K > 1$ such that

$$g = \begin{bmatrix} \sqrt{R} & 0 \\ 0 & \sqrt{R^{-1}} \end{bmatrix}$$

and

$$h = \frac{1}{\sqrt{K}(\xi_1 - \xi_2)} \begin{bmatrix} \xi_1 - K\xi_2 & \xi_1\xi_2(K-1) \\ 1 - K & K\xi_1 - \xi_2 \end{bmatrix}.$$

Further, if $\alpha \in \{1, 2, \dots,\} \cup \{1/2, 1/3, 1/4, \dots\}$, we have

$$h^\alpha = \frac{1}{\sqrt{K^\alpha}(\xi_1 - \xi_2)} \begin{bmatrix} \xi_1 - K^\alpha\xi_2 & \xi_1\xi_2(K^\alpha - 1) \\ 1 - K^\alpha & K^\alpha\xi_1 - \xi_2 \end{bmatrix}$$

and

$$g^\alpha = \begin{bmatrix} \sqrt{R^\alpha} & 0 \\ 0 & \sqrt{R^{-\alpha}} \end{bmatrix}.$$

Note that under our normalization a transformation has a multiplier greater than one and its inverse has one less than one.

Let the cross ratio of any four points, z_1, z_2, z_3, z_4, be denoted by $C[z_1, z_2, z_3, z_4]$. Then

$$C[z_1, z_2, z_3, z_4,] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

If A and B are any transformation with V_A and V_B their attracting fixed points and W_A and W_B their repelling fixed points, then one can form

$$C[V_A, V_B, W_A, W_B] \quad \text{and} \quad C[V_B, V_A, W_B, W_A]$$

and these are equal. We therefore write $C(A, B)$ for these numbers and call it the *cross ratio of A and B*. Then axes of A and B which we denote Ax_A and Ax_B intersect but are not parallel if and only if $C(A, B) < 0$; $C(A, B) = 1$ if A or B is parabolic (i.e., $V_A = W_A$ or $V_B = W_B$); if A and B have a common fixed point, then $C(A, B) = 0$ or ∞; in all other cases, $C(A, B) > 0$. When $C(A, B) > 0$, we set

$$C^\delta(A, B) = \begin{cases} C(A, B) & \text{if } 0 < C(A, B) < 1 \\ \dfrac{1}{C(A, B)} & \text{if } \infty > C(A, B) > 1. \end{cases}$$

Note that for g and h as above, $C(g, h) = \xi_1/\xi_2$ and if $C(g, h) > 0$, then

$$C^\delta(g, h) = \begin{cases} \xi_1/\xi_2 & 0 < \xi_1 < \xi_2 \\ \xi_2/\xi_1 & \xi_1 < \xi_2 < 0. \end{cases}$$

If R, K and α are as above, let

$$Q_\alpha^2(R, K) = \left(\frac{\sqrt{R^\alpha} + \sqrt{K}}{\sqrt{R^\alpha K} + 1}\right)^2$$

and

$$S_\alpha^2(R, K) = \left(\frac{\sqrt{R^\alpha} - \sqrt{K}}{\sqrt{R^\alpha K} - 1}\right)^2.$$

Note that since $R > 1$, $K > 1$, $S_\alpha^2(R, K)$ is finite. When R and K and α are determined by the matrices of g^α and h as above, we sometimes write $Q_\alpha^2(g, h)$ and $S_\alpha^2(g, h)$ or Q_α^2 and S_α^2 when g and h or K and R are clear from the context. We sometimes write Q and S respectively for Q_1 and S_1 respectively. Finally, define the function

$$f(x) = \frac{x}{(x-1)^2} \quad \text{and} \quad T_\alpha(g, h) = T_\alpha(R, K) = f(R^\alpha)f(K).$$

Remark. We note that $S_\alpha^2(g, h)$, $Q_\alpha^2(g, h)$ and $C(g, h)$ are conjugation invariant since R and K are. Thus, these quantities are independent of any normalization.

We have the following lemmas:

LEMMA 3.1. *For fixed g and h and for any $\theta, \delta \in \{1, 2, \ldots, \} \cup \{1/2, 1/3, \ldots\}$ if $\theta \leqq \delta$, assume $R > K$ with N_0 an integer satisfying $K^{N_0 - 1} \leqq R \leqq K^{N_0}$. Then*

$$S_\theta^2 \leqq S_\delta^2$$

$$Q_\theta^2 \geqq Q_\delta^2$$

$$T_\delta \leqq T_\theta$$

and

$$S_\theta^2 \leqq Q_\theta^2 \leqq 1 \ \forall\ \theta.$$

Also,

$$S_{\tilde{N}_0}^2(K, R) \leqq S_\theta^2(K, R) \ \forall\ \theta$$

$$S_\theta^2(K, R) \geqq S_\delta^2(K, R) \quad \text{if } \theta \leqq \delta \leqq \tilde{N}_0$$

$$S_\theta^2(K, R) \leqq S_\delta^2(K, R) \quad \text{if } \tilde{N}_0 \leqq \theta \leqq \delta$$

where $\tilde{N}_0 = N_0$ or $N_0 - 1$ is chosen so that

$$S_{\tilde{N}_0}^2(K, R) = \min\{S_{N_0}^2, S_{N_0 - 1}^2\}.$$

Proof. Using the fact that $K \geqq 1$ and $R \geqq 1$, it is straightforward to verify that S^2 is an increasing function of R for fixed K, etc. Here $\tilde{N}_0$ is chosen so that

$$S_{\tilde{N}_0}^2(K, R) \leqq S_{N_0}^2(K, R) \quad \text{and} \quad S_{\tilde{N}_0}^2(K, R) \leqq S_{N_0 - 1}^2(K, R).$$

LEMMA 3.2. *For g and h as above, the tables below give necessary and sufficient condition for the various trace inequalities. (E.g., the table is to be read as saying that if $\xi_1 < \xi_2 < 0$, then* tr $g^\alpha h \leqq -2$ *if and only if* $\xi_1/\xi_2 \leqq 1/Q_\alpha^2$.)

TABLE 1

	tr $g^\alpha h \geqq 2$	tr $g^\alpha h \leqq -2$
$0 < \xi_1 < \xi_2$	$\xi_1/\xi_2 \leqq \dfrac{1}{S_\alpha^2}$	$\xi_1/\xi_2 \geqq \dfrac{1}{Q_\alpha^2}$
$\xi_1 < \xi_2 < 0$	$\xi_1/\xi_2 \geqq \dfrac{1}{S_\alpha^2}$	$\xi_1/\xi_2 \leqq \dfrac{1}{Q_\alpha^2}$

TABLE 2

$0 < \xi_1 < \xi_2$	tr $g^{-\alpha}h \geqq 2$	tr $g^{-\alpha}h \leqq -2$
$\xi_1 < \xi_2$	$\xi_1/\xi_2 \leqq S_\alpha^2$	$\xi_1/\xi_2 \geqq Q_\alpha^2$
$\xi_1 < \xi_2 < 0$	$\xi_1/\xi_2 \geqq S_\alpha^2$	$\xi_1/\xi_2 \leqq Q_\alpha^2$

Proof. Again, these come from tedious but straightforward calculations.

Let $[A, B]$ denote the commutator of A and B. We have

LEMMA 3.3.

$$\begin{aligned}\mathrm{Tr}[g, h] &= 2 + \frac{\xi_1\xi_2}{(\xi_1 - \xi_2)^2} \cdot \frac{(K - 1)^2}{K} \cdot \frac{(R - 1)^2}{R} \\ &= 2 + \frac{C(K - 1)^2}{(C - 1)^2 K} \cdot \frac{(R - 1)^2}{R}.\end{aligned}$$

Proof. This is a calculation.

COROLLARY 3.4. *If $Ax_g \cap Ax_h \neq \emptyset$, then* $\mathrm{tr}[g, h] \leqq 2$.

$$\mathrm{Tr}[g, h] \leqq -2 \Leftrightarrow f(C) \leqq -4f(R)f(K).$$

Proof. We have $\xi_1 < 0$ and $\xi_2 > 0$. Thus $\xi_1\xi_2$ is negative. All other terms are positive. Since

$$\frac{\xi_1\xi_2}{(\xi_1 - \xi_2)^2} = \frac{\xi_1/\xi_2}{(\xi_1/\xi_2 - 1)^2} = \frac{C}{(C - 1)^2},$$

the corollary follows.

4. A summary of Rosenberger's results. Because we refer frequently to results of Rosenberger [**20**], we summarize some notation and state the results here.

Recall that an *elementary Nielsen transformation* sends $\{A, B\}$ to $\{A^N B, A\}$ for some integer $N \neq 0$ and a *free-Nielsen transformation of the*

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pair $\{A, B\}$ is any finite combination of elementary Nielsen transformation and their conjugates over the group $\langle A, B\rangle$. (See [**14**]).

Following Rosenberger [**20**], we write

$$\{A, B\} \overset{N}{\sim} \{U, V\}$$

if there is a free-(Nielsen) transformation from $\{A, B\}$ to $\{U, V\}$ and call the two pairs *Nielsen equivalent*. If A is of finite order $n \geqq 2$, the transformation which sends $\{A, B\}$ to the pair $\{A^m, B\}$ with $1 \leqq m \leqq n$ where m and n are relatively prime is called an *E-transformation*. If $\{U, V\}$ is obtained from $\{A, B\}$ by a finite sequence of E-transformations and free-(Nielsen) transformations, we write

$$\{A, B\} \overset{eN}{\sim} \{U, V\}.$$

Rosenberger obtains the following results [**20**].

THEOREM 4.1R (Rosenberger). *A two-generator Fuchsian group G has one and only one of the following description in terms of generators and relations*:

(1.1) $G = \langle A, B\rangle$, *that means G is a free group of rank two.*

(1.2) $G = \langle A, B | A^P = 1\rangle$ *for* $2 \leqq p$.

(1.3) $G = \langle A, B | A^P = B^q = 1\rangle$ *for* $2 \leqq p \leqq q$ *and* $p + q \geqq 5$.

(1.4) $G = \langle A, B | A^P = B^q = (AB)^r = 1\rangle$ *for*

$$2 \leqq p \leqq q \leqq r \text{ and } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

(1.5) $G = \langle A, B | [A, B]^P = 1\rangle$ *for* $2 \leqq p$.

(1.6) $G = \langle A, B, C | A^2 = B^2 = C^2 = (ABC)^p = 1\rangle$ *for*

$$p = 2k + 1, k \geqq 1.$$

Remark. In (1.6) is $G = \langle AB, CA\rangle$ and $[AB, CA] = (ABC)^2$.

THEOREM 4.2R (Rosenberger). *Let* $G = \langle U, V\rangle$ *be a two-generator Fuchsian group.*

(2.1) *If G is of type* (1.1) *or* (1.5), *then*

$$\{U, V\} \overset{N}{\sim} \{A, B\}.$$

(2.2) *If G is of type* (1.6), *then*

$$\{U, V\} \overset{N}{\sim} \{AB, CA\}.$$

(2.3) *If G is of type* (1.2) *or* (1.3), *then*

$$\{U, V\} \overset{eN}{\sim} \{A, B\}.$$

(2.4) *If G is of type* (1.4), *then one* (*and only one*) *of the following cases holds*:

a) $\{U, V\} \overset{eN}{\sim} \{A, B\}$.

b) *G is a* (2, 3, *r*)-*triangle group with* $(r, 6) = 1$ *and*

$$\{U, V\} \overset{N}{\sim} \{ABAB^2, B^2ABA\}.$$

c) *G is a* (2, 4, *r*)-*triangle group with* $(r, 2) = 1$ *and*

$$\{U, V\} \overset{N}{\sim} \{AB^2, B^3AB^3\}.$$

d) *G is a* (3, 3, *r*)-*triangle group with* $(r, 3) = 1$ *and*

$$\{U, V\} \overset{N}{\sim} \{AB^2, B^2A\}.$$

e) *G is a* (2, 3, 7)-*triangle group and*

$$\{U, V\} \overset{N}{\sim} \{AB^2ABAB^2AB^2AB, B^2ABAB^2ABABA\}.$$

5. Discreteness and two generator subgroups. It is well known that a group F is discrete if and only if every two generator subgroup is discrete. We will establish the slightly stronger result that it suffices to consider those non-elementary subgroups generated by two hyperbolics (Theorem 5.2).

We are interested in two generator subgroups of F when F contains no elliptic elements. In that case if F is Fuchsian, it is a one relator group and from the Freiheitsätz (see [**10**, p. 252]) one would expect most, but not all, two generator subgroups of F to be free. However, it turns out that the stronger result, that all two generator subgroups are free, holds (Theorem 5.3).

The aim of this section is to prove

THEOREM 5.1. *Let F be a non-elementary subgroup of* $PSL(2, \mathbf{R})$. *F is discrete and contains no elliptic elements if and only if for every hyperbolic* $g, h \in F$, $\langle g, h\rangle$ *is discrete and free when it is non-elementary.*

This will follow from Theorems 5.2, 5.3 and 5.4.

THEOREM 5.2. *Let F be a non-elementary subgroup of* $PSL(2, \mathbf{R})$. *F is discrete if and only if every non-elementary subgroup generated by two hyperbolics is discrete.*

THEOREM 5.3. *If F is a non-elementary Fuchsian group without elliptic elements, then for any two g and h in F,* $G = \langle g, h\rangle$ *is a free group.*

THEOREM 5.4. *If F is non-elementary and discrete and every non-elementary subgroup generated by two hyperbolics is free, then F contains no elliptic elements.*

We begin with

LEMMA 5.5. *If F is a non-elementary subgroup of $PSL(2, \mathbf{R})$, then given any $g \in F$, there exists a hyperbolic $h \in F$ such that $\langle g, h\rangle$ is non-elementary.*

Proof. If F is non-elementary, then F contains at least two hyperbolics h_1 and h_2 without a common fixed point since F is generated by hyperbolics. (See [**19**] or [**2**] for the latter fact.) We may assume that the axes of h_1 and h_2 are disjoint by replacing h_1 by $h_2h_1h_2^{-1}$ if necessary for

$$Ax_{h_2h_1h_2^{-1}} = h_2(Ax_{h_1})$$

and is disjoint from Ax_{h_1}. Let g be any element of F. Look at $G = \langle g, h_1\rangle$. If this group is elementary, then by Beardon's work (see Sections 5.1 and 5.2 of [**1**]), we know that either G contains only elliptics, is conjugate to a group all of whose elements fix ∞, or is conjugate to a group all of whose elements are either elliptic fixing i or hyperbolic fixing the imaginary axis. Since h_1 is hyperbolic, case 1 is ruled out. In the second case, since h_1 and h_2 have different fixed points, if $\langle g, h_1\rangle$ is elementary then g and h_1 have a common fixed point. If g is parabolic, then g and h_2 have no common fixed point and $\langle g, h_2\rangle$ is a non-elementary. If g is elliptic, $\langle g, h_1\rangle$ is not case 2. If g is hyperbolic and $\langle g, h_1\rangle$ and $\langle g, h_2\rangle$ are elementary, then if $h_3 = h_1h_2$, $\mathrm{tr}[h_1, h_3] \neq 2$ and $\mathrm{tr}[h_2, h_3] \neq 2$. Then h_3 and g have no common fixed points, so $\langle g, h_3\rangle$ is non-elementary. If $\langle g, h_1\rangle$ is the third case and g is elliptic, then the fixed point of g lies on the axis of h_1 and this property is preserved under conjugation. Since

$$Ax_{h_1} \cap Ax_{h_2} = \emptyset,$$

$\langle g, h_2\rangle$ is non-elementary. If g is hyperbolic, then if $\langle g, h\rangle$ is always elementary, $\mathrm{tr}[g, h] = 2$ $\forall h$ in F and every hyperbolic h fixes either ξ_1 or ξ_2 where ξ_1 and ξ_2 are the fixed points of g. Since h_1 and h_2 have no common fixed points, h_1h_2 moves both ξ_1 and ξ_2. Thus $\langle g, h_1h_2\rangle$ is non-elementary.

We next prove Theorem 5.2.

Proof of Theorem 5.2. If F is discrete then clearly every two generator subgroup is discrete. Conversely, assume that every two generator non-elementary subgroup generated by hyperbolics is discrete. If suffices by Jørgensen's work [**6**] or by Rosenberger's work [**20**] (which does not use Jørgensen's inequality) to show that every cyclic subgroup is discrete. Let g be any element of F. Then by Lemma 5.5 $\exists h \in F$ such that $G = \langle g, h\rangle$ is a non-elementary group. As such it is generated by hyperbolics. G is either cyclic hyperbolic so that $\langle g\rangle = \langle g, h\rangle$ is discrete or G has rank 2. (See [**19**] for a definition of rank.) Further, if G has rank 2, by [**19**] again, it is generated by two hyperbolics. Thus, $\langle g\rangle \subset G = \langle g, h\rangle$. Since G is discrete by hypothesis, $\langle g\rangle$ is discrete.

Proof of Theorem 5.3. Let $G = \langle g, h\rangle$. Either G is cyclic and, therefore, free or G is a discrete two generator group. If G is elementary and discrete by [**11**, p. 563], it is either cyclic (and thus free) or dihedral. Since F contains no elliptics, G cannot be dihedral. If G is non-elementary then it must be one of the groups in the Rosenberger list in Theorem 4.1R or Theorem 4.2R. Every group in cases 1.2-1.6 contains non-identity elements of finite order. Since F contains no elements of finite order, we must be in case 1.1, where G is a free group.

Proof of Theorem 5.4. Suppose to the contrary that t is an elliptic element of G. Then by Lemma 5.5 there is a hyperbolic element h in F such that $G = \langle t, h\rangle$ is non-elementary. The argument in the proof of Theorem 5.3 shows that $G = \langle t, h\rangle = \langle h, f\rangle$ for some other hyperbolic f. By hypothesis then G is free. Thus $t \in G$ cannot be of finite order.

Proof of Theorem 5.1. The forward implication follows from Theorem 5.3. For the reverse implication observe first that by Theorem 5.2 if $\forall$ hyperbolic $g, h \in F$, $\langle g, h\rangle$ is discrete when it is not elementary, then F is discrete. Then apply Theorem 5.4 to conclude that F contains no elliptics.

6. The main theorem. The purpose of this section is to prove the following.

THEOREM 6.1. *Let F be a non-elementary subgroup of $PSL(2, \mathbf{R})$. Then F is a discrete group and contains no elliptic elements if and only if for each pair of hyperbolic elements g and h in F which generate a non-elementary subgroup of F one of the following conditions hold*:

(i) $-\infty < C(g, h) < 0$ *and* $f(C) \leqq -4f(R)f(K)$,

(ii) $0 < C^{\delta}(g, h) < 1$ *and* $C^{\delta}(g, h) \geqq Q^2$,

(iii) $0 < C^{\delta}(g, h) < 1$ *and* $C^{\delta}(g, h) \leqq S^2$.

Here, $C = C(g, h)$ denotes the cross ratio of the fixed points of g and h, $C^{\delta}(g, h) = C$ when $0 < C < 1$ and $1/C$ if $C > 1$,

$$f(x) = \frac{x}{(x-1)^2},$$

R and K are the multipliers of g and h respectively, $R > 1$ and $K > 1$,

$$Q^2 = Q^2(g, h) = Q^2(R, K) = \left(\frac{\sqrt{R} + \sqrt{K}}{\sqrt{RK} + 1}\right)^2$$

and

$$S^2 = S^2(g, h) = S^2(R, K) = \left(\frac{\sqrt{R} - \sqrt{K}}{\sqrt{RK} - 1}\right)^2.$$

We begin with

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LEMMA 6.2. *If F contains an elliptic e, e can always be written as the product of two hyperbolics that generate a non-elementary subgroup.*

Proof. By Lemma 5.5, given e $\exists h$ such that $\langle e, h\rangle$ is non-elementary. We claim that $\exists n$ such that $h^n e$ is hyperbolic. To see this, conjugate e and h so that h is a matrix of the form $\left[\begin{smallmatrix} T & 0 \\ 0 & T^{-1}\end{smallmatrix}\right]$ and

$$e = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\operatorname{tr} he = Ta + \frac{d}{T} \quad \text{and} \quad \operatorname{tr} h^n e = T^n a + \frac{d}{T^n}.$$

We may assume $T > 1$. If a and d are both zero, then $bc = -1$ and $e^2 = -I$ with e interchanging 0 and ∞. Then $\langle e, h\rangle$ would be elementary, contrary to assumption. We analyze each case: a and d are both positive; a is negative and d is positive; or a is positive and d is zero; etc. In all cases as a function of T, tr he is either appropriately decreasing or increasing so that $\exists n$ with tr $h^n e \geqq 2$ or $\leqq -2$. Finally, note that if $\langle h^n e, h\rangle$ is non-elementary, so is $\langle h^n e, h^{-n}\rangle$ and e is thus the product of two hyperbolics which generate a non-elementary group.

LEMMA 6.3. *If g and h are hyperbolic with disjoint axes, then $(g^i h^j)$ is not elliptic for $i = \pm 1, j = \pm 1$ if and only if either*

$$C^\delta(g, h) \geqq Q^2 \quad \textit{or} \quad C^\delta(g, h) \leqq S^2.$$

Proof. It is necessary and sufficient to show that

$$|\operatorname{tr} g^i h^j| \geqq 2.$$

We use the table of Lemma 3.2. If $0 < \xi_1 < \xi_2$, note that since $\xi_1/\xi_2 = C^\delta < 1$ and $1/S^2 \geqq 1$, tr $gh \geqq 2$ always and tr $g^{-1}h \geqq 2$ or $\leqq -2$ depending upon whether $C^\delta \leqq S^2$ or $C^\delta \geqq Q^2$. If $\xi_1 < \xi_2 < 0$,

$$C^\delta = \xi_2/\xi_1 < 1.$$

Thus since $\xi_1/\xi_2 > 1$ and $S^2 < 1$, we have tr $g^{-1}h \geqq 2$ always and tr $gh \geqq 2$ or $\leqq -2$ depending upon whether $\xi_1/\xi_2 \geqq 1/S^2$ or $\xi_1/\xi_2 \leqq 1/Q^2$. Since $\xi_1/\xi_2 = 1/C^\delta$, this reduces to $C^\delta \leqq S^2$ or $C^\delta \geqq Q^2$. Similarly, tr hg and tr $h^{-1}g$ are both greater than two in absolute value if and only if these inequalities hold. Use the fact that for a two by two matrix with determinant one, tr A = tr A^{-1} to conclude that

$$|\operatorname{tr} h^{\pm 1} g| \geqq 2 \text{ if and only if } |\operatorname{tr} g h^{\pm 1}| \geqq 2.$$

Note that the proof of Lemma 6.3 also shows the following.

COROLLARY 6.4. *If g and h are hyperbolic with disjoint axes, then* $\operatorname{tr} g^i h^j \leqq -2$ *for some* $i \in \{\pm 1\}$ *and* $j \in \{\pm 1\}$ *if and only if* $C^\delta(g, h) \geqq Q^2$.

We are now able to prove Theorem 6.1.

Proof. Assume that (i), (ii) or (iii) holds for all non-elementary two generator groups. If suffices for discreteness to prove that F contains no elliptics and by Lemma 6.2 it thus suffices to show that whenever $\langle g, h\rangle$ is non-elementary, $g^{\pm 1}h^{\pm 1}$ is never elliptic. By Lemma 6.3, (ii) and (iii) imply this. If (i) holds, then by Corollary 3.4, $\operatorname{tr}[g, h] \leqq -2$ and thus by Theorem 8 of **[12]**, $\langle g, h\rangle$ is a discrete free group and contains no elliptics.

To prove the converse, note that if F is discrete and $C < 0$, then by Corollary 3.4 $\operatorname{tr}[g, h] \leqq -2$, implies (i). By Lemma 3.3,

$$\operatorname{tr}[g, h] = 2 \Leftrightarrow C = 0.$$

This means that g and h have a common fixed point which cannot happen if $\langle g, h\rangle$ is non-elementary. Finally, if F is discrete, when $C > 0$, Lemma 6.3 implies that (ii) or (iii) hold.

Note that since g and h have no common fixed point (since $\langle g, h\rangle$ is non-elementary), C is never 0 or ∞.

Remark 6.5. Note that Theorem 1 of **[12]** taken together with Corollary 6.4 shows that (ii) actually implies that $\langle g, h\rangle$ is a discrete free group. (To see this, either apply Theorem 1 directly to $\langle g, h\rangle$ or to $\langle g^{-1}, h\rangle$ as appropriate.)

Example. Let

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}, \quad B = \begin{bmatrix} 7/2 & -3 \\ 3/2 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 13/2 & -9 \\ 3 & -4 \end{bmatrix}.$$

(Here we have chosen so that $R = 9$, $K = 4$, B fixes 1 and 2, and D fixes 3/2 and 2.) Then $C(A, B) = 1/2$, $C(A, D) = 3/4$, $Q(A, B) = Q(A, D) = 25/49$, $S(A, B) = S(A, D) = 1/25$. Then since $S(A, B) < C(A, B) < Q(A, B)$, A and B can never lie inside a discrete group containing no elliptics but A and D might.

Note that for this example $\langle A, B^2\rangle$ and $\langle A^2, B\rangle$ are both discrete free groups. To see this compute that

$$C(A, B) = 1/2 > Q^2(A^2, B) = 121/361, \quad \text{and}$$

$$C(A, B) = 1/2 > Q^2(A, B^2) = 49/169.$$

Then use Remark 6.5 that in this case, (ii) implies discreteness.

7. Purely hyperbolic groups. In this section we want to improve Theorem 6.1 to tell us when F contains no parabolics. We will show

THEOREM 7.1. *Let F be a non-elementary subgroup of $PSL(2, \mathbf{R})$. Then F is a discrete group containing only hyperbolics if and only if for all pairs of hyperbolic elements g and h in F which generate a non-elementary subgroup of F one of the following hold.*

(i) $-\infty < C(g, h) < 0$ *and* $f(C) < -4f(R)f(K)$

(ii) $0 < C^{\delta}(g, h) < 1$ *and* $C^{\delta}(g, h) > Q^2$ *or* $C^{\delta}(g, h) < S^2$.

The notation is explained in Theorem 6.1.

We begin with

LEMMA 7.2. *Let A and B be hyperbolics with no common fixed points. Let $\epsilon = \pm 1$ and $\rho = \pm 1$. Then $A^{\rho}B^{\epsilon}$ and $[A^{\rho}, B^{\epsilon}]$ are parabolic if and only if either*

$$C^{\delta}(A, B) = Q^2 \text{ and } 0 < C^{\delta}(A, B) < 1$$

$$C^{\delta}(A, B) = S^2 \text{ and } 0 < C^{\delta}(A, B) < 1 \quad \text{or}$$

$$f(C(A, B)) = -4f(R)f(K) \text{ and } -\infty < C(A, B) < 0.$$

Proof. Let A and B be normalized. In the case $C(A, B) > 0$, compute $\operatorname{tr} AB = \pm 2$ if and only if

$$(**) \quad \xi_1(\sqrt{R} \mp \sqrt{K})^2 = \xi_2(\sqrt{RK} \mp 1)^2.$$

If $\sqrt{R} = \sqrt{K}$, then $\xi_2 = 0$ contrary to the assumption that A and B have no common fixed points. Also $\sqrt{RK} - 1 \neq 0$ since $R > 1$, $K > 1$. Thus,

$$\frac{\xi_1}{\xi_2} = \frac{1}{Q^2} \quad \text{or} \quad \frac{\xi_1}{\xi_2} = \frac{1}{S^2}.$$

If we want to compute $\operatorname{tr} A^{-1}B$, we replace R by $1/R$ in (**). We obtain

$$\xi_1\left(\frac{1}{\sqrt{R}} \mp \sqrt{K}\right)^2 = \xi_2\left(\frac{\sqrt{K}}{\sqrt{R}} \mp 1\right)^2$$

which simplifies to

$$\xi_1\left(\frac{1}{\sqrt{R}}\right)\left(\frac{1 \mp \sqrt{RK}}{\sqrt{R}}\right)^2 = \xi_2\left(\frac{\sqrt{K} \mp \sqrt{R}}{\sqrt{R}}\right)^2.$$

Thus

$$\frac{\xi_1}{\xi_2} = Q^2 \quad \text{or} \quad \frac{\xi_1}{\xi_2} = S^2.$$

Since tr $A^{-1}B$ = tr AB^{-1} and tr AB = tr BA, the first part of the lemma follows for $A^\rho B^\epsilon$.

Note that tr$[A, B]$ = tr$[A^\rho, B^\epsilon]$. Since A fixes 0 and ∞, compute that

$$\operatorname{tr}[A, B] = 2 + \frac{\xi_1\xi_2}{(\xi_1 - \xi_2)^2} \cdot \frac{(K-1)^2}{K} \cdot \frac{(R-1)^2}{R}.$$

Thus $[A, B]$ is parabolic if and only if

$$\frac{\xi_1\xi_2}{(\xi_1 - \xi_2)^2} \cdot \frac{(K-1)^2}{K} \cdot \frac{(R-1)^2}{R} = 0 \quad \text{or}$$

$$\frac{C}{(C-1)^2} = -4\frac{K}{(K-1)^2} \cdot \frac{R}{(R-1)^2}.$$

Since $K \neq 1$ and $R \neq 1$, the former happens if and only if

$$\frac{\xi_1\xi_2}{(\xi_1 - \xi_2)^2} = 0.$$

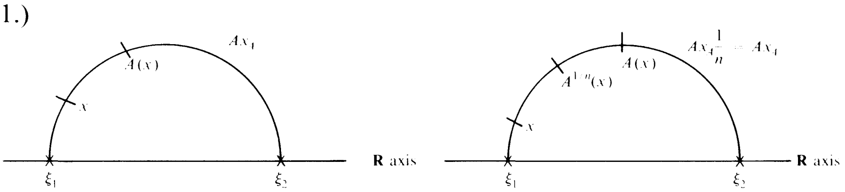
Since $\xi_1 \neq \xi_2$, either $\xi_1 = 0$ or $\xi_2 = 0$ or $\xi_1 = \infty$ or $\xi_2 = \infty$. But A and B have no common fixed points.

COROLLARY 7.3. *If we replace the inequalities in Theorem* 6.1 *part* (i), (ii) *and* (iii) *by strict inequalities, then we obtain necessary and sufficient condition for F to be a discrete group containing only hyperbolics.*

Proof. If P were a parabolic element of F, then we can write $P = h_1, \dots, h_r$ where the h_i are hyperbolic since F is generated by hyperbolics. By induction we may assume that $h_1, \dots, h_{r-1} = t$ is hyperbolic and that if $h = h_r$, $\langle h, t\rangle$ is non-elementary since it contains $\langle P, t\rangle$. The lemma shows that ht can be parabolic if and only if either we have equality in 6.1(i), (ii) or (iii) or h and t have a common fixed point. If $\langle h, t\rangle$ is non-elementary, our assumption that $-\infty < C(h, t) < 0$ implies h and t cannot have a common fixed point.

To prove Theorem 7.1 we note that Corollary 7.3 is just another formulation of Theorem 7.1.

8. Adjoining roots. For any hyperbolic transformation, by its n-th root we mean that transformation with the same fixed points that moves points along the fixed axis $1/n$ times the translation length of A. (See Figure 8.1.)



(Note that the translation length is measured in the hyperbolic metric.)

Figure 8.1

Thus if A is hyperbolic with multiplier $K > 1$, attracting fixed point ξ_2 and repelling fixed point ξ_1, by $A^{1/n}$ we mean the hyperbolic transformation with multiplier $K^{1/n}$, attracting fixed point ξ_2 and repelling fixed point ξ_1.

If A were parabolic, it would be conjugate to $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ and $A^{1/n}$ would be conjugate by the same transformation to $\left[\begin{smallmatrix}1 & 1/n\\ 0 & 1\end{smallmatrix}\right]$.

If F is a discrete group, $h_1, \ldots, h_r$ hyperbolic or parabolic elements of F, and $m_1, \ldots, m_r$ integers, one would like to understand necessary and sufficient conditions for

(1) $E = \langle F, h^{1/m_1}, \ldots, h^{1/m_r}\rangle$ to be discrete,

(2) for F to be a normal subgroup of E and

(3) for E to be a finite extension of F.

Why this is of interest and the relationship between some answers to (1), (2) and (3) is discussed in the last section of [**4**].

Because there is a relationship between E being discrete and a normal extension and the surface U/F having conformal automorphisms, one expects answers to (1), (2) and (3) to say that there exists a deformation of F for which the sufficient conditions are satisfied.

The best result occurs in the case of a torus group.

PROPOSITION 8.2. *If F is a torus group so that*

$$F = \langle z \rightarrow z + 1, z \rightarrow z + \tau\rangle$$

where τ is not real, then for any r and any $h_1, \ldots, h_r \in F$ and any positive integers $m_1, \ldots, m_r$, the group

$$E = \langle F, h_1^{1/m_1}, \ldots, h_r^{1/m_r}\rangle$$

is discrete and is a finite normal extension of F.

Proof. Since all parabolic transformations that fix ∞ commute, E is a finite normal extension of F and it is enough to see that $E_1 = \langle F, h_1^{1/n_1}\rangle$ is discrete. Let

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}1 & \tau\\ 0 & 1\end{bmatrix},$$

then

$$h_1 = (A^r B^s)^{1/n_1} \text{ for some integers } r \text{ and } s.$$

Any word C_i in E_1 can be written as

$$C_i = A^{x_i + r_i r/n_1} B^{t_i + s_i s/n_1} \text{ for some integers } x_i,\ r_i,\ t_i, \text{ and } s_i.$$

If $\{C_i\}$ is a sequence converging to the identity, then $\{C_i^{n_i}\}$ is a sequence in F converging to the identity. Thus if E_1 were not discrete, F would not be discrete.

The situation becomes already more complex when one considers Schottky groups.

PROPOSITION 8.3. *If* $F = \langle C_1, \ldots, C_g \rangle$ *is a Schottky group of genus* g, *and* n *any integer, then either*

$$E = \langle C_1, \ldots, C_{i-1}, C_i^{1/n}, C_{i+1}, \ldots, C_g \rangle$$

is discrete or $\exists F' = \langle C_1', \ldots, C_g' \rangle$ *such that*

$$\langle C_1', \ldots, C_{i-1}', (C_i')^{1/n}, C_{i+1}', \ldots, C_g' \rangle$$

is discrete.

Proof. If necessary we replace F by a classical Schottky group where each C_i maps the exterior of its isometric circle onto the interior of the isometric circle of its inverse. Since F is discrete and gives a surface of genus g, by [**3**] and [**8**] we may assume that these $2g$ isometric circles (those of each generator and its inverses) are disjoint. By [**3**] we see that the isometric circle of C_i is contained in that of $C_i^{1/n}$ and the same holds for that of C_i^{-1} and $C_i^{-1/n}$. If the isometric circles of $C_i^{1/n}$ and $C_i^{-1/n}$ are still disjoint from the other isometric circles, E is discrete. If not, replace C_i by C_i' such that the isometric circle of $(C_i')^{1/n}$ is contained in that of C_i and we are done.

Note that in Proposition 8.3 we have only adjoined a root of a generator and not of an arbitrary element, but we can still adjoin arbitrary roots of any number of generators.

Finally, we obtain

PROPOSITION 8.4. *If* A *and* B *are hyperbolic elements whose axes intersect but are not parallel and* $\langle A, B \rangle$ *is a discrete free group so that*

$$(f(R))^{-1} \geqq -4\frac{f(K)}{f(C)}$$

then $\langle A^{1/n}, B \rangle$ *is a discrete free group if and only if*

$$f(R^{1/n})^{-1} \geqq -4\frac{f(K)}{f(C)}.$$

Proof. This follows from Theorem 8 of [**14**], Corollary 3.4. Note also that since

$$\frac{1}{f(R)} \geqq \frac{1}{f(R^{1/n})},$$

$\langle A^{1/n}, B \rangle$ may not be discrete even if $\langle A, B \rangle$ is.

PROPOSITION 8.5. *If* A *and* B *are hyperbolic matrices whose axes do not intersect and* $\langle A, B \rangle$ *is a discrete free group, so that either* $C^{\delta}(A, B) > Q^2$ *or* $C^{\delta}(A, B) < S^2$, *then a necessary condition for* $\langle A^{1/n}, B \rangle$ *to be discrete is that either*

$$C^{\delta}(A, B) > Q_{1/n}^2 \quad \text{or} \quad C^{\delta}(A, B) < S_{1/n}^2.$$

If $C^{\delta}(A, B) > Q_{1/n}^2$, *then* $\langle A^{1/n}, B \rangle$ *is a discrete free group.*

Proof. This follows immediately from Proposition 6.1 and Remark 6.5.

Note that since $Q^2_{1/n} > Q^2$ and in most cases $S^2_{1/n} > S^2$ neither of the needed inequalities will automatically be true.

Remark. Using the results of **[9]** and **[20]**, one could improve Propositions 8.4 and 8.5 to give necessary and sufficient conditions for $\langle A^{1/n}, B\rangle$ to be discrete and not discrete free. It is not clear whether these conditions would be clean and useful enough. This will be discussed in a forthcoming paper which will also include a counterexample to case (6) of results claimed in **[9]**.

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