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On a Bernoulli shift with non-identical factor measures

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Abstract. There exists a Bernoulli shift with non-identical factor measures for which no invariant σ-finite equivalent measure exists.

1. *Introduction*

Our purpose is to give an example of a Bernoulli shift T acting on an infinite product measure space

$$(\Omega, \mathscr{F}, P) = \Big(\prod_{-\infty<k<\infty} \{0, 1\}, \bigvee_{-\infty<k<\infty} \mathscr{F}_k, \prod_{-\infty<k<\infty} P_k\Big)$$

such that the shift T,

$$(T\omega)_k = \omega_{k+1} \quad \text{for } \omega = (\omega_k)_{-\infty<k<\infty},$$

is non-singular, i.e. $P(A)=0$ if and only if $P(TA)=0$ for $A \in \mathscr{F} = \bigvee_{-\infty<k<\infty} \mathscr{F}_k$ such that T is ergodic and there exists no shift-invariant σ-finite measure equivalent to the infinite product measure $P = \prod_{-\infty<k<\infty} P_k$, where P_k is a probability measure on the set $\{0, 1\}$ and $\mathscr{F}_k$ is the smallest σ-algebra which makes the kth coordinate ω_k of $\omega \in \Omega$ measurable.

This problem was raised by U. Krengel at the symposium on Ergodic Theory at Oberwolfach 1978. He gave in [**3**] an ergodic Bernoulli shift without finite invariant measure.

In § 2 we give non-ergodic Bernoulli shifts which are dissipative. In § 3 we give an ergodic Bernoulli shift without σ-finite invariant measure.

2. *Dissipative Bernoulli shifts*

Let $\Omega = \prod_{-\infty<k<\infty} \{0, 1\}$, and T be the shift on Ω, i.e.

$$(T\omega)_k = \omega_{k+1}.$$

Take a probability measure

$$P = \prod_{-\infty<k<\infty} P_k$$

with

$$P_k(0) = P_k(1) = \tfrac{1}{2} \quad (k \geq 0)$$
$$0 < P_k(0) < 1, \quad P_k(1) = 1 - P_k(0) \quad (k < 0).$$

It follows from Kakutani's theorem on the equivalence of infinite product measures [**2**] that T is non-singular if and only if

$$\sum_{-\infty<k<\infty} P_{k-1}(0)P_k(0)((P_{k-1}(1)/P_{k-1}(0))^{\frac{1}{2}}-(P_k(1)/P_k(0))^{\frac{1}{2}})^2<\infty. \tag{1}$$

In this case we have that the Radon–Nikodym derivative dPT/dP of the measure PT,

$$(PT)(A)=P(TA) \quad \text{for } A\in\mathcal{F},$$

with respect to the measure P is given by

$$(dPT/dP)(\omega)=\prod_{-\infty<k<\infty} P_{k-1}(\omega_k)/P_k(\omega_k) \tag{2}$$

for a.e. ω, where the infinite product converges almost everywhere. In [**3**] Krengel claimed that the shift T is dissipative if

$$P_k(1)/P_k(0)=3 \quad (k<0).$$

In fact, more generally we have:

THEOREM 1. *Let* $P_k(0)=P_k(1)=\frac{1}{2}$ $(k\geq 0)$ *and let* $P_k(1)/P_k(0)=\lambda$ (λ *is a constant* >0) $(k<0)$. *If* $\lambda\neq 1$ *then the shift* T *on* $\Omega=\prod_{-\infty<k<\infty}\{0,1\}$ *is dissipative.*

Proof. The non-singularity condition (1) for the shift T,

$$\begin{aligned}
&\sum_{-\infty<k<\infty} P_{k-1}(0)P_k(0)((P_{k-1}(1)/P_{k-1}(0))^{\frac{1}{2}}-(P_k(1)/P_k(0))^{\frac{1}{2}})^2\\
&\quad=P_{-1}(0)P_0(0)((P_{-1}(1)/P_{-1}(0))^{\frac{1}{2}}-(P_0(1)/P_0(0))^{\frac{1}{2}})^2\\
&\quad=(\lambda^{\frac{1}{2}}-1)^2/(2(1+\lambda))\\
&\quad<\infty
\end{aligned}$$

is satisfied, and we have from (2)

$$(dPT^n/dP)(\omega)=(2/(1+\lambda))^n\lambda^{S_n(\omega)} \quad (n\geq 1),$$

where $S_n(\omega)=\omega_0+\omega_1+\cdots+\omega_{n-1}$.

What we are going to prove is that the infinite series

$$\sum_{n=1}^{\infty}(2/(1+\lambda))^n\lambda^{S_n(\omega)}$$

converges a.e. ω. Take $0<\theta<\frac{1}{2}$, then a standard fact says that

$$\lim_{n\to\infty}(S_n(\omega)-\tfrac{1}{2}n)/(n/4)^{\frac{1}{2}+\theta}=0 \quad \text{a.e. } \omega.$$

We assume $\lambda>1$. For any $\varepsilon>0$ and for a.e. ω, all but a finite number of n satisfy

$$-\varepsilon<(S_n(\omega)-\tfrac{1}{2}n)/(n/4)^{\frac{1}{2}+\theta}<\varepsilon.$$

Then we have for all large n

$$\begin{aligned}
(2/(1+\lambda))^n\lambda^{S_n(\omega)}&<\exp\{n\log(2/(1+\lambda))+n\log(\lambda)/2+\varepsilon\log(\lambda)(n/4)^{\frac{1}{2}+\theta}\}\\
&=\exp\{n^{\frac{1}{2}}(\log(2\lambda^{\frac{1}{2}}/(1+\lambda))n^{\frac{1}{2}}+\varepsilon\log(\lambda)(n/4)^{\theta})\}.
\end{aligned}$$

Since for all large n

$$\log(2\lambda^{\frac{1}{2}}/(1+\lambda))n^{\frac{1}{2}}+\varepsilon\log(\lambda)(n/4)^{\theta}<-1,$$

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we have for all large n

$$(2/(1+\lambda))^n \lambda^{S_n(\omega)} < \exp(-n^{\frac{1}{2}}).$$

Since the series

$$\sum_{n=1}^{\infty} \exp(-n^{\frac{1}{2}})$$

converges, the theorem is proved if $\lambda > 1$. If $\lambda < 1$, it is enough to see that for a.e. ω and for all large n

$$(2/(1+\lambda))^n \lambda^{S_n(\omega)} < \exp\{n^{\frac{1}{2}}[(\log(2\lambda^{\frac{1}{2}}/(1+\lambda))n^{\frac{1}{2}} - \varepsilon \log(\lambda)(n/4)^{\theta}]\}. \qquad \square$$

3. *Bernoulli shift without σ-finite invariant measure*

We are concerned with a class of infinite product measures

$$P = \prod_{-\infty<k<\infty} P_k$$

on

$$\Omega = \prod_{-\infty<k<\infty} \{0, 1\}$$

given by

$$P_k = \begin{cases} \mu & \text{if } k \geq 0, \\ \nu_t & \text{if } -N_t < k \leq -M_{t-1}, \\ \mu & \text{if } -M_t < k \leq -N_t \quad (t \geq 1), \end{cases}$$

where

$$N_t = M_{t-1} + n_t, \quad M_t = N_t + m_t, \quad M_0 = 1,$$

n_t and m_t are positive integers, and

$$\mu(0) = \mu(1) = \tfrac{1}{2}, \quad \nu_t(0) = 1/(1+\lambda_t), \quad \nu_t(1) = \lambda_t/(1+\lambda_t),$$

$$\lambda_1 > \lambda_2 > \cdots > \lambda_t > 1 \quad (t \geq 1).$$

We shall also consider the measure

$$Q = \prod_{-\infty<k<\infty} Q_k$$

with $Q_k = \mu$ $(k \in \mathbb{Z})$. The non-singular condition (1) for the shift T on (Ω, P) defined above is equivalent to the condition

$$\sum_{t=1}^{\infty} (\log(\lambda_t))^2 < \infty. \tag{3}$$

What we are going to do is to give inductively a sequence $(\lambda_t, n_t, m_t)_{t\geq 1}$ such that the shift T is non-singular, ergodic and admits no invariant σ-finite measures equivalent to P.

3.1. *Construction*. Take sequences $(p_t)_{t\geq 1}$ and $(\varepsilon_t)_{t\geq 1}$ such that

$$p_1 > p_2 > \cdots > 0, \quad p_t \to 0 \text{ (as } t \to \infty), \quad \sum_{t=1}^{\infty} p_t = \infty, \tag{4}$$

$$\varepsilon_1 > \varepsilon_2 > \cdots > 0, \quad \sum_{t=1}^{\infty} \varepsilon_t < \infty, \tag{5}$$

and write

$$\eta_t = \sum_{u=t}^{\infty} \varepsilon_u. \tag{6}$$

Let λ_1 be an arbitrary positive number >1, n_1 be an arbitrary integer >1 and m_1 be an arbitrary integer $>1+n_1$. Let

$$M_0 = 1, \quad N_1 = M_0 + n_1, \quad M_1 = N_1 + m_1.$$

We assume that (λ_u, n_u, m_u), $u = 1, 2, \ldots, t-1$ with $\lambda_{u-1} > \lambda_u > 1$, $m_u > N_u$ $(1 \le u \le t-1)$ are chosen.

First step: Choice of λ_t. Take λ_t such that

$$1 < \lambda_t < \lambda_{t-1}$$

and

$$(2\lambda_t/(1+\lambda_t))^{M_{t-1}} < \lambda_t^{M_{t-1}} < \exp(\varepsilon_t). \tag{7}$$

Take $\rho_t > 0$ such that

$$1 < (\lambda_1)^{2M_{t-1}} < (\lambda_t)^{\rho_t}. \tag{8}$$

Second step: Choice of n_t. Take $c_t > 0$ such that

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-c_t}^{c_t} \exp(-s^2/2)\, ds = p_t.$$

It follows from the central limit theorem that one can obtain a large integer $n_t > M_{t-1}$ such that

$$\sum_{U_t - 2\rho_t \le |k - n_t\lambda_t/(1+\lambda_t)| \le U_t} f_t(k) \le \tfrac{1}{4} p_t, \tag{9}$$

$$1 - 2p_t < \sum_{|k - n_t\lambda_t/(1+\lambda_t)| > U_t} f_t(k) \le 1 - \tfrac{1}{2} p_t, \tag{10}$$

where

$$f_t(k) = (1/(1+\lambda_t))^{n_t - k} (\lambda_t/(1+\lambda_t))^k \binom{n_t}{k},$$

and

$$U_t = n_t^{\frac{1}{2}} \lambda_t^{\frac{1}{2}} c_t/(1+\lambda_t)$$

for $k = 0, 1, \ldots, n_t$.

By (9) and (10) we have

$$\sum_{|k - n_t\lambda_t/(1+\lambda_t)| < U_t - 2\rho_t} f_t(k) > \tfrac{1}{4} p_t. \tag{11}$$

Last step: Choice of m_t. We write

$$F_t(\omega) = \prod_{u=1}^{t} \left(\frac{2}{1+\lambda_u}\right)^{n_u} (\lambda_u)^{\omega_{N_t - N_u + 1} + \omega_{N_t - N_u + 2} + \cdots + \omega_{N_t - M_u - 1}}, \tag{12}$$

and for $R < S$

$$H(\omega) = \chi_{[R,S]}(\omega_1 + \omega_2 + \cdots + \omega_{n_t}) \tag{13}$$

for $\omega \in \Omega$. It follows from Birkhoff's ergodic theorem that

$$\lim_{m \to \infty} \frac{\sum_{j=0}^{m} F_t(T^j\omega) H(T^j\omega)}{\sum_{j=0}^{m} F_t(T^j\omega)} = \frac{E_Q(F_t H)}{E_Q(F_t)}, \tag{14}$$

Q-a.e. ω, where E_Q is the integration with respect to the measure Q.

Since

$$(\lambda_u)^{\omega_{N_t-N_u+1}+\omega_{N_t-N_u+2}+\cdots+\omega_{N_t-M_{u-1}}}, \quad u=1,2,\ldots,t,$$

are independent random variables with respect to the measure Q, we have

$$\begin{aligned} E_Q(F_t) &= \prod_{u=1}^{t}\left(\frac{2}{1+\lambda_u}\right)^{n_u}\int(\lambda_u)^{\omega_{N_t-N_u+1}+\omega_{N_t-N_u+2}+\cdots+\omega_{N_t-M_{u-1}}}\,dQ(\omega) \\ &= \prod_{u=1}^{t} 1 \\ &= 1, \end{aligned} \tag{15}$$

and

$$\begin{aligned} E_Q(F_tH) &= \prod_{u=1}^{t-1}\left(\frac{2}{1+\lambda_u}\right)^{n_u}\int(\lambda_u)^{\omega_{N_t-N_u+1}+\omega_{N_t-N_u+2}+\cdots+\omega_{N_t-M_{u-1}}}\,dQ(\omega) \\ &\quad\times\left(\frac{2}{1+\lambda_t}\right)^{n_t}\int_{R\le\omega_1+\omega_2+\cdots+\omega_{n_t}\le S}(\lambda_t)^{\omega_1+\omega_2+\cdots+\omega_{n_t}}\,dQ(\omega) \\ &= \sum_{R\le k\le S} f_t(k). \end{aligned} \tag{16}$$

We write

$$H_t(\omega)=\chi_{[n_t\lambda_t/(1+\lambda_t)-U_t,\,n_t\lambda_t/(1+\lambda_t)+U_t]}(\omega_1+\omega_2+\cdots+\omega_{n_t}), \tag{17}$$

then we have from (10) and (16)

$$E_Q(F_tH_t)<2p_t. \tag{18}$$

It follows from (14), (15) and (18) that we have for all large integers m

$$Q\left(\omega:\frac{\sum_{j=0}^{m-1}F_t(T^j\omega)H_t(T^j\omega)}{\sum_{j=0}^{m-1}F_t(T^j\omega)}<2p_t\right)>1-\varepsilon_t. \tag{19}$$

We take a large integer m_t with

$$m_t>N_t,$$

such that for

$$A_t=\left\{\omega\in\Omega:\frac{\sum_{j=0}^{m_t-N_t-1}F_t(T^j\omega)H_t(T^j\omega)}{\sum_{j=0}^{m_t-N_t-1}F_t(T^j\omega)}<2p_t\right\}, \tag{20}$$

we have

$$Q(A_t)>1-\varepsilon_t \tag{21}$$

and

$$\frac{N_t\exp(2\eta_{t+1})(\lambda_1)^{3N_t}}{m_t-N_t}<\tfrac{1}{2}\varepsilon_t. \tag{22}$$

THEOREM 2. *Let T be the shift on the infinite product measure space*

$$(\Omega,P)=\left(\prod_{k=-\infty}^{\infty}\{0,1\},\prod_{k=-\infty}^{\infty}P_k\right)$$

constructed above. Then T is non-singular, ergodic and admits no σ-finite invariant measure equivalent to P.

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After some preparation we shall prove this theorem.

3.2. *Radon–Nikodym density* $(dPT/dP)(\omega)$. The sequence $(\lambda_t)_{t\geq 1}$ in § 3.1 satisfies the non-singularity condition (3)

$$\begin{aligned}\sum_{t=1}^{\infty}(\log\lambda_t)^2&<\sum_{t=1}^{\infty}\varepsilon_t^2\quad(\text{by }(7))\\&<\infty\quad(\text{by }(5)).\end{aligned}$$

Thus the shift T constructed in § 3.1 is non-singular.

LEMMA 1. *Let T be the shift in* § 3.1 *and put*

$$K_{t,i}(\omega)=\prod_{u=t+1}^{\infty}(\lambda_u)^{-\{\omega_{-N_u+1}+\omega_{-N_u+2}+\cdots+\omega_{-N_u+i}\}+\{\omega_{-M_{u-1}+1}+\omega_{-M_{u-1}+2}+\cdots+\omega_{-M_{u-1}+i}\}},$$

then we have

$$\frac{dPT^i}{dP}(\omega)=K_{t,i}(\omega)\times\prod_{k=-N_t+1}^{i-1}\frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}\quad\textit{if } 0\leq i<N_t,\tag{23}$$

and

$$\begin{aligned}\frac{dPT^i}{dP}(\omega)=K_{t,i}(\omega)\times\prod_{u=1}^{t}\left(\frac{1+\lambda_u}{2}\right)^{n_u}(\lambda_u)^{-\{\omega_{-N_u+1}+\omega_{-N_u+2}+\cdots+\omega_{-M_{u-1}}\}}\\\times F_t(T^{i-N_t}\omega)\quad\textit{if } N_t\leq i<m_t\end{aligned}\tag{24}$$

for P-a.e. ω, where $F_t(\omega)$ is the random variable defined in (12).

Proof. For $0<i<N_t$ it follows from (2) that

$$\begin{aligned}\frac{dPT^i}{dP}(\omega)&=\prod_{u=t+1}^{\infty}\left\{\prod_{k=-N_u+1}^{-N_u+i}\left(\frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}\right)\times\prod_{k=-M_{u-1}+1}^{-M_{u-1}+i}\left(\frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}\right)\right\}\\&\quad\times\prod_{k=-N_t+1}^{i-1}\frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}\\&=\prod_{u=t+1}^{\infty}\left\{\left(\frac{1+\lambda_u}{2}\right)^{i}\times(\lambda_u)^{-\{\omega_{-N_u+1}+\omega_{-N_u+2}+\cdots+\omega_{-N_u+i}\}}\right.\\&\quad\left.\times\left(\frac{2}{1+\lambda_u}\right)^{i}\times(\lambda_u)^{\{\omega_{-M_{u-1}+1}+\omega_{-M_{u-1}+2}+\cdots+\omega_{-M_{u-1}+i}\}}\right\}\\&\quad\times\prod_{k=-N_t+1}^{i-1}\frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}\end{aligned}$$

for P-a.e. ω.

For $N_t\leq i<m_t$, the second factor of (23) is

$$\begin{aligned}&\prod_{u=1}^{t}\left\{\prod_{k=-N_u+1}^{-M_{u-1}}\frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}\right\}\times\prod_{u=1}^{t}\left\{\prod_{k=i-N_u+1}^{i-M_{u-1}}\frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}\right\}\\&=\prod_{u=1}^{t}\left\{\left(\frac{1+\lambda_u}{2}\right)^{n_u}(\lambda_u)^{-\{\omega_{-N_u+1}+\omega_{-N_u+2}+\cdots+\omega_{-M_{u-1}}\}}\right\}\\&\quad\times\prod_{u=1}^{t}\left\{\left(\frac{2}{1+\lambda_u}\right)^{n_u}(\lambda_u)^{\{\omega_{i-N_u+1}+\omega_{i-N_u+2}+\cdots+\omega_{i-M_{u-1}}\}}\right\}\end{aligned}$$

for P-a.e. ω. □

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3.3. *Ratio ergodic theorem.* Krengel proved in [3] that if the shift on an infinite product measure

$$\left(\prod_{k=-\infty}^{\infty}\{0,1\},\ \prod_{k=-\infty}^{\infty}P_k\right)$$

with

$$P_k(0)=P_k(1)=\tfrac{1}{2}\quad\text{for } k>0$$

is non-singular and conservative then it is ergodic. The shift in § 3.1 satisfies the condition. This is because

$$F_t(\omega)>\prod_{u=1}^{t}\left(\frac{2}{1+\lambda_u}\right)^{n_u}$$

for every ω, so for $N_t\le i<m_t$ we have from (24)

$$\begin{aligned}\frac{dPT^i}{dP}(\omega)&\ge\prod_{u=t+1}^{\infty}(\lambda_u)^{-(i+1)}\times\prod_{u=1}^{t}(\lambda_u)^{-n_u} \qquad (25)\\ &\ge\prod_{u=t+1}^{\infty}\lambda_u^{-M_{u-1}}\times\prod_{u=1}^{t}\lambda_1^{-n_u}\\ &\ge\prod_{u=t+1}^{\infty}\exp(-\varepsilon_u)\times(\lambda_1)^{-N_t}\quad\text{(by (7))}\\ &=\exp(-\eta_{t+1})\times(\lambda_1)^{-N_t}.\end{aligned}$$

Then we have for a.e. ω,

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{dPT^i}{dP}(\omega)&\ge\sum_{t=1}^{\infty}\sum_{i=N_t}^{m_t-1}\frac{dPT^i}{dP}(\omega)\\ &\ge\sum_{t=1}^{\infty}(m_t-N_t)\exp(-\eta_{t+1})(\lambda_1)^{-N_t}\\ &\ge\sum_{t=1}^{\infty}\frac{2N_t\exp(\eta_{t+1})(\lambda_1)^{2N_t}}{\varepsilon_t}\quad\text{(by (22))}\\ &=\infty.\end{aligned}$$

It follows from the Chacon–Ornstein ratio ergodic theorem [1] that for any measurable set E with $P(E)>0$,

$$\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\frac{dPT^i}{dP}(\omega)\chi_E(T^i\omega)}{\sum_{i=0}^{n-1}\frac{dPT^i}{dP}(\omega)}=P(E) \qquad (26)$$

for P-a.e. ω.

PROPOSITION 1. *Let T be the shift constructed in § 3.1 and, for a measurable set E with $P(E)>0$ and $t>1$, we write*

$$\mathrm{I}_t(\omega)=\frac{\sum_{i=0}^{m_t-1}\frac{dPT^i}{dP}(\omega)\chi_E(T^i\omega)}{\sum_{i=0}^{m_t-1}\frac{dPT^i}{dP}(\omega)},$$

$$\mathrm{II}_t(\omega)=\frac{\sum_{i=N_t}^{m_t-1}\frac{dPT^i}{dP}(\omega)\chi_E(T^i\omega)}{\sum_{i=N_t}^{m_t-1}\frac{dPT^i}{dP}(\omega)},$$

$$\mathrm{III}_t(\omega)=\frac{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)\chi_E(T^i\omega)}{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)},$$

$$\mathrm{IV}_t(\omega)=\frac{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)\{1-H_t(T^{i-N_t}\omega)\}\chi_E(T^i\omega)}{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)},$$

where $H_t(\omega)$ is the random variable defined in (17). *Then we have*

$$\lim_{t\to\infty}\mathrm{I}_t(\omega)=\lim_{t\to\infty}\mathrm{II}_t(\omega)=\lim_{t\to\infty}\mathrm{III}_t(\omega)=\lim_{t\to\infty}\mathrm{IV}_t(\omega)=P(E)$$

for P-a.e. ω.

Proof. We already mentioned in (26) that the limit of $(\mathrm{I}_t)_{t\geq 1}$ exists for P-a.e. ω, and is equal to the constant $P(E)$. Since

$$\begin{aligned}|\mathrm{I}_t(\omega)-\mathrm{II}_t(\omega)|&\leq 2\times\frac{\sum_{i=0}^{N_t-1}\frac{dPT^i}{dP}(\omega)}{\sum_{i=N_t}^{m_t-1}\frac{dPT^i}{dP}(\omega)}\\&\leq 2\times\frac{N_t\exp(\eta_{t+1})\lambda_1^{2N_t}}{(m_t-N_t)\exp(-\eta_{t+1})\lambda_1^{-N_t}}\quad\text{(by (23) and (25))}\\&<\varepsilon_t\quad\text{(by (22))},\end{aligned}$$

we have

$$\lim_{m\to\infty}\mathrm{I}_t(\omega)=\lim_{m\to\infty}\mathrm{II}_t(\omega)$$

for P-a.e. ω.

It follows from (24) that

$$\mathrm{II}_t(\omega)=\frac{\sum_{i=N_t}^{m_t-1}K_{t,i}(\omega)F_t(T^{i-N_t}\omega)\chi_E(T^i\omega)}{\sum_{i=N_t}^{m_t-1}K_{t,i}(\omega)F_t(T^{i-N_t}\omega)}.$$

Since

$$\exp(-\eta_{t+1})<K_{t,i}(\omega)<\exp(\eta_{t+1}),$$

we have

$$\exp(-2\eta_{t+1})<\frac{\mathrm{II}_t(\omega)}{\mathrm{III}_t(\omega)}<\exp(2\eta_{t+1}).$$

Thus

$$\lim_{t\to\infty}\mathrm{II}_t(\omega)=\lim_{t\to\infty}\mathrm{III}_t(\omega)$$

for P-a.e. ω.

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Let us reconsider the set A_t defined in (21). Since this set is $\bigvee_{k=1}^{M_t-1}\mathcal{F}_k$-measurable, we have

$$P(A_t)=Q(A_t)>1-\varepsilon_t.$$

Since

$$\sum_{t=1}^{\infty} P(A_t^c)\leq \sum_{t=1}^{\infty}\varepsilon_t<\infty,$$

by the Borel–Cantelli lemma (in the general case), it holds that P-a.e. ω is in A_t for all but a finite number of t. This means that for P-a.e. ω and for all large numbers t

$$\frac{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)H_t(T^{i-N_t}\omega)\chi_E(T^i\omega)}{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)}\leq\frac{\sum_{j=0}^{m_t-N_t-1}F_t(T^j\omega)H_t(T^j\omega)}{\sum_{j=0}^{m_t-N_t-1}F_t(T^j\omega)}$$
$$<2p_t.$$

Since p_t converges to 0, we have that for P-a.e. ω

$$\lim_{t\to\infty}\mathrm{III}_t(\omega)=\lim_{t\to\infty}\frac{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)H_t(T^{i-N_t}\omega)\chi_E(T^i\omega)}{\sum_{i=N_t}^{m_t-1}F_t(T^{i-N_t}\omega)}+\lim_{t\to\infty}\mathrm{IV}_t(\omega)$$
$$=\lim_{t\to\infty}\mathrm{IV}_t(\omega).\qquad\square$$

3.4. *Recurrence*

PROPOSITION 2. *Let T be the shift constructed in* § 3.1. *For any measurable set E with $P(E)>0$, there exist for P-a.e. ω an infinite number of t such that $T^i\omega\in E$ for some i, $N_t\leq i<m_t$, and that for such i, either*

$$\frac{dPT^i}{dP}(\omega)>(\lambda_1)^{2M_{t-1}}\exp(-\eta_{t+1}),$$

or

$$\frac{dPT^i}{dP}(\omega)<(\lambda_1)^{-2M_{t-1}}\exp(\eta_{t+1})$$

holds.

Proof. We define a set B_t by

$$B_t=\left\{\omega\in\Omega:\frac{n_t\lambda_t}{1+\lambda_t}-U_t+2\rho_t<\omega_{-N_t+1}+\omega_{-N_t+2}+\cdots+\omega_{-M_{t-1}}<\frac{n_t\lambda_t}{1+\lambda_t}+U_t-2\rho_t\right\}. \tag{27}$$

By (11) we have

$$P(B_t)>\tfrac{1}{4}p_t.$$

Since the set B_t is $\bigvee_{j=-N_t+1}^{-M_{t-1}}\mathcal{F}_j$-measurable, the sets $B_1, B_2,\ldots$ are independent with respect to the measure P. It follows from the Borel–Cantelli lemma (in the independent case) that for P-a.e. ω there exists an infinite number of t such that $\omega\in B_t$.

On the other hand we have from proposition 1 that

$$\lim_{t\to\infty} \frac{\sum_{i=N_t}^{m_t-1} F_t(T^{i-N_t}\omega)\{1-H_t(T^{i-N_t}\omega)\}\chi_E(T^i\omega)}{\sum_{i=N_t}^{m_t-1} F_t(T^{i-N_t}\omega)} = P(E)$$

for P-a.e. ω. Then we have that for P-a.e. ω and for all large numbers t, $T^i\omega \in E$, for some i with $N_t \le i < m_t$, and that for such i either

$$\omega_{i-N_t+1}+\omega_{i-N_t+2}+\cdots+\omega_{i-M_{t-1}} > \frac{n_t\lambda_t}{1+\lambda_t}+U_t$$

or

$$\omega_{i-N_t+1}+\omega_{i-N_t+2}+\cdots+\omega_{i-M_{t-1}} < \frac{n_t\lambda_t}{1+\lambda_t}-U_t$$

holds.

Combining these results, we see that there exists for P-a.e. ω an infinite number of t such that $T^i\omega \in E$ for some i with $N_t \le i < m_t$, and that for such i,

$$(\lambda_t)^{\{\omega_{i-N_t+1}+\omega_{i-N_t+2}+\cdots+\omega_{i-M_{t-1}}\}-\{\omega_{-N_t+1}+\omega_{-N_t+2}+\cdots+\omega_{-M_{t-1}}\}} > (\lambda_t)^{2\rho_t} \tag{28}$$

or

$$(\lambda_t)^{\{\omega_{i-N_t+1}+\omega_{i-N_t+2}+\cdots+\omega_{i-M_{t-1}}\}-\{\omega_{-N_t+1}+\omega_{-N_t+2}+\cdots+\omega_{-M_{t-1}}\}} < (\lambda_t)^{-2\rho_t} \tag{29}$$

holds. If (28) holds then it follows from (24) that

$$\begin{aligned}\frac{dPT^i}{dP}(\omega) &> (\lambda_t)^{\{\omega_{i-N_t+1}+\omega_{i-N_t+2}+\cdots+\omega_{i-M_{t-1}}\}-\{\omega_{-N_t+1}+\omega_{-N_t+2}+\cdots+\omega_{-M_{t-1}}\}}\\ &\quad\times(\lambda_1)^{-2M_{t-1}}\exp(-\eta_{t+1})\\ &> (\lambda_t)^{2\rho_t}(\lambda_1)^{-2M_{t-1}}\exp(-\eta_{t+1})\\ &> (\lambda_1)^{2M_{t-1}}\exp(-\eta_{t+1}) \quad \text{(by (8))}.\end{aligned}$$

If (29) holds then it follows from (24) that

$$\begin{aligned}\frac{dPT^i}{dP}(\omega) &< (\lambda_t)^{\{\omega_{i-N_t+1}+\omega_{i-N_t+2}+\cdots+\omega_{i-M_{t-1}}\}-\{\omega_{-N_t+1}+\omega_{-N_t+2}+\cdots+\omega_{-M_{t-1}}\}}\\ &\quad\times(\lambda_1)^{2M_{t-1}}\exp(\eta_{t+1})\\ &< (\lambda_t)^{-2\rho_t}(\lambda_1)^{2M_{t-1}}\exp(\eta_{t+1})\\ &< (\lambda_1)^{-2M_{t-1}}\exp(-\eta_{t+1}).\end{aligned}$$

□

3.5. *Proof of theorem* 2. We assume that T admits a σ-finite invariant measure equivalent to P. Then there exists a positive measurable function $f(\omega)$ such that

$$\frac{dPT^i}{dP}(\omega) = \frac{f(T^i\omega)}{f(\omega)}$$

for $i \in \mathbb{Z}$ and P-a.e. ω. Take $a > b > 0$ such that

$$P(\omega \in \Omega : b < f(\omega) < a) > 0$$

and set $E = \{\omega \in \Omega : b < f(\omega) < a\}$. Then we have

$$\frac{b}{a} < \frac{dPT_E^i}{dP} < \frac{a}{b}$$

for all $i \in \mathbb{Z}$ and P-a.e. ω, that is, the functions $(dPT_E^i/dP)(\omega)$, $i \in \mathbb{Z}$, have a uniformly positive lower bound and a uniform upper bound, where T_E is the induced transformation on the set E of T. However, this contradicts proposition 2. $\square$

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