Canad. Math. Bull. Vol. **57** (2), 2014 pp. 245–253
http://dx.doi.org/10.4153/CMB-2013-024-8

Assouad–Nagata Dimension of Wreath Products of Groups

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Abstract. Consider the wreath product $H \wr G$, where $H \neq 1$ is finite and G is finitely generated. We show that the Assouad–Nagata dimension $\dim_{AN}(H \wr G)$ of $H \wr G$ depends on the growth of G as follows: if the growth of G is not bounded by a linear function, then $\dim_{AN}(H \wr G) = \infty$; otherwise $\dim_{AN}(H \wr G) = \dim_{AN}(G) \leq 1$.

1 Introduction

Asymptotic dimension was introduced by Gromov in [11] as a large scale invariant of a metric space. Any finitely generated group can be equipped with a word metric. The idea of Gromov was that asymptotic dimension is an invariant of the finitely generated group; *i.e.*, it does not depend on the word metric. An additional asymptotic invariant of the group of asymptotic dimension n introduced by Gromov is the asymptotic type of a certain function associated with the given asymptotic dimension (we call it an *n-dimensional control function*). The Assouad–Nagata dimension of a metric space X is the smallest integer n such that X has an n-dimensional control function that is a dilation.

Spaces of finite asymptotic Assouad–Nagata dimension have some extra properties that spaces of finite asymptotic dimension do not necessarily have. For example, if a metric space is of finite asymptotic Assouad–Nagata dimension, then it satisfies nice Lipschitz extension properties (see [2, 13]). It was proved in [6] that the asymptotic Assouad–Nagata dimension bounds the topological dimension of every asymptotic cone of a metric space. Also, every metric space of finite asymptotic Assouad–Nagata dimension has Hilbert space compression one [9].

P. Nowak [15] proved that the Assouad–Nagata dimension of some wreath products $H \wr G$ is infinite, where H is finite and G is a finitely generated amenable group whose Folner function grows sufficiently fast and satisfies some other conditions suitable for applying Erschler's result [8]. That result states that the Folner function $F(H \wr G)$ of $H \wr G$ is comparable to $F(H)^{F(G)}$ and the passage from it to Assouad–Nagata dimension of $H \wr G$ is fairly complicated as it includes Property A. Thus, the results of [15] apply only to amenable groups G and do not apply either to lamplighter groups (as the Folner function of $\mathbf{Z}$ is linear) or to wreath products with free non-Abelian groups (as those are not amenable).

Received by the editors June 6, 2012.
Published electronically August 10, 2013.
The second-named author was partially supported by the Center for Advanced Studies in Mathematics at Ben Gurion University of the Negev (Beer-Sheva, Israel).
AMS subject classification: **54F45**, 55M10, 54C65.
Keywords: Assouad–Nagata dimension, asymptotic dimension, wreath product, growth of groups.

In this paper we show that the Assouad–Nagata dimension of $H \wr G$ completely depends on the linearity of the growth of G. If G is finite, then $H \wr G$ is also finite and $\dim_{AN}(H \wr G) = 0 = \dim_{AN}(G)$. If G is virtually cyclic (*i.e.*, has linear growth), then $\dim_{AN}(H \wr G) = \dim_{AN}(G) = 1$. If the growth of G is not bounded by a linear function and $H \neq 1$, then $\dim_{AN}(H \wr G) = \infty$.

In particular, the lamplighter groups are not finitely presented and are of Assouad–Nagata dimension 1, which answers positively the following question of [6].

Question 1.1 Is there a finitely generated group of Assouad–Nagata dimension 1 that is not finitely presented?

2 Assouad–Nagata Dimension

Let X be a metric space and $n \geq 0$. An *n-dimensional control function of X* is a function $D^n_X \colon \mathbf{R}_+ \to \mathbf{R}_+ \cup \infty$ with the following property. For any $r > 0$ there is a cover $\{X_0, \dots, X_n\}$ of X whose Lebesgue number is at least r (that means every open r-ball $B(x, r)$ is contained in some X_i) and every r-component of X_i is of diameter at most $D^n_X(r)$. Two points x and y belong to the same r-component of X_i if there is a sequence $x_0 = x, x_1, \dots, x_k = y$ in X_i such that $\operatorname{dist}(x_j, x_{j+1}) < r$ (such a sequence will be called an *r-path*).

The *asymptotic dimension* $\operatorname{asdim}(X)$ is the smallest integer such that X has an n-dimensional control function whose values are finite.

The *Assouad–Nagata dimension* $\dim_{AN}(X)$ of a metric space X is the smallest integer n such that X has an n-dimensional control function that is a dilation (*i.e.,* $D^n_X(r) = C \cdot r$ for some $C > 0$).

The *asymptotic Assouad–Nagata dimension* $\operatorname{asdim}_{AN}(X)$ of a metric space X is the smallest integer n such that X has an n-dimensional control function that is linear (*i.e.,* $D^n_X(r) = C \cdot r + C$ for some $C > 0$).

In the case of metrically discrete spaces X (that means there is $\epsilon > 0$ such that every two distinct points have the distance at least ϵ) $\operatorname{asdim}_{AN}(X) = \dim_{AN}(X)$ (see [2]). In particular, in case of finitely generated groups we can talk about Assouad–Nagata dimension instead of asymptotic Assouad–Nagata dimension.

A countable group G is called *locally finite* if every finitely generated subgroup of G is finite. A group G has asymptotic dimension 0 if and only if it is locally finite [16].

Notice that $\dim_{AN}(X) = 0$ if and only if there is $C > 0$ such that for any $r > 0$ and for every r-path the distance between its end-points is less than $C \cdot r$. In the case of groups one has the following useful criterion of being 0-dimensional.

Proposition 2.1 *Let (G, d_G) be a group equipped with a proper left-invariant metric d_G (that means bounded sets are finite). If G is locally finite, then the following conditions are equivalent:*

(i) $\dim_{AN}(G, d_G) = 0$*;*

(ii) *there is a constant $c > 0$ such that for each $r > 0$ the subgroup of G generated by $B(1, r)$ is contained in $B(1, c \cdot r)$.*

Proof (i) $\Rightarrow$ (ii). Consider a constant $K > 0$ such that for each $r > 0$ all r-components of G have diameter less than $K \cdot r$. Notice that if $g \in G$ belongs to r-component of 1 and $h \in B(1,r)$, then $d_G(g,gh) = d_G(1,h) < r$, so gh lies in the r-component of 1. Therefore the subgroup generated by $B(1,r)$ is contained in $B(1,K \cdot r)$.

(ii) $\Rightarrow$ (i). Let G_r be the subgroup of G generated by $B(1,r)$. Consider two different left cosets $y \cdot G_r$ and $z \cdot G_r$ of G_r in G. If $d_G(yg,zh) < r$ for some $g,h \in G_r$, then $f = h^{-1}z^{-1}yg \in B(1,r) \subset G_r$, so $y = z(hfg^{-1})$, a contradiction. That means each r-component of G is contained in a left coset of G_r and its diameter is less than $2cr$, *i.e.*, $\dim_{AN}(G,d_G) = 0$. ∎

Let us generalize r-paths as follows. By an *r-cube* in a metric space X we mean an injective function $f\colon \{0,1,\dots,k\}^n \to X$ with the property that the distance between $f(x)$ and $f(x+e_i)$ is less than r for all $x \in \{0,1,\dots,k\}^n$ such that $x + e_i \in \{0,1,\dots,k\}^n$. Here e_i belongs to the standard basis of $\mathbf{R}^n$.

A sufficient condition for $\dim_{AN}(X)$ being positive is the existence for every $C > 0$ of an r-path joining points of distance at least $C \cdot r$. The purpose of the remainder of this section is to find a similar sufficient condition for $\dim_{AN}(X) \geq n$.

Lemma 2.2 *Consider the set $X = \{0,1,\dots,k\}^n$ equipped with the l_1-metric. Suppose $X = X_1 \cup \dots \cup X_n$. If the open $(n+1)$-ball of every point of X is contained in some X_i, then a 2-component of some X_i contains two points whose i-coordinates differ by k.*

Proof Let us proceed by contradiction and assume that all 2-components of each X_i do not contain points whose i-coordinates differ by k. Create the cover A_i, $1 \leq i \leq n$, of the solid cube $[0,k]^n$ by adding unit cubes to A_i whenever all of its vertices are contained in X_i. Given $i \in \{1,\dots,n\}$ consider the two faces L_i and R_i of $[0,k]^n$ consisting of points whose i-th coordinates are 0 and k, respectively. Let B_i be the complement of the $\frac{1}{4}$-neighborhood of $A_i \cup L_i \cup R_i$. Notice that B_i separates between L_i and R_i. Indeed, if $L_i \cup R_i$ belongs to the same component of the $\frac{1}{4}$-neighborhood of $A_i \cup L_i \cup R_i$, then one can find a $\frac{1}{2}$-path in A_i between points in X_i whose i-coordinates differ by k. Picking points in X_i in the same unit cubes as vertices of the path, one gets a 2-path in X_i between points in X_i whose i-th coordinates differ by k.

Now we get a contradiction, as $\bigcap_{i=1}^n B_i = \varnothing$ in violation of the well-known result in dimension theory about separation (see [7, Theorem 1.8.1]). ∎

Corollary 2.3 *Suppose X is a metric space with an $(n-1)$-dimensional control function $D_X^{n-1}\colon \mathbf{R}_+ \to \mathbf{R}_+ \cup \infty$. For any r-cube*

$$f\colon \{0,1,\dots,k\}^n \to X$$

there exist two points a and b in $\{0,1,\dots,k\}^n$ whose i-th coordinates differ by k for some i and $\operatorname{dist}(f(a),f(b)) \leq D_X^{n-1}(n \cdot r)$.

Proof Consider a cover $X = X_1 \cup \dots \cup X_n$ of X of Lebesgue number at least $n \cdot r$ such that $(n\cdot r)$-components of each X_i are of diameter at most $D_X^{n-1}(n\cdot r)$. The cover $\{0,1,\dots,k\}^n = f^{-1}(X_1) \cup \dots \cup f^{-1}(X_n)$ has the property that the open $(n+1)$-ball of every point is contained in some $f^{-1}(X_i)$, so by Lemma 2.2 a 2-component (in the l_1-metric) of some $f^{-1}(X_i)$ contains two points a and b whose i-coordinates differ by k. Therefore $f(a)$ and $f(b)$ belong to the same r-component of X_i and $\operatorname{dist}(f(a),f(b)) \leq D_X^{n-1}(n \cdot r)$. ∎

We need an upper bound on the size of r-cubes f in terms of dimension control functions and the Lipschitz constant of f^{-1}. One should view the next result as a discrete analog of the fact that one cannot embed I^n into an $(n-1)$-dimensional topological space.

Corollary 2.4 *Suppose X is a metric space with an $(n-1)$-dimensional control function $D_X^{n-1}\colon \mathbf{R}_+ \to \mathbf{R}_+ \cup \infty$. If $f\colon \{0,1,\dots,k\}^n \to X$ is an r-cube, then $k \le D_X^{n-1}(n\cdot r)\cdot \operatorname{Lip}(f^{-1})$.*

Proof By Corollary 2.3 there is an index $i \le n$ and points a and b whose i-coordinates differ by k such that $\operatorname{dist}(f(a), f(b)) \le D_X^{n-1}(n\cdot r)$. Since

$$k \le \operatorname{dist}(a,b) \le \operatorname{Lip}(f^{-1})\cdot \operatorname{dist}\big(f(a), f(b)\big) \le D_X^{n-1}(n\cdot r)\cdot \operatorname{Lip}(f^{-1}),$$

we are done. ∎

3 Wreath Products

Let A and B be groups. Define the action of B on the direct product A^B (functions have finite support) by

$$bf(\gamma) := f(b^{-1}\gamma),$$

for any $f \in A^B$ and $\gamma \in B$. The *wreath product* of A and B, denoted $A \wr B$, is the semidirect product $A^B \rtimes B$ of groups A^B and B. That means it consists of ordered pairs $(f,b) \in A^B \times B$ and $(f_1,b_1)\cdot(f_2,b_2) = (f_1(b_1 f_2), b_1 b_2)$.

We will identify $(1,b)$ with $b \in B$ and $(f_a,1)$ with $a \in A$, where f_a is the function sending $1 \in B$ to a and $B \setminus \{1\}$ to 1. This way both A and B are subgroups of $A \wr B$, which is generated by B and elements of the form $b\cdot a\cdot b^{-1}$. That way the union of generating sets of A and B generates $A \wr B$.

The *lamplighter group* L_n is the wreath product $\mathbf{Z}/n \wr \mathbf{Z}$ of $\mathbf{Z}/n$ and $\mathbf{Z}$.

Consider the wreath product $H \wr G$, where H is finite and G is finitely generated. Let K be the kernel of $H \wr G \to G$. The group K is locally finite (the direct product of $|G|$ copies of H). In case H is finite we choose as a set of generators of $H \wr G$ the union of $H \setminus \{1\}$ and a set of generators of G. A *length* of an element of a finitely generated group (with a fixed set of generators) is the smallest number of the generating elements needed to make the given element of the group.

If $g \in G$ and $a \in H \setminus \{1\}$, then $g\cdot a\cdot g^{-1} \in K$ will be called the *a-bulb indexed by g* or the *(g,a)-bulb*. A *bulb* is a (g,a)-bulb for some $a \in H$ and some $g \in G$.

Lemma 3.1 *Suppose $n > 1$. Any product of bulbs indexed by mutually different elements $g_i \in G$, $i \in \{1,\dots,n\}$, has length at least n.*

Proof Consider $x = (g_1 a_1 g_1^{-1})\cdot\dots\cdot(g_n a_n g_n^{-1}) \in K$. If its length is smaller than n, then $x = x_1\cdot b_1\cdot x_2\cdot b_2\cdot\dots\cdot x_k\cdot b_k\cdot x_{k+1}$, where $k < n$ and $b_i \in H$, $x_i \in G$ for all i. We can rewrite x as $(y_1\cdot b_1\cdot y_1^{-1})\cdot(y_2\cdot b_2\cdot y_2^{-1})\cdot\dots\cdot(y_k\cdot b_k\cdot y_k^{-1})\cdot y$, where $y_1 = x_1$. Since $x \in K$, $y = 1$. Now we arrive at a contradiction by looking at projections of K onto its summands. ∎

Lemma 3.2 *Suppose $r > 1$. Any element of K of length less than r is a product of bulbs indexed by elements of G of length less than r.*

Proof Consider a minimal representation $x_1a_1x_2a_2\cdots x_ka_kz$ of an element of K of length less than r, where $x_i \in G$ and $a_j \in H \setminus \{1\}$. One can write this element as

$$(x_1a_1x_1^{-1})(x_1x_2a_2x_2^{-1}x_1^{-1})\cdots(x_1\cdots x_ka_kx_k^{-1}\cdots x_1^{-1}).$$

Therefore the bulbs involved are indexed by elements of G of length less than r. ■

In case of the lamplighter group L_2 there is a precise calculation of length of its elements in [4]. We need a generalization of those calculations.

Lemma 3.3 *Let H be finite and let G be virtually cyclic. Suppose the subgroup $\mathbf{Z}$ generated by $t \in G$ is of finite index n and there are generators $\{t, g_1, \dots, g_n\}$ of G such that every element g of G can be expressed as $g_i \cdot t^{e(g)}$ for some i.*

(i) *Every element of K can be expressed as a product of (h_i, a_i)-bulbs, $i = 1, \dots, k$, such that $h_i \neq h_j$ for $i \neq j$.*
(ii) *The length of such product is at most $n(k + 2 + 4\max\{|e(h_i)|\})$.*

Proof Observe that the product of the (g, a)-bulb and the (g, b)-bulb is the $(g, a\cdot b)$-bulb, so every product of bulbs can be represented as a product of (h_i, a_i)-bulbs, $i = 1, \dots, k$, such that $h_i \neq h_j$ for $i \neq j$. We will divide those bulbs into classes determined by $h_i \cdot t^{-e(h_i)}$. Since there are at most n classes, it suffices to show that if $h_i \cdot t^{-e(h_i)} = g$ for all i, then the length of the product x of (h_i, a_i)-bulbs is at most $k + 2 + 4\max\{|e(h_i)|\}$. We may order h_i so that the function $i \to e(h_i)$ is strictly increasing. Now,

$$g^{-1}\cdot x\cdot g = \prod_{i=1}^{k} t^{e(h_i)}\cdot a_i\cdot t^{-e(h_i)} = t^{e(h_1)}\cdot a_1\cdot t^{-e(h_1)+e(h_2)}\cdot a_2\cdot\dots\cdot a_k\cdot t^{-e(h_k)},$$

and its length is at most $k + |e(h_1)| + e(h_k) - e(h_1) + |e(h_k)| \leq k + 4\max\{|e(h_i)|\}$. Therefore the length of x is at most $k + 2 + 4\max\{|e(h_i)|\}$. ■

4 Dimension Control Functions of Wreath Products

Recall that the *growth* γ of G is the function counting the number of points in the open ball $B(1, r)$ of G for all $r > 0$. Notice that γ being bounded by a linear function is independent of the choice of generators of G.

The next result relates the growth function of G to dimension control functions of the kernel of the projection $H \wr G \to G$.

Theorem 4.1 *Suppose G and H are finitely generated and K is the kernel of the projection $H \wr G \to G$ equipped with the metric induced from $H \wr G$. If γ is the growth function of G and D_K^{n-1} is an $(n-1)$-dimensional control function of K, then the integer part of $\frac{\gamma(r)}{n}$ is at most $D_K^{n-1}(3nr)$.*

https://doi.org/10.4153/CMB-2013-024-8 Published online by Cambridge University Press

Proof Given $k \geq 1$ we will construct a $3r$-cube $f\colon \{0,k\}^n \to K$ similarly to the way paths in the Cayley graph of K are constructed. There it suffices to label the beginning vertex and all the edges, since that induces labeling of all the vertices. In the case of our $3r$-cube we label the origin by $1 \in K$ and each edge from x to $x + e_i$, e_i being an element of the standard basis of $\mathbf{R}^n$, will be labeled by $x(j,i)$, where j is the i-th coordinate of x. It remains to choose $x(j,i)$, $1 \leq i \leq n$ and $0 \leq j \leq k-1$. Given $r > 0$, consider mutually different elements $g(j,i)$, $1 \leq i \leq n$ and $0 \leq j \leq k-1$ of G whose length is smaller than r, where k is the integer part of $\frac{\gamma(r)}{n}$. Pick $u \in H \setminus \{1\}$ and put $x(j,i) = g(j,i) \cdot u \cdot g(j,i)^{-1}$. By Lemma 3.1 one has $\operatorname{Lip}(f^{-1}) \leq 1$, so $k \leq D_K^{n-1}(3nr)$ by Corollary 2.4. ∎

If H is finite, then the kernel K of the projection $H \wr G \to G$ is locally finite and it has a 0-dimensional control function D_K^0 attaining finite values (K is equipped with the metric induced from $H \wr G$). Let us relate D_K^0 to the growth of G.

Theorem 4.2 *Suppose G is finitely generated and $H \neq \{1\}$ is finite. Let K be the kernel of the projection $H \wr G \to G$ equipped with the metric induced from $H \wr G$. If γ is the growth function of G, then $D_K^0(r) := (2r+1)\gamma(r)$ is a 0-dimensional control function of K.*

Proof It suffices to show that r-component of 1 in K is of diameter at most $(2r+1)\gamma(r)$, as any r-component of K is a shift of the r-component containing 1. By Lemma 3.2 any element of $B(1,r)$ in K is a product of bulbs indexed by elements of G of length less than r. Therefore any product of elements in $B(1,r)$ is a product of bulbs indexed by elements of G of length less than r, and such product can be reduced to a product of at most $\gamma(r)$ such bulbs. Each of them is of length at most $2r+1$, so the length of the product is at most $(2r+1)\cdot\gamma(r)$. ∎

Theorem 4.3 (*cf.* [5, Proposition 4.2]) *Suppose G is finitely generated and $\pi\colon G \to I$ is a retraction onto its subgroup I with kernel K. Assume that K is equipped with the metric induced from a word metric on G such that generators of I are included in the set of generators of G. If D_I^n is an n-dimensional control function of I and D_K^0 is a 0-dimensional control function of K, then*

$$D_I^n(r) + D_K^0\big(r + 2D_I^n(r)\big)$$

is an n-dimensional control function of G.

Proof Given $r > 0$ express I as $I_0 \cup \cdots \cup I_n$ so that r-components of I_i have diameter at most $D_I^n(r)$. Consider $G_i = \pi^{-1}(I_i)$. If $g_1 \cdot 1, \dots, g_1 \cdot x_m$ is an r-path in G_i, then $h_1 = \pi(g_1)\cdot 1, \dots, h_m = \pi(g_1)\cdot y_m$ form an r-path in I_i (here $y_j = \pi(x_j)$), so $l(y_j) \leq D_I^n(r)$ for all j. Consider $z_j = x_j \cdot y_j^{-1} \in K$. Notice that $\operatorname{dist}(z_j, z_{j+1}) < r + 2D_I^n(r)$. Therefore, $\operatorname{dist}(1, z_m) \leq D_K^0(r + 2D_I^n(r))$, resulting in $l(x_m) \leq D_K^0(r+2D_I^n(r)) + D_I^n(r)$ and $\operatorname{dist}(g_1, g_1 \cdot x_m) \leq D_I^n(r) + D_K^0(r + 2D_I^n(r))$, which completes the proof. ∎

Definition 4.4 (*cf.* [12, Section VI.B]) Let f and g be functions from $\mathbf{R}_+$ to $\mathbf{R}_+$. We say that f *weakly dominates* g if there exist constants $\lambda \geq 1$ and $C \geq 0$ such that $g(t) \leq \lambda f(\lambda t + C) + C$ for all $t \in \mathbf{R}_+$.

Two functions are *weakly equivalent* if each weakly dominates the other.

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Notice that the functions 2^t and $t2^t$ are weakly equivalent.

Theorem 4.5 *Suppose G is finitely generated infinite group and $H \neq \{1\}$ is finite. Let γ be the growth function of G and D^n_G be an n-dimensional control function of G. Then for any $k \geq n$ there is a k-dimensional control function of $H \wr G$ that is weakly dominated by $(D^n_G(t)+t)\cdot\gamma(D^n_G(t)+t)$. Also, for any $k \geq n$ every k-dimensional control function of $H \wr G$ weakly dominates the function γ.*

Proof Notice that γ dominates a linear function and combine Theorems 4.2 and 4.3. To get the estimate from below, notice that a k-dimensional control function of $H \wr G$ works as a k-dimensional control function of the kernel K and apply Theorem 4.1. ∎

Our next result gives a better solution to Question 2 in [15].

Corollary 4.6 *Suppose G is a finitely generated group of exponential growth and $H \neq \{1\}$ is finite. If $\dim_{AN}(G) \leq n$, then for any $k \geq n$ the k-dimensional control function of $H \wr G$ is weakly equivalent to the function 2^t (*i.e.*, there is a k-dimensional control function of $H \wr G$ weakly dominated by 2^t, and every such control function weakly dominates 2^t).*

Corollary 4.7 *Let F_2 be the free non-Abelian group of two generators. For every $n \geq 1$ the n-dimensional control function of $\mathbf{Z}/2 \wr F_2$ is weakly equivalent to the function 2^t (*i.e.*, there is an n-dimensional control function of $\mathbf{Z}/2 \wr F_2$ weakly dominated by 2^t, and every such control function weakly dominates 2^t).*

Proof Notice that the function $f(t) = 2^t$ is weakly equivalent to the growth function of F_2 and $\dim_{AN}(F_2) = 1$. ∎

5 Assouad–Nagata Dimension of Wreath Products

Suppose that G is finitely generated and $H \neq 1$ is finite. If $\dim_{AN}(G) = 0$, then G is finite and so is $H \wr G$. In such a case $\dim_{AN}(H \wr G) = 0 = \dim_{AN}(G)$. Therefore it remains to consider the case of infinite groups G.

Theorem 5.1 *Suppose G is an infinite finitely generated group and H is a finite group. Let K be the kernel of $H \wr G \to G$. If the growth of G is bounded by a linear function, then $\dim_{AN}(K) = 0$ and $\dim_{AN}(H \wr G) = \dim_{AN}(G) = 1$.*

Proof Notice that Theorem 4.2 provides a 0-dimensional control function for K. However, it may not be bounded by a linear function, so we have to do more precise calculations.

The group G is a virtually nilpotent group by Gromov's Theorem (see [10] or [14, Theorem 97]). Let F be a nilpotent subgroup of G of finite index. Pick elements a_i, $i = 1, \dots, k$, of G such that $G = \bigcup_{i=1}^k a_i \cdot F$ and pick a natural n satisfying $|a_i| \leq n$ for all $i \leq k$. Every two elements of F can be connected in G by a 2-path. From each point of the path (other than initial and terminal points) one can move to F by a distance at most n (by representing that point as $a_i \cdot x$ for some $x \in F$). Therefore we

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can create a $(2n+2)$-path in F joining the original points. That means F is generated by its elements of length at most $2n+1$.

Let $\{F_i\}$ be the lower central series of F and let d_i be the rank of F_i/F_{i+1}, $i \geq 0$. Since the growth of F is also linear, Bass' Theorem (see [1] or [14, Theorem 103]) stating that the growth of F is polynomial of degree $d = \sum_{i=0}^{\infty}(i+1)\cdot d_i$ implies that $d_0 = 1$ and all the other ranks d_i are 0. Hence the abelianization of F is of the form $\mathbf{Z} \times A$, A being a finite group, and the commutator group of F is finite. Therefore F is virtually $\mathbf{Z}$, and that means G is virtually $\mathbf{Z}$ as well.

Now let n be the index of $\mathbf{Z}$ in G and pick elements $g_1, \ldots, g_n$ of G such that any element of G can be expressed as $g_i \cdot t^k$ for some $i \leq n$ and some k, where t is the generator of $\mathbf{Z} \subset G$. Without loss of generality we may assume that the set of generators of G chosen to compute the word length $l(w)$ of elements $w \in H \wr G$ is $t, g_1, \ldots, g_n$. For H we choose all of $H \setminus \{1\}$ as the set of generators.

We need the existence of $C > 0$ such that $\frac{|k|}{C} \leq l(t^k) \leq |k|$ for all k. It suffices to consider $k > 0$. Since the number of points in $B(1_G, 4)$ is finite, there is $C > 0$ such that $t^u \in B(1_G, 4)$ implies $|u| \leq C$. Now, if $l(t^k) = m$ and $t^k = x_1 \cdots x_m$, where $l(x_i) = 1$, then there are $u(i)$ such that $\operatorname{dist}(x_1 \cdots x_i, t^{u(i)}) \leq 1$ for all $i \leq k$ (we choose $u(m) = k$ obviously). Therefore $\operatorname{dist}(t^{u(i)}, t^{u(i+1)}) \leq 3$ and $u(i+1) - u(i) \leq C$. Now $k = u(m) = (u(m) - u(m-1)) + \cdots + (u(2) - u(1)) + u(1) \leq C \cdot m$, implying $l(t^k) = m \geq \frac{k}{C}$.

By Lemma 3.2 any element of K of length less than r is a product of bulbs indexed by elements of G of length less than $r > 1$. If $l(g_i \cdot t^k) < r$, then $l(t^k) < r + 1 < 2r$ and $|k| \leq C \cdot l(t^k) \leq 2Cr$. Therefore there are at most $n \cdot 4Cr$ such words and any product of such bulbs is of length at most $n(4Crn + 2 + 2Cr) \leq r(4Cn^2 + 2n + 2Cn)$ by Lemma 3.3.

Therefore the group generated by $B(1, r)$ in K is contained in $B(1, Lr)$, where $L = 4Cn^2+2n+2Cn$, and $\dim_{AN}(K) = 0$ by Proposition 2.1. Using the Hurewicz Theorem for Assouad–Nagata dimension from [3] we get $\dim_{AN}(H \wr G) \leq \dim_{AN}(G) = 1$ (one can also use Theorem 4.3). Since $H \wr G$ is infinite, its Assouad–Nagata dimension is positive and $\dim_{AN}(H \wr G) = \dim_{AN}(G) = 1$. ■

Corollary 5.2 *If the growth of G is not bounded by a linear function and $H \neq 1$, then* $\dim_{AN}(H \wr G) = \infty$.

Proof Let γ be the growth of G in some set of generators. Suppose $\dim_{AN}(K) < n < \infty$, so it has an $(n-1)$-dimensional function of the form $D_K^{n-1}(r) = C \cdot r$ for some $C > 0$. By Theorem 4.1 one has $\gamma(r)/n \leq C \cdot 3nr + 1$. Thus $\gamma(r) \leq n \cdot (3nCr + 1)$, and the growth of G is bounded by a linear function, a contradiction. ■

Problem 5.3 Suppose G is a locally finite group equipped with a proper left-invariant metric d_G. If $\dim_{AN}(G, d_G) > 0$, is $\dim_{AN}(G, d_G)$ infinite?

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