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THE FRATTINI p-SUBALGEBRA OF A SOLVABLE LIE p-ALGEBRA

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In this paper we continue our study of the Frattini p-subalgebra of a Lie p-algebra L. We show first that if L is solvable then its Frattini p-subalgebra is an ideal of L. We then consider Lie p-algebras L in which L^2 is nilpotent and find necessary and sufficient conditions for the Frattini p-subalgebra to be trivial. From this we deduce, in particular, that in such an algebra every ideal also has trivial Frattini p-subalgebra, and if the underlying field is algebraically closed then so does every subalgebra. Finally we consider Lie p-algebras L in which the Frattini p-subalgebra of every subalgebra of L is contained in the Frattini p-subalgebra of L itself.

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1. Introduction

In this paper we continue our study of the Frattini p-subalgebra of a Lie p-algebra which was started in [3]. Recall that a Lie algebra L over a field K of characteristic $p > 0$ is called a *Lie p-algebra* if, in addition to the usual compositions, there is a "p-map" $a \mapsto a^p$ such that

$$(\alpha a)^p = \alpha^p a^p \text{ for all } \alpha \in K, a \in L,$$
$$a(ad\, b^p) = a(ad\, b)^p \text{ for all } a, b \in L, \text{ and}$$
$$(a+b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a,b) \text{ for all } a, b \in L,$$

where $is_i(a,b)$ is the coefficient of X^{i-1} in the expansion of $a(ad(Xa+b))^{p-1}$. Throughout, unless stated otherwise, L will denote a finite-dimensional Lie p-algebra over a field K.

A subalgebra (respectively, ideal) of L is a *p-subalgebra* (respectively, *p-ideal*) of L if it is closed under the p-map. A proper p-subalgebra M of L is a *maximal p-subalgebra* of L if there are no proper p-subalgebras of L strictly containing M. The *Frattini p-subalgebra,* $F_p(L)$, of L is the intersection of the maximal p-subalgebras of L, and the *Frattini p-ideal,* $\phi_p(L)$, is the largest p-ideal of L inside $F_p(L)$. We shall denote by $F(L)$, $\phi(L)$ the usual Frattini subalgebra and ideal of L (see, for example, [6]).

In Section 2 we shall show that $F_p(L) = \phi_p(L)$ when L is solvable. In Sections 3, 4 we seek analogues for $\phi_p(L)$ of the results of Stitzinger on $\phi(L)$ when the derived algebra $L^{(1)}$ is nilpotent, which were obtained in [5]. The following notation will be used:

$[x, y]$	the product of x, y in L
$L^{(1)}$	the derived algebra of L
$L^{(n)} =$	$(L^{(n-1)})^{(1)}$ for all $n \geq 2$
(H)	the subalgebra generated by the subset H of L
$(H)_p =$	$(\{x^{p^n} : x \in (H), n \in \mathbb{N}\})$ where $x^{p^m} = (x^{p^{m-1}})^p$
$A^p =$	$(\{x^p : x \in A\})$, where A is a subalgebra of L
$A^{p^n} =$	$(A^{p^{n-1}})^p$
$L_1 =$	$\cap_{i=1}^{\infty} L^{p^i}$
$L_0 =$	$\{x \in L : x^{p^n} = 0$ for some $n\}$
$Z(L)$	the centre of L
$\oplus$	algebra direct sum
$\dot{+}$	vector space direct sum
$\subseteq$	is a subset of
$\subset$	is a proper subset of

2. Normality of $F_p(L)$

We show here that $F_p(L) = \phi_p(L)$ when L is solvable. The proof is modelled on that of Theorem 3.27 of [1]. First we need a lemma.

Lemma 2.1. *Let A be an abelian ideal of L. Then $A^p \subseteq Z(L)$.*

Proof. Let $\ell \in L, a \in A$. Then

$$[\ell, a^p] = \ell(ada)^p = [\ell, a](ada)^{p-1} \in A^{(1)} = 0.$$

□

Corollary 2.2. *If L is solvable and A is a minimal p-ideal of L, then A is abelian.*

Proof. Let B be a minimal ideal of L contained in A. Then $B + Z(L)$ is p-closed (by Lemma 2.1 and the fact that $Z(L)$ is p-closed), and so

$$A \cap (B + Z(L)) = B + A \cap Z(L) = A.$$

Thus, $A^{(1)} \subseteq B^{(1)} = 0$. □

Theorem 2.3. *If L is solvable then $F_p(L)$ is an ideal of L; that is; $F_p(L) = \phi_p(L)$.*

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Proof. Let L be a minimal counter-example, and suppose that A is a p-ideal of L. Put

$$F_p(L : A) = \cap\{M : A \subseteq M, M \text{ is a maximal } p\text{-subalgebra of } L\}.$$

Then $F_p(L : A)/A = F_p(L/A)$, which is an ideal of L/A if $A \neq 0$. We consider two cases.

Case (i): For each maximal p-subalgebra M of L there is a non-zero p-ideal A of L contained in M. Then

$$F_p(L) = \cap\{F_p(L : A) : A \text{ is a minimal } p\text{-ideal of } L\},$$

which is an ideal of L.

Case (ii): Suppose now that there is a maximal p-subalgebra M of L which contains no non-zero p-ideals of L. Let A be a minimal p-ideal of L. Then $L = A \dot{+} M$. But $A^{(1)} = 0$, by Corollary 2.2, and so $A \subseteq C_L(A) = \{x \in L : [x, A] = 0\}$. Also, $C_L(A) \cap M$ is a p-ideal of L, since it is p-closed, $[A, C_L(A) \cap M] = 0$ and $C_L(A) \cap M$ is an ideal of M. As M contains no proper p-ideals of L, we have $C_L(A) \cap M = 0$. It follows that $C_L(A) = A$ and hence that $Z(L) \subseteq A$. But $Z(L)$ is a p-ideal of L and so $Z(L) = A$ or $Z(L) = 0$. The former implies that $L = A$ is abelian and the result is clear, so assume the latter holds. Then $a^p = 0$ for all $a \in A$, by Lemma 2.1, and so A is a minimal ideal of L. Thus $[M, A] = A$ or $[M, A] = 0$. The latter implies that $A = C_L(A) = L$ is abelian, a contradiction. Hence $A = [M, A] \subseteq L^{(1)}$ and $L^{(1)} = A \dot{+} M^{(1)}$.

Let $0 \neq m \in M$. Then there is an $a \in A$ such that $[m, a] \neq 0$. Define $\theta : L \to L$ by putting $\theta = 1 + ada$. Then it is easily checked that θ is an automorphism of L.

Suppose that M is not a maximal subalgebra of L. Then there is a maximal subalgebra K of L properly containing M, and K is an ideal of L, by Lemma 3.9 of **[3]**. But this implies that $L^{(1)} \subseteq K$ and thus that $L = M + A \subseteq K$, a contradiction. Hence M is maximal in L, and so $\theta(M)$ is maximal in L.

Suppose that $A \subseteq \theta(M)$. Then, if $b \in A$, there exists an $n \in M$ such that $b = n + [n, a]$, and so $n \in M \cap A = 0$, a contradiction. Thus, $A \not\subseteq \theta(M)$. It follows that $L^{(1)} \not\subseteq \theta(M)$ and hence that $\theta(M)$ is not an ideal of L. We conclude from Lemma 3.9 of **[3]** that $\theta(M)$ is a p-subalgebra of L.

Finally suppose that $m \in \theta(M)$. Then there is an $m' \in M$ such that $m = m' + [m', a]$ and so $[m, a] = [m', a] + [[m', a], a] = [m', a] = 0$, a contradiction. Hence $m \notin \theta(M)$, and so $m \notin F_p(L)$. It follows that $F_p(L) = 0$. □

3. ϕ_p-free algebras

We aim first to prove an analogue of Proposition 1 of **[5]**. This is Theorem 3.2 below.

Lemma 3.1. $(L^{(1)})_p \cap Z(L) \subseteq \phi_p(L)$.

Proof. Note first that $Z(L)$ is clearly p-closed. Let M be a maximal p-subalgebra of L and suppose that $Z(L) \not\subseteq M$. Then $L = M + Z(L)$, so $L^{(1)} = M^{(1)} \subseteq M$ and hence $(L^{(1)})_p \subseteq (M)_p \subseteq M$. □

By the *abelian socle* (respectively, *abelian p-socle*) of L, denoted by $AsocL$ (respectively, $ApsocL$), we shall mean the sum of the minimal abelian ideals (respectively, p-ideals) of L. We shall say that L *splits* (respectively, *p-splits*) over an ideal (respectively, p-ideal) I if there is a subalgebra (respectively, p-subalgebra) B of L such that $L = I \dot{+} B$; in these circumstances we call B a *complement* (respectively, p-complement) of A.

Theorem 3.2. *Suppose that $L^{(1)} \neq 0$ and that $L^{(1)}$ is nilpotent. Then the following are equivalent:*

(i) $\phi_p(L) = 0$;

(ii) $ApsocL = N(L)$, *the nilradical of L, and L p-splits over $N(L)$;*

(iii) $L^{(1)}$ *is abelian,* $(L^{(1)})^p = 0$, *L p-splits over* $L^{(1)} \oplus Z(L)$, *and* $ApsocL = L^{(1)} \oplus Z(L)$.

Under these circumstances, the Cartan subalgebras of L are exactly those subalgebras which p-complement $L^{(1)}$. If K is perfect then the maximal toral subalgebras are precisely those subalgebras which p-complement $L^{(1)} \oplus Z(L)_0$.

Proof. (i) $\Leftrightarrow$ (ii): These implications are immediate from Theorems 4.1, 4.2 of [**3**].
(iii) $\Rightarrow$ (i): This also follows from Theorem 4.1 of [**3**].
(i) $\Rightarrow$ (iii): Suppose that $\phi_p(L) = 0$. Then $\phi(L) = 0$ by Theorem 3.5 of [**3**], and so $L^{(1)}$ is abelian, by Proposition 1 of [**5**]. Now $(L^{(1)})^p \subseteq Z(L)$ by Lemma 2.1, and so

$$(L^{(1)})^p \subseteq (L^{(1)})^p \cap Z(L) \subseteq (L^{(1)})_p \cap Z(L) \subseteq \phi_p(L) = 0$$

by Lemma 3.1. Clearly $L^{(1)} \oplus Z(L) \subseteq N(L) = ApsocL$. Now let A be a minimal (and hence abelian) p-ideal of L. Then $[L, A] = A$ is an ideal of L and

$$\begin{aligned}[L, A]^p \subseteq (L^{(1)})^p \cap A^p \subseteq (L^{(1)})^p \cap Z(L) &\text{ by Lemma 2.1} \\ = 0 &\text{ by Lemma 3.1.}\end{aligned}$$

Hence $[L, A]$ is p-closed, and so $[L, A] = A$ or $[L, A] = 0$. The former implies that $A \subseteq L^{(1)}$, and the latter that $A \subseteq Z(L)$, whence $ApsocL = L^{(1)} \oplus Z(L)$ and (iii) follows.

The assertion that the Cartan subalgebras are exactly those subalgebras which p-complement $L^{(1)}$ follows from Proposition 1 of [**5**], or from Theorem 4.4.1.1 of [**7**], and the fact that Cartan subalgebras are p-closed.

So assume now that K is perfect. Write $L = (L^{(1)} \oplus Z(L)) \dot{+} B$ where $B^{(1)} = 0$ and B is p-closed, and let $B = B_0 \oplus B_1$ be the Fitting decomposition of B relative to the p-map. (See, for example, Theorem 4.5.8 of [**7**]). Then $L^{(1)} \oplus Z(L) = ApsocL = N(L)$ from (ii), (iii). But $L^{(1)} \oplus Z(L) \dot{+} B_0$ is a nilpotent ideal of L, and so $B_0 \subseteq N(L) \cap B = 0$. Hence $B = B_1$ is toral. It is clear then that $B_1 + Z(L)_1$ is a maximal toral subalgebra of L. Finally, let T be any maximal torus of L, and let $C = C_L(T)$. Then C is a Cartan subalgebra of L, by Theorem 4.5.17 of [7], and so $L = L^{(1)} \dot{+} C$ as above. Clearly we can write $C = C_0 \oplus T$. But now $L^{(1)} + C_0$ is a nilpotent ideal of L, and so $C_0 \subseteq N(L) \cap C = Z(L)$, making T a p-complement of $L^{(1)} \oplus Z(L)_0$. □

The condition '$ApsocL = L^{(1)} \oplus Z(L)$' in (iii) above cannot be weakened to '$Z(L) \subseteq ApsocL$', as is shown by the following example.

Example 3.1. Consider $L = B + V$ where $B = Ka + Kb$, $V = Kv_1 + Kv_2$, $v_1^p = v_2^p = b^p = 0$, $a^p = a$, $[V, V] = 0$, $[a, b] = 0$, $[a, v_1] = v_1$, $[a, v_2] = v_2$, $[b, v_1] = v_2$, $[b, v_2] = 0$. Then $L^{(1)} = V$ is abelian, $(L^{(1)})^p = 0, Z(L) = 0$. Now $N(L) = Kb + Kv_1 + Kv_2$. Also Kv_2 is a minimal p-ideal. Let J be a minimal p-ideal contained in $N(L)$. Since $[N(L), N(L)] = Kv_2$, either $J = Kv_2$ or $[N(L), J] = 0$. Suppose that $J \neq Kv_2$. Then $[b, J] = 0$ so $J \subseteq Kb + Kv_2$, and $[v_1, J] = 0$ so $J \subseteq Kv_1 + Kv_2$. Thus $J \subseteq Kv_2$, a contradiction. Hence $N(L) \neq ApsocL$.

In [5] it was shown that for any Lie algebra L, over any field K, such that $L^{(1)}$ is nilpotent, L is ϕ-free (that is, $\phi(L) = 0$) if and only if every subalgebra of L is ϕ-free ([5, Theorem 1]). The complete analogue of this result does not hold when $\phi(L)$ is replaced by $\phi_p(L)$ throughout, as the following example shows.

Example 3.2. Let $L = Ka + Kb + Kv_1 + Kv_2$ where $K = \mathbb{Z}_2$, $a^2 = a, b^2 = a + b$, $[a, v_1] = v_1$, $[a, v_2] = v_2$, $[b, v_1] = v_2$, $[b, v_2] = v_1 + v_2$, $[a, b] = [v_1, v_2] = 0$, $v_1^2 = v_2^2 = 0$. Put $B = Ka + Kb$. Then $\phi_p(L) = 0$ whereas $\phi_p(B) = Ka$.

Nevertheless partial results in this direction can be obtained. We will deduce these from the following result.

Theorem 3.3. *The following are equivalent:*

(i) $L^{(1)}$ *is nilpotent and* $\phi_p(L) = 0$;

(ii) $L = A \dot{+} B$ *where B is an abelian subalgebra, A is an abelian p-ideal, the (adjoint) action of B on A is faithful and completely reducible, $Z(L)$ is completely reducible under the p-map, and the p-map is trivial on* $[B, A]$.

Proof. (i) ⇒ (ii): By Theorem 3.2, $L = A \dot{+} B$ where $A = ApsocL = A_1 \oplus \ldots \oplus A_n$ with A_i a minimal abelian p-ideal of L for $i = 1, \ldots, n$, and B is p-subalgebra of L. Now $C_B(A) = \{x \in B : [x, A] = 0\}$ is an ideal in the solvable Lie algebra L. If $C_B(A) \neq 0$ it follows that

$$0 \neq C_B(A) \cap ApsocL \subseteq B \cap A = 0,$$

which is a contradiction. Hence $C_B(A) = 0$ and the action of B on A is faithful.

Now suppose that $A_i \not\subseteq Z(L)$. Then $A_i \cap Z(L) \subset A_i$ and so, as $A_i \cap Z(L)$ is a p-ideal, $A_i \cap Z(L) = 0$. If $a \in A_i$ then $(ada)^2 = 0$, and so $ada^p = 0$; that is, $a^p \in Z(L)$. Thus,

$a^p \in A_i \cap Z(L) = 0$, and the minimality of A_i implies that A_i is an irreducible B-module. But, of course, if $A_i \subseteq Z(L)$ then A_i is a completely reducible B-module, so $A = A_1 \oplus \ldots \oplus A_n$ is a completely reducible B-module.

Now $L^{(1)}$ is nilpotent, so adx is nilpotent for every $x \in B^{(1)}$. It follows from Engel's Theorem that $[B^{(1)}, A_i] \subset A_i$ for every $i = 1, \ldots, n$. If $A_i \not\subseteq Z(L)$ this implies that $[B^{(1)}, A_i] = 0$, since A_i is an irreducible B-module. If $A_i \subseteq Z(L)$ then, clearly, $[B^{(1)}, A_i] = 0$ also. Thus $[B^{(1)}, A_i] = 0$, and so $B^{(1)} = 0$, as $C_B(A) = 0$. Moreover, $Z(L) \subseteq A$, since $C_B(A) = 0$. If $a \in Z(L)$ and $a = a_1 + \ldots + a_n$, with $a_i \in A_i$, then $0 = [x, a_1] + \ldots + [x, a_n]$ for all $x \in L$, so each $a_i \in Z(L)$. Hence $Z(L) = \sum A_i$, where the sum is over all A_i contained in $Z(L)$. Since each $A_i \subseteq Z(L)$ is a minimal p-ideal, $Z(L)$ must be irreducible under the p-map.

(ii) $\Rightarrow$ (i): In view of Theorem 4.1 of [3] it suffices to show that $A = ApsocL$. Now we have that $A = [B, A] \oplus Z(L)$, $[B, A]$ is a direct sum of irreducible B-modules (each of which is a minimal p-ideal) and $Z(L)$ is a direct sum of irreducible subspaces for the p-map (each of which is a minimal p-ideal). Thus, $A \subseteq ApsocL$. But, as B acts faithfully on L, A is a maximal abelian ideal. Hence $A = ApsocL$, as required. □

Corollary 3.4. *Suppose that $L^{(1)}$ is nilpotent and that $\phi_p(L) = 0$. Let S be a p-subalgebra of L with $ApsocL \subseteq S$. Then $\phi_p(S) = 0$.*

Proof. Write $L = A \dotplus B$ as in Theorem 3.3 (ii). Then $S = A \dotplus (B \cap S)$ since $A = ApsocL \subseteq S$. Now B acts completely reducibly on $[B, A]$, and hence so does $B \cap S$. It follows that $B \cap S$ acts completely reducibly on $[B \cap S, A]$. Moreover, $Z(S) = Z(L) \oplus C_{[B,A]}(B \cap S)$ and the p-map is trivial on $[B, A]$, so $Z(S)$ is completely reducible under the p-map. The result now follows from Theorem 3.3. □

Corollary 3.5. *Suppose that $L^{(1)}$ is nilpotent and $\phi_p(L) = 0$. If I is an ideal of L, then $\phi_p(I) = 0$.*

Proof. If suffices to show this for maximal ideals. By Corollary 3.4 we may assume that $A_1 \not\subseteq I$, where $ApsocL = A_1 \oplus \ldots \oplus A_n$ with $A_1, \ldots, A_n$ minimal abelian p-ideals. Then $L = I + A_1$, since I is maximal, and $I \cap A_1 = 0$. Thus $L = I \oplus A_1$, $I \cong L/A_1 \cong B \dotplus (A_2 \oplus \ldots \oplus A_n)$, and $A_1 \subseteq Z(L)$. Hence $C_B(A_2 \oplus \ldots \oplus A_n) = C_B(A) = 0$, and it is clear that all of the conditions of Theorem 3.3 (ii) hold. □

Corollary 3.6. *If L is abelian then $\phi_p(L) = 0$ if and only if L is completely reducible under the p-map.*

Proof. Simply apply Theorem 3.3, noting that $B = 0$ and $L = Z(L)$.

Corollary 3.7. *Suppose that $L = ApsocL \dotplus B$ and that the conditions of Theorem 3.3 (ii) are satisfied. Assume in addition that B is completely reducible under the p-map; that is, $ApsocB = B$. Then if S is any p-subalgebra of L, $S = ApsocS \dotplus B'$, the conditions of Theorem 3.3 (ii) are satisfied and B' is completely reducible under the p-map.*

Proof. If $ApsocL \subseteq S$, then $ApsocS = ApsocL$, and taking $B' = B \cap S$ gives the result.

It suffices to prove the corollary for maximal p-subalgebras. So assume that S is maximal and that $A_1 \not\subseteq S$, where $ApsocL = A_1 \oplus \ldots \oplus A_n$ with $A_1, \ldots, A_n$ minimal abelian p-ideals. Then $L = A_1 + S$ with $S \cap A_1 = 0$. Hence

$$S \cong B \dot{+} (A_2 \oplus \ldots \oplus A_n).$$

As B is completely reducible under the p-map we have $B = B' \oplus C_B(A_2 \oplus \ldots \oplus A_n)$. Then

$$ApsocS = C_B(A_2 \oplus \ldots \oplus A_n) \oplus A_2 \oplus \ldots \oplus A_n,$$

$S = ApsocS \dot{+} B'$, the conditions of Theorem 3.3. (ii) are satisfied and B' is completely reducible under the p-map. □

We shall call L *p-elementary* if $\phi_p(S) = 0$ for every p-subalgebra S of L.

Corollary 3.8. *Suppose that $L^{(1)}$ is nilpotent and that $\phi_p(L) = 0$. Let $L = ApsocL \dot{+} B$ as in Theorem 3.3 (ii). Then L is p-elementary if and only if $B = ApsocB$.*

Proof. ($\Rightarrow$) Corollary 3.7.
($\Leftarrow$) Corollary 3.6. □

Corollary 3.9. *Let L be a Lie p-algebra over an algebraically closed field K of characteristic $p > 0$, and suppose that $L^{(1)}$ is nilpotent. Then $\phi_p(L) = 0$ if and only if L is p-elementary.*

Proof. Suppose that $\phi_p(L) = 0$ and write $L = ApsocL \dot{+} B$ as in Theorem 3.3 (ii). Then B has a faithful completely reducible representation on $ApsocL$. This is equivalent to the fact that B has no non-zero nil ideals (see, for example, [**4**, Section 1.5, p. 27]). As B is abelian this is equivalent to the injectivity of the p-map. Since K is algebraically closed, this is equivalent to $ApsocB = B$ (see [**2**, Theorem 13, p. 192]). It follows from Corollary 3.8 that L is p-elementary.

The converse is immediate. □

The above result cannot be extended to the case where K is a perfect field (as perhaps we might have hoped) as is shown by the following examples.

Example 3.3. Let B be any abelian Lie p-algebra for which the p-map is injective but B is not completely reducible under the p-map. Then B has a faithful completely reducible module A. Make A into an abelian Lie p-algebra with trivial p-map. Then $\phi_p(A \dot{+} B) = 0$ but $\phi_p(B) \neq 0$. Examples of such B can be produced as follows.

If K is not perfect, let $\lambda \in K \backslash K^p$ and take $B = Ka + Kb$ with $a^p = a$, $b^p = \lambda a$. If

$\lambda \in K$ and $\mu^p - \mu + \lambda = 0$ has no solution in K, take $B = Ka + Kb$ with $a^p = a, b^p = b + \lambda a$. Here we can take A to be p-dimensional with a represented by the identity matrix and b represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ -\lambda & 1 & 0 & \dots & 0 \end{pmatrix}$$

(the companion matrix of $\mu^p - \mu + \lambda$). If $F = \mathbb{Z}_p$ we may take $\lambda = -1$. (Putting $p = 2$ gives Example 3.2.)

4. E-p-algebras

We say that L is an *E-algebra* (respectively, *E-p-algebra*) if, for every subalgebra (respectively, p-subalgebra) S of L we have $\phi(S) \subseteq \phi(L)$ (respectively, $\phi_p(S) \subseteq \phi_p(L)$).

The following result is the restricted version of Proposition 2 of [**5**].

Theorem 4.1. *L is an E-p-algebra if and only if $L/\phi_p(L)$ is p-elementary.*

Proof. Suppose first that L is an E-p-algebra, and let $S/\phi_p(L)$ be a subalgebra of $L/\phi_p(L)$. Choose a p-subalgebra U of L which is minimal with respect to $\phi_p(L) + U = S$ (so U could be equal to S). Let T be a p-ideal of S such that $T/\phi_p(L) = \phi_p(S/\phi_p(L))$, and suppose that $T \neq \phi_p(L)$.

Then

$$T = T \cap S = T \cap (\phi_p(L) + U) = \phi_p(L) + T \cap U,$$

and $T \cap U \not\subseteq \phi_p(L)$. It follows that $T \cap U \not\subseteq \phi_p(U)$ since L is an E-p-algebra. But $T \cap U$ is a p-ideal of U, so $T \cap U \not\subseteq F_p(U)$. Hence there is a maximal p-subalgebra M of U such that $T \cap U \not\subseteq M$, and $U = T \cap U + M$.

By the minimality of U we must have $\phi_p(L) + M \neq S$. We claim that $\phi_p(L) + M$ is a maximal p-subalgebra of S. Suppose that $\phi_p(L) + M \subset J \subset S$. Then $M \subseteq J \cap U \subseteq U$ and so, by the maximality of M, either $J \cap U = M$ or $J \cap U = U$. The former implies that

$$\phi_p(L) + M = \phi_p(L) + J \cap U = J \cap (\phi_p(L) + U) = J \cap S = J,$$

a contradiction. The latter gives $U \subseteq J$ and hence $J \supseteq U + \phi_p(L) = S$, also a contradiction. Hence the maximality of $\phi_p(L) + M$ in S. Thus

$$(\phi_p(L) + M)/\phi_p(L) \supseteq \phi_p(S/\phi_p(L)) = T/\phi_p(L),$$

and so $T \subseteq \phi_p(L) + M$. But now $T \cap U \subseteq T \subseteq \phi_p(L) + M$ and so

$$S = \phi_p(L) + U = \phi_p(L) + T \cap U + M = \phi_p(L) + M,$$

contradicting the minimality of U. We conclude that $T = \phi_p(L)$, whence $\phi_p(S/\phi_p(L)) = 0$ and $L/\phi_p(L)$ is p-elementary.

Conversely, suppose that $L/\phi_p(L)$ is p-elementary and let S be a p-subalgebra of L. Then

$$(\phi_p(S) + \phi_p(L))/\phi_p(L) \subseteq \phi_p((S + \phi_p(L))/\phi_p(L)) = 0,$$

and so $\phi_p(S) \subseteq \phi_p(L)$. □

Corollary 4.2. *Let L be a Lie p-algebra over an algebraically closed field K of characteristic $p > 0$, and suppose that $L^{(1)}$ is nilpotent. Then L is an E-p-algebra.*

Proof. This is immediate from Corollary 3.9 and Theorem 4.1. □

We finish by noting the relationship between elementary and p-elementary Lie p-algebras (respectively E-algebras and E-p-algebras) given by Corollary 4.4 below.

Theorem 4.3. *Let S be a subalgebra (not necessarily p-closed) of the Lie p-algebra L. Then*
(i) $\phi(S) \subseteq \phi((S)_p)$, *and*
(ii) $\phi(S) \subseteq \phi_p(L) \Rightarrow \phi(S) \subseteq \phi(L)$.

Proof. (i) Let M be a maximal subalgebra of $(S)_p$, and suppose that $\phi(S) \nsubseteq M$. Then $(S)_p = M + \phi(S)$, and so $S = M \cap S + \phi(S) = M \cap S$ (Lemma 2.1 of [**6**]). Hence $S \subseteq M$ and so $\phi(S) \subseteq M$, contrary to our assumption. Thus $\phi(S) \subseteq F((S)_p)$, whence $\phi(S) \subseteq \phi((S)_p)$.

(ii) Suppose that $\phi(S) \subseteq \phi_p(L)$, and let M be a maximal subalgebra of L such that $\phi(S) \nsubseteq M$. Then $L = M + \phi(S) = M + \phi_p(L)$. Thus

$$\begin{aligned} L^{(1)} = M^{(1)} + L\phi_p(L) &\subseteq M^{(1)} + \phi(L) \text{ by Corollary 3.11 of [3]} \\ &\subseteq M. \end{aligned}$$

But now $\phi(S) \subseteq S^{(1)} \subseteq L^{(1)} \subseteq M$, a contradiction. □

Corollary 4.4. (i) *If L is p-elementary, then L is elementary.*
(ii) *If L is an E-p-algebra, then L is an E-algebra.*

Proof. (i) Let L be p-elementary and let S be a subalgebra of L. Then

$$\phi(S) \subseteq \phi((S)_p) \subseteq \phi_p((S)_p) = 0,$$

so L is elementary.

(ii) Let L be an E-p-algebra and let S be a subalgebra of L. Then

$$\phi(S) \subseteq \phi((S)_p) \subseteq \phi_p((S)_p) \subseteq \phi_p(L),$$

and so $\phi(S) \subseteq \phi(L)$. □

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