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Integrable geodesic flows on homogeneous spaces

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Abstract. A method is exposed which allows the construction of families of first integrals in involution for Hamiltonian systems which are invariant under the Hamiltonian action of a Lie group G. This is applied to invariant Hamiltonian systems on the tangent bundles of certain homogeneous spaces $M = G/K$. It is proved, for example, that every such invariant Hamiltonian system is completely integrable if M is a real or complex Grassmannian manifold or SU $(n+1)/$SO $(n+1)$ or a distance sphere in $\mathbb{C}P^{n+1}$. In particular, the geodesic flows of these homogeneous spaces are integrable.

1. *Introduction*

During the last 15 years the study of completely integrable Hamiltonian systems has regained interest among mathematicians and physicists. One calls a Hamiltonian system (completely) integrable if it admits a maximal number of independent first integrals in involution. A geometrical motivation for this definition is a theorem of Arnold–Jost which states that the integral curves of an integrable Hamiltonian system (under a certain additional hypothesis) are quasiperiodic, i.e. they are the orbits of a constant vector field on an invariant torus.

In the nineteenth century, mechanical problems described by integrable Hamiltonian systems were the only ones which could be treated successfully. The reason for this is a classical theorem of Liouville which says that, if a maximal system of independent first integrals in involution is known, then the Hamiltonian equations can be solved explicitly ('by quadratures'). After Poincaré had recognized that integrability is an exceptional phenomenon of Hamiltonian systems and began the study of their qualitative properties, the interest in integrable Hamiltonian systems vanished nearly completely and has only recently been revived. This new interest is based to a large extent on the discovery that certain partial differential equations can be considered as infinite dimensional integrable Hamiltonian systems; the most famous example of such an equation is the Korteweg–de Vries equation.

Today the study of integrable Hamiltonian systems has developed on its own, leading to the discovery of new examples and to a fruitful interplay between different mathematical disciplines such as differential geometry, algebraic geometry and the theory of Lie algebras.

The geodesic flows of Riemannian manifolds are special examples of Hamiltonian systems. The question arises, for which Riemannian manifolds will the geodesic flows, considered as Hamiltonian systems, be integrable?

Further, if the manifold is compact and the geodesic flow integrable, the theorem of Arnold–Jost applies. This is of special interest since it allows the possibility of a global qualitative study of the flow.

There are four classical examples of compact Riemannian manifolds with integrable geodesic flow:

(*a*) compact surfaces of revolution ('Clairaut's first integral');

(*b*) SO(3) with left invariant metric (the geodesic flow of a left invariant metric on SO(3) is, as a mechanical problem, equivalent to the equation of motion of a rigid body around a fixed point without external forces. This differential equation was first integrated by Euler, 1765);

(*c*) n-dimensional ellipsoids with different principal axes (Jacobi, 1838);

and, trivially,

(*d*) flat tori.

The only new examples are semi-simple Lie groups with certain left invariant metrics (generalized Euler equations), see [**6**].

The purpose of this paper is to find further examples of integrable geodesic flows. For this it seems reasonable to consider manifolds with large isometry groups, in particular Riemannian homogeneous spaces. By a theorem of E. Noether the geodesic flow of such a manifold M admits a whole Lie algebra of first integrals. However, these first integrals in general are not in involution and cannot be used directly to prove the integrability of the flow. In this paper integrability is proved using a more general class of first integrals. These are defined in the following way (cf. § 2). The Lie algebra of first integrals, arising from Noether's theorem, can be described equivalently by a '$\mathfrak{g}^*$-valued first integral', the so-called moment map. The moment map is a first integral not only for the geodesic flow but for all G-invariant Hamiltonian systems on T^*M. By composing the moment map with arbitrary functions on $\mathfrak{g}^*$ one thereby obtains a large class of real-valued first integrals for invariant Hamiltonian systems.

Furthermore, by this method, there are several possible ways to construct families of first integrals which are in involution. The difficult part in proving the complete integrability of G-invariant Hamiltonian systems on T^*M then consists in proving the independence of the functions in question. This is accomplished in several cases by considering explicit expressions for the Hamiltonian vector fields of these functions. In § 3 a class of homogeneous spaces $M = G/K$ is introduced, characterized by the property that they admit a G-invariant metric which is induced by a bi-invariant (possibly indefinite) metric of G. Using this metric a simple formula for the symplectic structure on T^*M is derived, which makes it possible to compute the Hamiltonian vector fields of G-invariant Hamiltonians on T^*M and of the first integrals defined in § 2. A corollary of these computations is that, for a Riemannian symmetric space, the invariant functions on T^*M are all in involution.

The particular construction of the first integrals, on which are based the main results of this paper, is explained in § 4. These first integrals are defined by projecting the moment map to non-degenerate subalgebras $\mathfrak{g}' \subset \mathfrak{g}$ and composing this projection with invariant functions on $\mathfrak{g}'$. For two such subalgebras $\mathfrak{g}_1, \mathfrak{g}_2$ one finds simple conditions guaranteeing that the first integrals generated by $\mathfrak{g}_1, \mathfrak{g}_2$ are in involution; for example $[\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_2$, which holds in particular if $\mathfrak{g}_1 \subset \mathfrak{g}_2$. This makes it possible to consider chains $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_n \hookrightarrow \mathfrak{g}_{n+1} = \mathfrak{g}$ of non-degenerate subalgebras. The whole family of first integrals defined by $\mathfrak{g}_1, \ldots, \mathfrak{g}_{n+1}$ is in involution. At the end of this section some geometric properties of the first integrals generated by a non-degenerate subalgebra are discussed. For example, the integral curves of their Hamiltonian vector fields are images of 1-parameter subgroups.

In §§ 5–7 these ideas are applied to concrete examples of homogeneous spaces. It is shown that every G-invariant Hamiltonian on T^*M defines a completely integrable Hamiltonian system if M is one of the following compact homogeneous spaces: (*a*) $G_{p,q}(\mathbb{R})$, (*b*) $G_{p,q}(\mathbb{C})$ (i.e. real and complex Grassmannians), (*c*) $\mathrm{SU}(n+1)/\mathrm{SO}(n+1)$, (*d*) M is a distance sphere in $\mathbb{C}P^{n+1}$ (these can be written as homogeneous spaces $\mathrm{SU}(n+1)\times\mathbb{R}/\mathrm{SU}(n)\times\mathbb{R}$). In particular, the geodesic flows of these Riemannian manifolds are integrable. As corollaries one obtains, for example, the complete integrability of the geodesic flow of $\mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ and the classical results, already mentioned, about left invariant metrics on SO (3).

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2. *First integrals induced by a Lie group invariance*

In this section we recall some basic facts about symplectic group actions. Detailed expositions of the theory can be found for example in [**1**] or [**2**].

Let (N, ω) be a symplectic manifold and G a Lie group acting on N by symplectic diffeomorphisms. If $\mathfrak{g}$ denotes the Lie algebra of G, then, for any $\xi \in \mathfrak{g}$, the 1-parameter subgroup $\exp t\xi$ induces a symplectic vector field H_ξ on N.

We assume that the action of G on N is Hamiltonian. This means that, for any $\xi \in \mathfrak{g}$, H_ξ is a Hamiltonian vector field generated by a Hamiltonian function $f_\xi \in C^\infty(N)$ and such that the mapping $\mathfrak{g} \to C^\infty(N)$, $\xi \mapsto f_\xi$ is a homomorphism of Lie algebras. Here the Lie algebra structure on $C^\infty(N)$ is given by the Poisson bracket:

$$(f, g) = \omega(H_f, H_g).$$

The Hamiltonian group actions which are considered in this paper arise in the following way. Let $\tau: G \times M \to M$ be the action of a Lie group G on a manifold M. T^*M, with the canonical 2-form, is a symplectic manifold, and the lift of the G-action to T^*M is Hamiltonian. For $\xi \in \mathfrak{g}$ the Hamiltonian function $f_\xi \in C^\infty(T^*M)$ is given as follows. Let ξ^* be the vector field on M induced by ξ, then f_ξ is the evaluation of cotangent vectors on ξ^*:

$$f_\xi(\alpha) = \alpha(\xi^*(m)) \quad \text{for } \alpha \in T^*_m M.$$

We want to study G-invariant Hamiltonians $f \in C^\infty(N)$ and try to construct a large set of first integrals of the Hamiltonian system H_f. By a first integral we mean a second function $g \in C^\infty(N)$ which is constant along the integral curves of H_f, or equivalently $(f, g) = 0$. Moreover, we require these first integrals to be in involution, i.e. the Poisson brackets of any two of them vanish identically. The final goal is to find a maximal number (that is $\frac{1}{2}\dim N$) of first integrals which are in involution and are functionally independent almost everywhere on N. If such a maximal system of first integrals exists, H_f is called (completely) integrable.

First integrals of H_f can be constructed by using the G-action on N. G-invariance of f implies that each f_ξ is a first integral (Noether's theorem). One obtains a larger class of first integrals by considering polynomial combinations of the f_ξ, $\xi \in \mathfrak{g}$. This amounts to an extension of the mapping $\mathfrak{g} \to C^\infty(N)$ to $S(\mathfrak{g}) \to C^\infty(N)$, where $S(\mathfrak{g})$ is the symmetric algebra over $\mathfrak{g}$. A further extension can be given by making use of the moment map.

The moment map $P: N \to \mathfrak{g}^*$ is defined as

$$P(n)(\xi) := f_\xi(n), \quad n \in N, \xi \in \mathfrak{g}.$$

P is a $\mathfrak{g}^*$-valued first integral of H_f. Therefore any $h \in C^\infty(\mathfrak{g}^*)$ generates a first integral f_h of H_f by setting $f_h := h \circ P \in C^\infty(N)$. If $\xi \in \mathfrak{g} = (\mathfrak{g}^*)^*$ this definition of f_ξ coincides with the previous one and, since $S(\mathfrak{g})$ is generated by $\mathfrak{g}$, the mapping $C^\infty(\mathfrak{g}^*) \to C^\infty(N)$, $h \mapsto f_h$ extends $S(\mathfrak{g}) \to C^\infty(N)$. (Here $S(\mathfrak{g})$ is identified with the algebra of polynomials on $\mathfrak{g}^*$.)

Now we turn to the question of how one can, by this method, find first integrals of H_f which are in involution. One defines a Poisson bracket on $C^\infty(\mathfrak{g}^*)$ which is derived from the Kostant–Kirillov symplectic structure on coadjoint orbits. For $h_1, h_2 \in C^\infty(\mathfrak{g}^*)$ define $(h_1, h_2) \in C^\infty(\mathfrak{g}^*)$ as

$$(h_1, h_2)(\alpha) := \alpha([dh_1(\alpha), dh_2(\alpha)]),$$

where $dh_i(\alpha) \in \mathfrak{g}$ denotes the derivative of h_i at the point $\alpha \in \mathfrak{g}^*$. Note that for $\xi_1, \xi_2 \in \mathfrak{g}$ this coincides with the ordinary Lie bracket of $\mathfrak{g}$. One can show that this Poisson bracket satisfies the same algebraic identities as the Poisson bracket on $C^\infty(N)$, see [**2**]. Moreover, it follows that:

(2.1) LEMMA. *The mapping $C^\infty(\mathfrak{g}^*) \to C^\infty(N)$, $h \mapsto f_h = h \circ P$ is compatible with the Poisson brackets, i.e.*

$$f_{(h_1,h_2)} = (f_{h_1}, f_{h_2}) \quad \textit{for all } h_1, h_2 \in C^\infty(\mathfrak{g}^*).$$

Proof. Let $n \in N$, $\alpha = P(n)$ and $\xi_i = dh_i(\alpha)$, $i = 1, 2$. Since both Poisson brackets only depend on first derivatives it follows that

$$\begin{aligned}(f_{h_1}, f_{h_2})(n) &= (f_{\xi_1}, f_{\xi_2})(n) = f_{[\xi_1,\xi_2]}(n) \\ &= \alpha([\xi_1, \xi_2]) = (h_1, h_2)(\alpha) = f_{(h_1,h_2)}(n). \qquad \square\end{aligned}$$

This gives a method of constructing first integrals of H_f which are in involution.

(2.2) PROPOSITION. *Let $f \in C^\infty(N)$ be a G-invariant Hamiltonian. For each $h_i \in C^\infty(\mathfrak{g}^*)$, $f_i := h_i \circ P$ is a first integral of H_f, and $(f_1, f_2) = 0$ if $(h_1, h_2) = 0$.* $\square$

Now one has to find families of functions in involution in $C^\infty(\mathfrak{g}^*)$. Such families can be obtained by using the invariant functions on $\mathfrak{g}^*$. These functions are those

$h \in C^\infty(\mathfrak{g}^*)$ invariant under the coadjoint action of G on $\mathfrak{g}^*$, which means

$$h(\mathrm{Ad}\,(g^{-1})^*\alpha) = h(\alpha) \quad \text{for all } g \in G \text{ and } \alpha \in \mathfrak{g}^*.$$

Inserting a 1-parameter subgroup $\exp t\xi$ into this equation and taking the derivative one finds for invariant functions:

$$\alpha([dh(\alpha), \xi]) = 0 \quad \text{for all } \xi \in \mathfrak{g} \text{ and } \alpha \in \mathfrak{g}^*.$$

This means that the Poisson bracket of an invariant function and an arbitrary function vanishes.

Actually, invariant functions can be used to generate even larger sets of functions in involution. With such a construction Mishchenko and Fomenko proved the complete integrability of the system of Euler equations on the regular coadjoint orbits of semi-simple Lie algebras [**6**]. A similar construction is given in [**10**], using an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g}$.

The results of this paper are based on a simple enlargement of the set of invariant functions which will be described in § 4.

3. *Symplectic geometry on $T(G/K)$*

We consider homogeneous spaces $M = G/K$, where G is a Lie group and K a closed subgroup; $\pi: G \to M$ denotes the canonical projection. Furthermore, we restrict ourselves to homogeneous spaces of the following special kind:

(3.1) *Property.* On the Lie algebra $\mathfrak{g}$ of G there exists an Ad (G)-invariant, symmetric, non-degenerate bilinear form B such that the restriction of B to the Lie algebra $\mathfrak{k}$ of K is likewise non-degenerate.

This property is not only used to simplify the notation but it will enable us to derive a simple formula for the symplectic structure on T^*M. We list some consequences.

Let $\mathfrak{m}$ be the B-complement of $\mathfrak{k}$ in $\mathfrak{g}$. $\mathfrak{m}$ is canonically identified with $T_{\pi(e)}M$ by means of $\pi_{*|e}$. $B_{|\mathfrak{m}\times\mathfrak{m}}$ is non-degenerate and Ad (K)-invariant and thus defines a (possibly indefinite) G-invariant metric on M. One has an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ (for $\xi \in \mathfrak{g}$ we write $\xi = \xi_{\mathfrak{k}} + \xi_{\mathfrak{m}}$) and, in the Lie algebra, the relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}.$$

An important property of homogeneous spaces satisfying (3.1) is that they are naturally reductive. This implies

$$\mathrm{Exp}_{\pi(e)} = \pi \circ \exp_{|\mathfrak{m}},$$

where Exp denotes the exponential map of M. Thus geodesics on M are images of 1-parameter subgroups of G, cf. [**5**, ch. X, thm. 3.5].

Examples of homogeneous spaces of this kind are:

Lie groups with bi-invariant (possibly indefinite) metric;

normal homogeneous spaces (i.e. B is positive definite on $\mathfrak{g}$);

Riemannian symmetric spaces.

Henceforth we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of B, and T^*M with TM by means of the invariant metric $\langle\ ,\ \rangle$ on M. We shall study the symplectic manifold TM. For the moment map P, one derives a very simple expression.

(3.2) LEMMA. *Let $M = G/K$ be a homogeneous space satisfying* (3.1). *The moment map $P: TM \to \mathfrak{g}$ is given as $P(g\xi) = \mathrm{Ad}\,(g)\xi$, where $g\xi \in T_{\pi(g)}M$, $\xi \in \mathfrak{m}$, $g \in G$.*

Proof. For all $\zeta \in \mathfrak{g}$ one has

$$B(P(g\xi), \zeta) = \left\langle g\xi, \frac{d}{dt}_{|0} \tau(\exp t\zeta, \pi(g))\right\rangle$$

$$= B\left(\xi, \frac{d}{dt}_{|0} \tau(g^{-1} \exp t\zeta g, \pi(e))\right) = B\left(\xi, \frac{d}{dt}_{|0} \pi(\exp t(\mathrm{Ad}\,(g^{-1})\zeta))\right)$$

$$= B(\xi, (\mathrm{Ad}\,(g^{-1})\zeta)_{\mathfrak{m}}) = B(\mathrm{Ad}\,(g)\xi, \zeta). \qquad \square$$

The problem we are concerned with in this paper is the construction of first integrals in involution for G-invariant Hamiltonian systems on TM. These first integrals shall be functionally independent almost everywhere. The proof of this independence turns out to be the difficult part in such constructions. We shall be able to do this in certain cases by using explicit expressions for their Hamiltonian vector fields. To derive such expressions we now develop a formula for the symplectic structure of $T(G/K)$.

We identify a neighbourhood W of $\pi(e)$ with a certain submanifold of G as follows. Let $U \subset \mathfrak{g}$ be a neighbourhood of 0 such that $\exp_{|U}: U \to \exp U$ is a diffeomorphism and such that there exists a neighbourhood V of 0 in $\mathfrak{m}$, $V \subset U \cap \mathfrak{m}$, which is mapped by $\pi \circ \exp$ diffeomorphically onto a neighbourhood W of $\pi(e)$ in M. By means of $\pi_{|\exp V}$ we identify $\exp V \subset G$ with W and regard

$$\exp_{*|TV}: TV = V \times \mathfrak{m} \to T(\exp V)$$

as a coordinate system for TW.

Using left translations one has the identifications

$$TG = G \times \mathfrak{g} \quad \text{and} \quad TTG = G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}.$$

Now, $T_\xi TM$, $\xi \in \mathfrak{m} = T_{\pi(e)}M$ can be looked at as a subspace of $G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. To determine this subspace define a tangent vector $(v, w) \in T_{(0,\xi)}(V \times \mathfrak{m})$ by the curve $t \mapsto (tv, \xi + tw)$. This is mapped in $T(\exp V)$ to $\exp_{*|tv}(\xi + tw)$ which becomes, by the identification $TG = G \times \mathfrak{g}$,

$$(\exp tv, L^{-1}_{\exp tv *} \circ \exp_{*|tv}(\xi + tw)).$$

The derivative at time 0 of this curve can be calculated using the formula:

$$L^{-1}_{\exp tv *} \circ \exp_{*|tv} = \frac{1 - \exp\{-\mathrm{ad}\,(tv)\}}{\mathrm{ad}\,(tv)} = \mathrm{Id} - \tfrac{1}{2} t\,\mathrm{ad}\,v + O(t^2),$$

cf. [**3**, ch. II, thm 1.7]. It follows that

$$(e, \xi, v, -\tfrac{1}{2}[v, \xi] + w) \in T_\xi TM \subset G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}.$$

M is naturally reductive and so $\mathrm{Exp} = \pi \circ \exp_{|V}$ defines normal coordinates centred at $\pi(e)$. This means, if one chooses a basis $\xi_1, \ldots, \xi_n \in \mathfrak{m}$ and defines local coordinates by

$$(x_1, \ldots, x_n) \mapsto \mathrm{Exp} \sum_{i=1}^{n} x_i \xi_i,$$

that the coefficients (g_{ij}) of the metric tensor $\langle\ ,\ \rangle$ satisfy

$$\frac{\partial g_{ij}}{\partial x_r}(0)=0 \quad \text{for } i, j, r=1, \ldots, n.$$

In such coordinates the symplectic structure of $T_\xi TM$ becomes the standard symplectic structure defined by the bilinear form B; so we have

$$\omega_{(\xi)}((v_1, -\tfrac{1}{2}[v_1, \xi]+w_1), (v_2, -\tfrac{1}{2}[v_2, \xi]+w_2))=B(v_1, w_2)-B(v_2, w_1).$$

Using the symplectic G-action on TM, one obtains a formula for the symplectic structure of TM at arbitrary points $g\xi \in TM$, $\xi \in \mathfrak{m}$, $g \in G$. Summarizing we have:

(3.3) PROPOSITION. *Let $M=G/K$ be a homogeneous space satisfying* (3.1). *Identify $T_\xi TM$, $\xi \in \mathfrak{m}$ by means of the exponential map with the subspace*

$$\{(v, -\tfrac{1}{2}[v, \xi]+w)_{(e,\xi)} : v, w \in \mathfrak{m}\}$$

of $G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. Then the symplectic structure of TM in $g\xi$ is given by

$$\omega_{g\xi}(g_{*|\xi}(v_1, -\tfrac{1}{2}[v_1, \xi]+w_1), g_{*|\xi}(v_2, -\tfrac{1}{2}[v_2, \xi]+w_2))=B(v_1, w_2)-B(v_2, w_1).$$

The horizontal subspace of $T_{g\xi}TM$ is

$$g_{*|\xi}\{(v, -\tfrac{1}{2}[v, \xi]) : v \in \mathfrak{m}\}$$

and the vertical subspace is $g_{*|\xi}\{(0, w)/w \in \mathfrak{m}\}$. □

As a first application let us compute the Hamiltonian vector field of a G-invariant function $f: TM \to \mathbb{R}$. G-invariant functions f on TM are in one–one correspondence with Ad (K)-invariant functions $h: \mathfrak{m} \to \mathbb{R}$, the correspondence being $f(g\xi)=h(\xi)$. For $h: \mathfrak{m} \to \mathbb{R}$ define grad $h(\xi)$, $\xi \in \mathfrak{m}$ using $B_{|\mathfrak{m}\times\mathfrak{m}}$.

(3.4) PROPOSITION. *Let $h: \mathfrak{m} \to \mathbb{R}$ be* Ad (K)*-invariant and $f: TM \to \mathbb{R}$ the G-invariant Hamiltonian defined by h. The Hamiltonian vector field H_f of f is given by*

$$H_f(g\xi)=g_{*|\xi}(v_1, -\tfrac{1}{2}[v_1, \xi]+w_1),$$

where $v_1=\operatorname{grad} h(\xi)$ and $w_1=-\tfrac{1}{2}[\operatorname{grad} h(\xi), \xi]_{\mathfrak{m}}$.

If $f_1, f_2: TM \to \mathbb{R}$ are two invariant Hamiltonians defined by $h_1, h_2: \mathfrak{m} \to \mathbb{R}$, then their Poisson bracket is

$$(f_1, f_2)(g\xi)=-B([\operatorname{grad} h_1(\xi), \operatorname{grad} h_2(\xi)], \xi).$$

Proof. For all $v, w \in \mathfrak{m}$ one has

$$\begin{aligned}
&df_{|g\xi}(g_* \circ (\pi_*)_{*|\xi}(v, -\tfrac{1}{2}[v, \xi]+w)) \\
&\quad=\frac{d}{dt_{|0}}(f \circ g \circ \pi_* \circ \exp_{*|tv}(\xi+tw)) \\
&\quad=\frac{d}{dt_{|0}}(f \circ (g \exp tv) \circ \pi_{*|e}(\xi+tw-\tfrac{1}{2}t[v, \xi]+O(t^2))) \\
&\quad=\frac{d}{dt_{|0}}(h(\xi+tw-\tfrac{1}{2}t[v, \xi]_{\mathfrak{m}}+O(t^2))) \\
&\quad=B(\operatorname{grad} h(\xi), w)-\tfrac{1}{2}B(\operatorname{grad} h(\xi), [v, \xi]) \\
&\quad=B(\operatorname{grad} h(\xi), w)-B(-\tfrac{1}{2}[\operatorname{grad} h(\xi), \xi], v).
\end{aligned}$$

The formula for the Poisson bracket follows by setting

$$v = \operatorname{grad} h_2(\xi), \quad w = -\tfrac{1}{2}[\operatorname{grad} h_2(\xi), \xi]_{\mathfrak{m}}. \qquad \square$$

As a simple example, take

$$h(\xi) = \tfrac{1}{2}B(\xi, \xi)$$

and obtain the equation of the geodesic flow on TM:

$$H_f(g\xi) = g_{*|\xi}(\xi, 0).$$

The corresponding Hamiltonian is in involution with all G-invariant functions on TM.

Symmetric spaces are characterized by $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, so we have the corollary:

(3.5) COROLLARY. *Let $M = G/K$ be a Riemannian symmetric space and $f: TM \to \mathbb{R}$ a G-invariant Hamiltonian. Then H_f is purely horizontal. Any two such Hamiltonians are in involution.* $\square$

For Riemannian symmetric spaces the algebra of $\operatorname{Ad}(K)$-invariant polynomials on $\mathfrak{m}$ is generated by r polynomials, where $r = \operatorname{rank} M$. These polynomials define on TM a family of r independent functions in involution.

This shows that one cannot expect to find integrable Hamiltonian systems on TM by considering only the invariant functions (except trivially where M is a symmetric space of Euclidian type). However, as we saw in § 2, G-invariant Hamiltonian systems on TM have many first integrals, namely all functions

$$f_h = h \circ P, \qquad h \in C^\infty(\mathfrak{g}).$$

We compute the Hamiltonian vector field of f_h.

(3.6) PROPOSITION. *Let $M = G/K$ be a homogeneous space satisfying* (3.1) *and $h: \mathfrak{g} \to \mathbb{R}$. Then the Hamiltonian vector field H_f of $f = h \circ P$ is given by*

$$H_f(g\xi) = g_{*|\xi}(v, -\tfrac{1}{2}[v, \xi] + w),$$

where

$$v = (\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}} \quad \text{and} \quad w = [\operatorname{Ad}(g^{-1})\zeta, \xi]_{\mathfrak{m}} - \tfrac{1}{2}[(\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}}, \xi]_{\mathfrak{m}}$$

with $\zeta = \operatorname{grad} h(\operatorname{Ad}(g)\xi)$.

Proof. Using (3.1) we compute as in proposition (3.4) that, for all $v', w' \in \mathfrak{m}$,

$$\begin{aligned} df_{|g\xi}&(g_* \circ (\pi_*)_{*|\xi}(v', -\tfrac{1}{2}[v', \xi] + w')) \\ &= \frac{d}{dt_{|0}}(h \circ \operatorname{Ad}(g \exp tv')\{\xi + tw' - \tfrac{1}{2}t[v', \xi]_{\mathfrak{m}} + O(t^2)\}) \\ &= B(\zeta, \operatorname{Ad}(g)\{[v', \xi] + w' - \tfrac{1}{2}[v', \xi]_{\mathfrak{m}}\}) \\ &= B(\operatorname{Ad}(g^{-1})\zeta, w') + B(\operatorname{Ad}(g^{-1})\zeta, [v', \xi] - \tfrac{1}{2}[v', \xi]_{\mathfrak{m}}). \end{aligned}$$

This gives $v = (\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}}$ and the second term equals

$$B(\operatorname{Ad}(g^{-1})\zeta - \tfrac{1}{2}(\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}}, [v', \xi]) = -B([\operatorname{Ad}(g^{-1})\zeta - \tfrac{1}{2}(\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}}, \xi], v')$$

which gives the expression for w. $\square$

4. *Integrals in involution defined by non-degenerate subalgebras*

Let G be a Lie group with an Ad (G)-invariant, non-degenerate, symmetric bilinear form B on $\mathfrak{g}$ its Lie algebra. As usual, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of B. The Poisson bracket in $C^\infty(\mathfrak{g}^*)$ of § 2 transforms to the following Poisson bracket in $C^\infty(\mathfrak{g})$:

$$(h_1, h_2)(\xi) = B(\xi, [\text{grad } h_1(\xi), \text{grad } h_2(\xi)]), \qquad h_1, h_2 \in C^\infty(\mathfrak{g}).$$

(Here grad $h(\xi)$ is defined using B.)

The equation characterizing Ad (G)-invariant functions h on $\mathfrak{g}$ is

$$[\xi, \text{grad } h(\xi)] = 0 \quad \text{for all } \xi \in \mathfrak{g}.$$

We now introduce a class of functions on $\mathfrak{g}$ which generalize the invariant functions.

Suppose $\mathfrak{g}' \subset \mathfrak{g}$ is a non-degenerate subalgebra, i.e. $B_{|\mathfrak{g}' \times \mathfrak{g}'}$ is non-degenerate. Then the orthogonal projection $\pi' : \mathfrak{g} \to \mathfrak{g}'$ is defined. Let $G' \subset G$ be the connected subgroup corresponding to $\mathfrak{g}'$. We consider functions $h' \circ \pi' \in C^\infty(\mathfrak{g})$, where h' is an Ad (G')-invariant function on $\mathfrak{g}'$.

Let now $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{g}$ be two such non-degenerate subalgebras with orthogonal projections $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ and invariant functions $h_i \in C^\infty(\mathfrak{g}_i)$, $i = 1, 2$. When are $h_1 \circ \pi_1$ and $h_2 \circ \pi_2$ in involution on $\mathfrak{g}$?

Their Poisson bracket can be computed as follows:

$$\begin{aligned}(h_1 \circ \pi_1, h_2 \circ \pi_2)(\xi) &= B(\xi, [\text{grad } h_1(\pi_1 \xi), \text{grad } h_2(\pi_2 \xi)]) \\ &= B([\xi, \text{grad } h_1(\pi_1 \xi)], \text{grad } h_2(\pi_2 \xi)) \\ &= B([\xi_{\mathfrak{g}_1^\perp}, \text{grad } h_1(\pi_1 \xi)], \text{grad } h_2(\pi_2 \xi)) \\ &= B(\xi_{\mathfrak{g}_1^\perp}, [\text{grad } h_1(\pi_1 \xi), \text{grad } h_2(\pi_2 \xi)]).\end{aligned}$$

(4.1) PROPOSITION. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two non-degenerate subalgebras of $\mathfrak{g}$ with orthogonal projections $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ and invariant functions $h_i \in C^\infty(\mathfrak{g}_i)$, $i = 1, 2$. The Poisson bracket of $h_1 \circ \pi_1$ and $h_2 \circ \pi_2$ vanishes identically on $\mathfrak{g}$ if $[\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_1$. This holds in particular if $\mathfrak{g}_2 \subset \mathfrak{g}_1$ or $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$.* □

Clearly, one obtains further conditions for the vanishing of the Poisson brackets by interchanging $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Another condition, which is symmetric in $\mathfrak{g}_1, \mathfrak{g}_2$, is that $[\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_1 + \mathfrak{g}_2$ and $B(\mathfrak{g}_1, \mathfrak{g}_2) = 0$.

Using these conditions it now becomes possible to construct large sets of functions in involution in $C^\infty(\mathfrak{g})$. Consider, for example, a chain of non-degenerate subalgebras in $\mathfrak{g}$:

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_n \hookrightarrow \mathfrak{g}_{n+1} = \mathfrak{g}$$

and invariant functions $h_i \in C^\infty(\mathfrak{g}_i)$. It follows that the $h_i \circ \pi_i$ are all in involution in $C^\infty(\mathfrak{g})$.

Let (N, ω) be a symplectic manifold and assume that there exists a Hamiltonian action of G on (N, ω). By proposition (2.2) such a chain of non-degenerate subalgebras in $\mathfrak{g}$ generates a large set of first integrals in involution for arbitrary G-invariant Hamiltonian systems on N.

This idea will be applied in the next sections to $N = T^*M$, where $M = G/K$ is a homogeneous space satisfying (3.1). By considering appropriate chains of

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non-degenerate subalgebras in $\mathfrak{g}$ and using the expressions given in proposition (3.6) we shall, in several cases, be able to prove that every G-invariant Hamiltonian on TM defines a completely integrable system.

In the rest of this section we discuss some special properties of the first integrals which arise from non-degenerate subalgebras in $\mathfrak{g}$. Again, we restrict ourselves to homogeneous spaces satisfying (3.1), although (4.2), (4.3) and (4.4) (depending only on the equivariance of the moment map) hold for general Hamiltonian G-actions.

(4.2) LEMMA. *Let $M = G/K$ be a homogeneous space satisfying* (3.1), *$P: TM \to \mathfrak{g}$ the moment map, $\mathfrak{g}' \subset \mathfrak{g}$ a non-degenerate subalgebra with orthogonal projection $\pi': \mathfrak{g} \to \mathfrak{g}'$, and $h' \in C^\infty(\mathfrak{g}')$ an invariant function. Then $f' = h' \circ \pi' \circ P$ is invariant under the G'-action on TM.* □

We next show that the integral curves of Hamiltonian systems generated by Hamiltonians f' as in lemma (4.2) are images of 1-parameter subgroups.

Recall that, for any $\zeta \in \mathfrak{g}$, the 1-parameter subgroup $\exp t\zeta$ induces the Hamiltonian vector field H_ζ on TM with Hamiltonian function f_ζ. Consider $h \in C^\infty(\mathfrak{g})$ and $f = h \circ P \in C^\infty(TM)$. It follows from proposition (3.6) that

$$H_f(g\xi) = H_\zeta(g\xi) \quad \text{with} \quad \zeta = \operatorname{grad} h(\operatorname{Ad}(g)\xi).$$

This and the following fact were first noted by Guillemin & Sternberg [**2**] in the context of general Hamiltonian group actions.

(4.3) PROPOSITION. *Let $h \in C^\infty(\mathfrak{g})$. The solution curve of the Hamiltonian vector field $H_f, f = h \circ P$, on TM passing through $g\xi$ can be written locally in the form $g(t)g\xi$. Here $g(t)$ is the solution of the following ordinary differential equation on G:*

$$\dot g(t) = R_{g(t)*}(\operatorname{grad} h(P(g(t)g\xi))), \qquad g(0) = e.$$

Proof. Let $\zeta = \operatorname{grad} h(P(g(t)g\xi))$. Then we have

$$\begin{aligned}\frac{d}{dt}(g(t)g\xi) &= \frac{d}{ds}_{|0}(\exp s\zeta g(t)g\xi)\\ &= H_\zeta(g(t)g\xi) = H_f(g(t)g\xi).\end{aligned}$$ □

In particular, if $h \in C^\infty(\mathfrak{g})$ is Ad (G)-invariant, the equation above reduces to

$$\dot g(t) = L_{g(t)*}(\operatorname{grad} h(\operatorname{Ad}(g)\xi))$$

and its solution is the 1-parameter subgroup

$$\exp t(\operatorname{grad} h(\operatorname{Ad}(g)\xi)).$$

The orbit of H_f passing through $g\xi$ becomes

$$g(t)g\xi = g \exp t(\operatorname{grad} h(\xi)) \cdot \xi.$$

In the examples below we shall have grad $h(\xi) \in \mathfrak{m}$ and then the projection of this orbit to M is the geodesic on M with initial direction

$$g \cdot (\operatorname{grad} h(\xi)) \in T_{\pi(g)}M.$$

For first integrals generated by a non-degenerate subalgebra $\mathfrak{g}'$, the vector field given by the differential equation of proposition (4.3) is tangent to G' and we have:

(4.4) PROPOSITION. *Let $\mathfrak{g}'$ be a non-degenerate subalgebra of $\mathfrak{g}$ with corresponding connected subgroup $G' \subset G$, orthogonal projection $\pi': \mathfrak{g} \to \mathfrak{g}'$ and* Ad(G')*-invariant function $h' \in C^\infty(\mathfrak{g}')$. The solution curve of the Hamiltonian vector field H_f, $f = h' \circ \pi' \circ P$, on TM passing through $g\xi$ is the image of the 1-parameter subgroup* exp $t\zeta$ *in G' with $\zeta = \operatorname{grad} h'(\pi' \operatorname{Ad}(g)\xi)$.* □

Let $M' := G'\pi(g)$ and f as in proposition (4.4); then H_f is tangent to the submanifold TM' of TM. Consider, for example,

$$h'(\zeta) = \tfrac{1}{2}B(\zeta, \zeta).$$

The corresponding Hamiltonian f on TM can be regarded as a quadratic form on M. Sometimes this quadratic form defines a metric on M'. For example:

(4.5) PROPOSITION. *Assume $B_{|\mathfrak{m}\times\mathfrak{m}}$ is definite and $\mathfrak{k}' := \operatorname{Ad}(g)\mathfrak{k} \cap \mathfrak{g}'$ a non-degenerate subalgebra. Then $f_{|TM'}$ defines a G'-invariant metric on M' whose geodesics are images of 1-parameter subgroups of G'. These geodesics coincide with geodesics of M if and only if $\mathfrak{k}' = \pi' \operatorname{Ad}(g)\mathfrak{k}$. If $\mathfrak{k}' = \pi' \operatorname{Ad}(g)\mathfrak{k}$ holds, the metric defined by f equals the metric on M' induced by the metric of M and M' is a totally geodesic submanifold of M.*

Proof. $M' = G'/K'$ with $K' = gKg^{-1} \cap G'$. Let $\mathfrak{m}'$ be the complement of $\mathfrak{k}'$ in $\mathfrak{g}'$, thus

$$\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}' \quad \text{and} \quad \dim \mathfrak{m}' = \dim M'.$$

On the other hand, the tangent space $T_{\pi(g)}M'$ equals

$$g \cdot ((\operatorname{Ad}(g)^{-1})\mathfrak{g}')_{\mathfrak{m}} \subset T_{\pi(g)}M.$$

We claim that

$$\pi' \circ \operatorname{Ad}(g) : ((\operatorname{Ad}(g)^{-1})\mathfrak{g}')_{\mathfrak{m}} \to \mathfrak{m}'$$

is an isomorphism. This follows from

$$\begin{aligned} &B(\pi' \operatorname{Ad}(g)(\operatorname{Ad}(g^{-1})\zeta_1)_{\mathfrak{m}}, \zeta_2) \\ &\quad = B((\operatorname{Ad}(g^{-1})\zeta_1)_{\mathfrak{m}}, (\operatorname{Ad}(g^{-1})\zeta_2)_{\mathfrak{m}}) \quad \text{for all } \zeta_1, \zeta_2 \in \mathfrak{g}', \end{aligned}$$

the definiteness of $B_{|\mathfrak{m}\times\mathfrak{m}}$ and the fact that both spaces have the same dimension. We therefore have

$$\mathfrak{m}' = \pi' \operatorname{Ad}(g)(\operatorname{Ad}(g^{-1})\mathfrak{g}')_{\mathfrak{m}} = \pi' \operatorname{Ad}(g)\mathfrak{m}.$$

Since

$$f(g\xi) = \tfrac{1}{2}B(\pi' \operatorname{Ad}(g)\xi, \pi' \operatorname{Ad}(g)\xi),$$

the bilinear form defined by f is non-degenerate on $T_{\pi(g)}M'$. The condition $\pi' \operatorname{Ad}(g)\mathfrak{k} \subset \mathfrak{k}'$ is equivalent to $\mathfrak{m}' \subset \operatorname{Ad}(g)\mathfrak{m}$. This means that the 1-parameter subgroups exp $t\zeta$, $\zeta \in \mathfrak{m}'$ generate geodesics on M with initial point $\pi(g)$.

Furthermore, it then follows for $\zeta \in \mathfrak{m}'$:

$$f(g \cdot (\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}}) = \tfrac{1}{2}B(\zeta, \zeta) = \tfrac{1}{2}B((\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}}, (\operatorname{Ad}(g^{-1})\zeta)_{\mathfrak{m}}).$$

This shows that this metric coincides with the metric induced by M. □

5. *Real Grassmannians*

The real Grassmann manifolds can be defined as

$$G_{p,q}(\mathbb{R}) = \mathrm{SO}\,(n+1)/\mathrm{SO}\,(p)\times \mathrm{SO}\,(q), \qquad p+q=n+1,\, q\le p.$$

On $\mathfrak{g} = \mathfrak{so}(n+1)$ one has the natural Ad (G)-invariant scalar product

$$B(\xi,\eta) = -\tfrac{1}{2}\,\mathrm{tr}\,(\xi\cdot\eta), \quad \xi,\eta\in\mathfrak{g}.$$

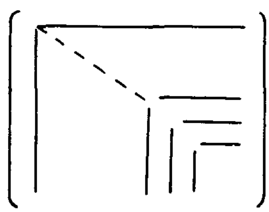
The complement of $\mathfrak{k} = \mathfrak{so}(p)\times\mathfrak{so}(q)$ in $\mathfrak{g}$ is

$$\mathfrak{m} = \left\{\begin{pmatrix} 0 & X \\ -{}^tX & 0\end{pmatrix} : X\in \mathfrak{m}(p,q;\mathbb{R})\right\}.$$

In $\mathfrak{g}$ we consider the chain of subalgebras

$$\mathfrak{so}(2)\hookrightarrow\mathfrak{so}(3)\hookrightarrow\cdots\hookrightarrow\mathfrak{so}(n)\hookrightarrow\mathfrak{so}(n+1)$$

embedded in this way:



Let $\pi_j:\mathfrak{g}\to\mathfrak{so}(j)$ denote the orthogonal projection. By the method described in the last section, every subalgebra $\mathfrak{so}(j)$ defines a set of first integrals for G-invariant Hamiltonian systems on $T(G_{p,q}(\mathbb{R}))$. First, we examine how many independent such integrals one obtains from a single subalgebra. The general theory of invariant polynomials and symmetric spaces (see, for example, [**3**], [**4**], [**8**]) guarantees that: there are exactly $q=\mathrm{rank}\,M$ invariant polynomials on $\mathfrak{g}$ whose gradients are independent at any regular $\xi\in\mathfrak{m}$. ($\xi\in\mathfrak{m}$ is called regular if the centralizer of ξ in $\mathfrak{m}$ is a Cartan algebra Ca (ξ) in $\mathfrak{m}$, i.e. this centralizer has dimension q.) It follows that, if $\xi\in\mathfrak{m}$ is regular, the gradients of these polynomials are a basis of Ca (ξ). These statements can also be proved by elementary means by considering the polynomials

$$h_k(\xi) = \frac{-1}{4k}\,\mathrm{tr}\,\xi^{2k}, \qquad k=1,\dots,q.$$

Proposition (3.6) implies that $\mathfrak{so}(n+1)$ defines q independent first integrals ($g = e$, $v_k = \mathrm{grad}\,h_k(\xi)$, $w_k=0$, $k=1,\dots,q$). The same is true for every $\mathfrak{so}(j), j\ge 2q$, since

$$\pi_j(\mathfrak{m}) = \mathfrak{so}(j)\cap\mathfrak{m}$$

is the Lie triple system of a Grassmannian $G_{j-q,q}(\mathbb{R})$ (again of rank q). $\pi_{2q-1}(\mathfrak{m})$ is the Lie triple system of a $G_{q-1,q}(\mathbb{R})$ of rank $q-1$, and so the subalgebra $\mathfrak{so}(2q-1)$ generates $q-1$ independent first integrals.

Now consider $\mathfrak{so}(2q-2)$. For a regular point $\pi_{2q-2}(\xi)$ in $\pi_{2q-2}(\mathfrak{m})$, the gradients of $q-2$ invariant polynomials on $\mathfrak{so}(2q-2)$ are independent. However, this subalgebra defines more independent first integrals. To see this, we consider a different point $\pi(g) \in G_{p,q}(\mathbb{R})$. Let

$$g_1 \in \mathrm{SO}\,(2q-1) \hookrightarrow \mathrm{SO}\,(n+1)$$

be the transformation which permutes e_{p-q+2} and $\pm e_{p+1}$, where $e_1, \ldots, e_{n+1}$ denote the canonical basis of $\mathbb{R}^{n+1}$. Ad (g_1) acts on $\mathfrak{g}$ by permuting the corresponding rows and columns (and perhaps changing some signs) and one finds that

$$\pi_{2q-2}(\mathrm{Ad}\,(g_1)\mathfrak{m}) = \mathrm{Ad}\,(g_1)\mathfrak{m} \cap \mathfrak{so}(2q-2)$$

is again the Lie triple system of a $G_{q-1,q-1}(\mathbb{R})$. Therefore, there are $q-1$ invariant polynomials h_j on $\mathfrak{so}(2q-2)$ such that

$$\mathrm{grad}\; h_j(\pi_{2q-2}(\mathrm{Ad}\,(g_1)\xi)), \qquad j = 1, \ldots, q-1$$

are independent for almost all $\xi \in \mathfrak{m}$. The horizontal parts of the corresponding Hamiltonian vector fields in $g_1\xi$ are given by

$$v_j = \mathrm{Ad}\,(g_1^{-1})\; \mathrm{grad}\; h_j(\pi_{2q-2}(\mathrm{Ad}\,(g_1)\xi)) \in \mathfrak{m}.$$

Clearly, they are also independent.

Note that all first integrals found before this last step are invariant under g_1. In particular, their Hamiltonian vector fields are described by the same expressions v of proposition (3.6) at the two points $g_1\xi$ and ξ.

Keeping the point $\pi(g_1)$, we obtain $q-2$ independent first integrals from the subalgebra $\mathfrak{so}(2q-3)$. Passing to $\mathfrak{so}(2q-4)$, we take the transformation $g_2 \in \mathrm{SO}\,(2q-3)$ which permutes e_{p-q+4} and $\pm e_{p+2}$. All previously constructed first integrals are g_2-invariant. Setting $g_3 = g_2 g_1$ we find that $\pi_{2q-4}(\mathrm{Ad}\,(g_3)\mathfrak{m})$ is the Lie triple system of a $G_{q-2,q-2}(\mathbb{R})$, thus obtaining $q-2$ independent first integrals. Proceeding in this way up to the subalgebras $\mathfrak{so}(3)$, $\mathfrak{so}(2)$, each of which defines one first integral, we have obtained the following total number of first integrals:

$$q(p-q+1)+(q-1)+(q-1)+(q-2)+(q-2)+\cdots+1+1 = pq.$$

Now we want to show that the union of all these first integrals is functionally independent almost everywhere.

The above construction has led to a $g \in \mathrm{SO}\,(n+1)$ with the property that the first integrals which are defined by a single $\mathfrak{so}(j)$ are independent (and purely horizontal) at $g\xi$ for almost all ξ. By induction on $j = 2, 3, \ldots, n+1$, we build up a specific $\xi \in \mathfrak{m}$ such that all the Hamiltonian vector fields of these functions are independent at $g\xi$.

We describe the induction first for $2q-1 \leq j \leq n$. Here we have to show the following. Assume there are $\xi_j \in \pi_j(\mathfrak{m})$ such that the Hamiltonian vector fields v at $g\xi_j$ coming from the subalgebras $\mathfrak{so}(i)$, $i \leq j$, span $\pi_j(\mathfrak{m})$. Then it is possible to supplement one of these ξ_j by $x \in \mathbb{R}^q$ to a regular $\xi_{j+1} \in \pi_{j+1}(\mathfrak{m})$ such that

$$\mathrm{Ca}\,(\xi_{j+1}) \oplus \pi_j(\mathfrak{m}) = \pi_{j+1}(\mathfrak{m}).$$

$$\xi_{j+1}=\left(\begin{array}{c|c|c} & 0 & x \\ \hline 0 & 0 & * \\ \hline -{}^t x & * & 0 \end{array}\right)\Big\}\ \mathfrak{so}(j)$$

(Recall that the regularity of ξ_{j+1} implies that the gradients of q invariant polynomials on $\mathfrak{so}(j+1)$ span Ca (ξ_{j+1}).) To prove this, begin with any x, assuming ξ_j regular in $\pi_j(\mathfrak{m})$ and ξ_{j+1} regular in $\pi_{j+1}(\mathfrak{m})$. This is possible, since one can change ξ_j slightly without disturbing the hypothesis on the independence of the previously constructed Hamiltonian vector fields. The other condition is

$$\text{Ca}\,(\xi_{j+1})\cap\pi_j(\mathfrak{m})=\{0\}$$

or that there is no α,

$$\alpha=\left(\begin{array}{c|c} & 0 \\ & A \\ \hline 0-{}^tA & \end{array}\right)\in\pi_j(\mathfrak{m})-\{0\},$$

with $[\xi_{j+1},\alpha]=0$. But $[\xi_{j+1},\alpha]=0$ is equivalent to $[\xi_j,\alpha]=0$ and $A^t x=0$, which means that $\alpha\in\text{Ca}\,(\xi_j)$ and $A^t x=0$.

However, it is true for x in an open dense set that $A^t x\neq 0$ for all $\alpha\in\text{Ca}\,(\xi_j)-\{0\}$. To see this one can transform Ca (ξ_j) by means of Ad $(K\cap\text{SO}\,(j))$ to the standard ('diagonal') Cartan algebra of $\pi_j(\mathfrak{m})$. Then one can choose any vector whose components are all different from 0.

Thus, after possibly changing x slightly, we have

$$\text{Ca}\,(\xi_{j+1})\oplus\pi_j(\mathfrak{m})=\pi_{j+1}(\mathfrak{m}).$$

Now we consider the cases $j=2,4,\ldots,2q-2$. For these we had passed by means of a transformation g_i, $i=2q-2-j$, from $\pi(g_{i-1})$ to $\pi(g_{i+1})$ in $G_{p,q}(\mathbb{R})$,

$$g_{i+1}=g_i g_{i-1}\quad(g_{-1}=e).$$

Just as with the first case one shows the following. If there exist $\zeta_j\in\pi_j(\text{Ad}\,(g_{i+1})\mathfrak{m})$ with the following property: the Hamiltonian vector fields v at $g\xi_j$, $\xi_j=\text{Ad}\,(g_{i+1}^{-1})\zeta_j\in\mathfrak{m}$, coming from the subalgebras $\mathfrak{so}(k)$, $k\le j$ span all of

$$\text{Ad}\,(g_{i+1}^{-1})\pi_j(\text{Ad}\,(g_{i+1})\mathfrak{m}),$$

then it is possible to supplement some Ad $(g_i^{-1})\zeta_j$ by a vector x to a regular

$$\zeta_{j+1}\in\pi_{j+1}(\text{Ad}\,(g_{i-1})\mathfrak{m})$$

such that

$$\text{Ca}\,(\zeta_{j+1})\oplus\text{Ad}\,(g_i^{-1})\pi_j(\text{Ad}\,(g_{i+1})\mathfrak{m})=\pi_{j+1}(\text{Ad}\,(g_{i-1})\mathfrak{m}).$$

Now set

$$\xi_{j+1}=\text{Ad}\,(g_{i-1}^{-1})\zeta_{j+1}.$$

Since the first integrals arising from $\mathfrak{so}(j+1)$ are invariant under all g_k, $k\ge i$, their Hamiltonian vector fields are described by the same expressions v at $g\xi_{j+1}$ and $g_{i-1}\xi_{j+1}$. Now the vectors v coming from the subalgebras $\mathfrak{so}(k)$, $k\le j+1$ span at $g\xi_{j+1}$ all of

$$A(g_{i-1}^{-1})\pi_{j+1}(\text{Ad}\,(g_{i-1})\mathfrak{m})=\text{Ad}\,(g^{-1})\pi_{j+1}(\text{Ad}\,(g)\mathfrak{m}).$$

In the remaining cases $j=3,5,\ldots,2q-3$, the situation is identical to the one which was considered first. Beginning with $\mathfrak{so}(2)$ one can thus build up inductively $\xi \in \mathfrak{m}$ such that, for each j, the horizontal components v of the Hamiltonian vector fields defined by $\mathfrak{so}(k)$, $k \le j$, span in $g\xi$ the subspace $\mathrm{Ad}\,(g^{-1})\pi_j(\mathrm{Ad}\,(g)\mathfrak{m})$ of $\mathfrak{m}$.

Finally, one has

$$\mathfrak{m} = \bigoplus_{j=2}^{n+1} \mathrm{Ad}\,(g^{-1})\,\mathrm{Ca}\,(\pi_j\,\mathrm{Ad}\,(g)\xi).$$

Moreover, this means that all pq Hamiltonian vector fields are independent at $g\xi$. By analyticity this has to hold almost everywhere on $T(G_{p,q}(\mathbb{R}))$. We therefore have:

(5.1) THEOREM. *Every G-invariant Hamiltonian on $T(G_{p,q}(\mathbb{R}))$ defines a completely integrable Hamiltonian system. In particular, the geodesic flow of $G_{p,q}(\mathbb{R})$ is integrable.* □

Applying proposition (4.5) one can give a geometrical interpretation of the construction above. To the chain of subalgebras in $\mathfrak{g}$ corresponds a chain of embedded totally geodesic Grassmannian submanifolds in $G_{p,q}(\mathbb{R})$. The proof of theorem 5.1 proceeded by induction over this sequence of Grassmannians.

Further note that our system of integrals in involution is defined by the moment map alone. Therefore it is clear that theorem 5.1 also holds for non-oriented Grassmannians $\mathrm{SO}\,(n+1)/\mathrm{S}\,(\mathrm{O}\,(p)\times \mathrm{O}\,(q))$.

As an illustration, we take a closer look at the first integrals we obtain in the simplest case: $q=1$, $G_{n,1}(\mathbb{R})=S^n$. These are

$$f_j(g\xi) = -\tfrac{1}{4}\,\mathrm{tr}\,(\pi_j\,\mathrm{Ad}\,(g)\xi)^2, \qquad j=2,\ldots,n+1.$$

They can be expressed using the standard embedding

$$TS^n \subset T\mathbb{R}^{n+1} = \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}.$$

Then the moment map becomes the classical angular momentum

$$P(x,y) = (x_k y_l - x_l y_k) \in \mathfrak{so}(n+1)$$

and it follows that

$$f_j(x,y) = -\tfrac{1}{4}\,\mathrm{tr}\,(\pi_j P(x,y))^2 = \tfrac{1}{4}\sum_{k,l\ge n+2-j} (x_k y_l - x_l y_k)^2, \qquad j=2,\ldots,n+1.$$

In [**10**] it is shown that these integrals can be deduced from a solution of the Hamilton–Jacobi equation by separation of variables in polar coordinates. They are first integrals for arbitrary $\mathrm{SO}\,(n+1)$-invariant Hamiltonian systems on $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ and actually they can already be recognized in the classical construction of action-angle variables for the Kepler problem (Delaunay variables).†

For S^n no isometries are needed to prove the independence of the first integrals. Therefore, the proof carries over immediately to the dual symmetric space $H^n = \mathrm{SO}_0\,(n,1)/\mathrm{SO}\,(n)$ of S^n.

(5.2) COROLLARY. *Every* $\mathrm{SO}_0\,(n,1)$*-invariant Hamiltonian on TH^n defines a completely integrable Hamiltonian system. In particular, the geodesic flow of H^n is integrable.* □

† This was pointed out to me by Professor J. Moser.

Finally, one can deduce a result for the normal homogeneous space $M = \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$.

(5.3) PROPOSITION. *The geodesic flow of* $\mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ *is completely integrable.*

Proof. The proof of theorem 5.1, especially for $G_{n-1,2}(\mathbb{R})$, shows that there are $2(n-1)$ independent first integrals in involution for arbitrary $\mathrm{SO}(n+1)$-invariant Hamiltonians on TM. Let

$$D = E_{n,n+1} - E_{n+1,n} \in \mathfrak{m}.$$

Then the linear map $h: \mathfrak{m} \to \mathbb{R}$,

$$h(\xi) = -\tfrac{1}{2}\operatorname{tr}(D \cdot \xi)$$

is $\mathrm{Ad}(K)$-invariant and thus defines a G-invariant function on TM. Since $\operatorname{grad} h(\xi) = D$, it is clear from proposition (3.4) that this Hamiltonian is independent from the other $2(n-1)$ first integrals. □

6. *Complex Grassmannians*

We define the complex Grassmannians as

$$G_{p,q}(\mathbb{C}) = \mathrm{U}(n+1)/\mathrm{U}(p) \times \mathrm{U}(q), \qquad p+q = n+1, q \le p.$$

On $\mathfrak{g} = \mathfrak{u}(n+1)$ one has the $\mathrm{Ad}(G)$-invariant scalar product

$$B(\xi, \eta) = -\tfrac{1}{2}\operatorname{Re}\operatorname{tr}(\xi \cdot \eta), \qquad \xi, \eta \in \mathfrak{g}.$$

The complement of $\mathfrak{k} = \mathfrak{u}(p) \times \mathfrak{u}(q)$ is

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & X \\ -{}^t\bar{X} & 0 \end{pmatrix} : X \in \mathfrak{m}(p, q; \mathbb{C}) \right\}.$$

In the same way as in the last section we consider the chain of subalgebras in $\mathfrak{g}$:

$$\mathfrak{u}(1) \hookrightarrow \mathfrak{u}(2) \hookrightarrow \cdots \hookrightarrow \mathfrak{u}(n) \hookrightarrow \mathfrak{u}(n+1)$$

with the corresponding embedding. Let $\pi_j : \mathfrak{g} \to \mathfrak{u}(j)$ be the orthogonal projection.

To determine the number of independent first integrals arising from a single subalgebra $\mathfrak{u}(j)$, we shall proceed as with the real Grassmannians: namely, we shall interpret $\pi_j(\mathfrak{m})$ (or $\pi_j(\mathrm{Ad}(g)\mathfrak{m})$ for a suitable $g \in G$) as the Lie triple system of another Grassmannian. We first have to answer the following question. Let

$$p' + q' = j \quad \text{and} \quad \mathfrak{m}' \subset \mathfrak{u}(j) = \mathfrak{g}'$$

be the Lie triple system of a corresponding Grassmannian. How many $\mathrm{Ad}(U(j))$-invariant polynomials on $\mathfrak{g}'$ are independent on $\mathfrak{m}'$?

The algebra of invariant polynomials on $\mathfrak{u}(j)$ is generated by

$$h_k(\zeta) = \frac{-\sigma}{2k}\operatorname{tr}\zeta^k, \qquad k = 1, \ldots, j,$$

where $\sigma = 1$ if k is even and $\sigma = i = \sqrt{(-1)}$ if k is odd. Let $\xi \in \mathfrak{m}'$ be regular. From the theory of invariant polynomials on semi-simple Lie algebras one then concludes, cf. [**3**], [**8**]:

(*a*) If $q' < p'$, then $\operatorname{grad} h_k(\xi)$, $k = 1, \ldots, 2q'+1$, are linearly independent.

(*b*) If $q' = p'$, then $\operatorname{grad} h_k(\xi)$, $k = 1, \ldots, 2q'$, are linearly independent.

Note that

$$\operatorname{grad} h_k(\xi) = \xi^{k-1} \in \mathfrak{m}' \quad \text{if } k \text{ is even,}$$

and

$$\operatorname{grad} h_k(\xi) = i\xi^{k-1} \in \mathfrak{k}' \quad \text{if } k \text{ is odd.}$$

Later we shall need some more information. We write

$$\xi = \begin{pmatrix} 0 & X \\ -{}^t\bar{X} & 0 \end{pmatrix} \in \mathfrak{m}', \quad X \in \mathfrak{m}(p', q'; \mathbb{C})$$

and

$$i\xi^2 = i\begin{pmatrix} -X{}^t\bar{X} & 0 \\ 0 & -{}^t\bar{X}X \end{pmatrix} \in \mathfrak{k}'.$$

The following two statements are trivial if one transforms ξ to the standard ('diagonal') Cartan algebra of $\mathfrak{m}'$. Let $s_0 \in \mathbb{N}$, then:

(*c*) The linear independence of the sequence of matrices ξ^{2s}, $s = 1, \ldots, s_0$, is equivalent to the linear independence of $({}^t\bar{X}X)^s$, $s = 1, \ldots, s_0$.

(*d*) If $p' \leq q'$, the linear independence of the matrices ξ^{2s}, $s = 0, 1, \ldots, s_0$, is equivalent to the linear independence of $({}^t\bar{X}X)^s$, $s = 0, 1, \ldots, s_0$.

Now we determine how many independent first integrals are generated by a single subalgebra. With $\mathfrak{u}(n+1)$ one obtains q independent first integrals using $h_2, h_4, \ldots, h_{2q}$. We claim that for $\mathfrak{u}(n)$ $(p \geq q+1)$ the first integrals defined by $h_1, h_2, \ldots, h_{2q}$ are independent. At first, it is clear that those first integrals induced by $h_2, h_4, \ldots, h_{2q}$ are independent at ξ for almost all $\xi \in \mathfrak{m}$. Their Hamiltonian vector fields are purely horizontal at these points. On the other hand, the Hamiltonian vector fields arising from $h_1, h_3, \ldots, h_{2q+1}$ are purely vertical at ξ. This vertical part is given by

$$w_k = i[\pi_n(\xi)^{k-1}, \xi] = i[\pi_n(\xi)^{k-1}, \xi - \pi_n(\xi)], \qquad k = 1, 3, \ldots, 2q+1.$$

Write

$$\xi = \left(\begin{array}{cc|c} & & x \\ & 0 & X \\ \hline -{}^t\bar{x} & -{}^t\bar{X} & 0 \end{array}\right).$$

If $p > q+1$, it follows from (*a*) and (*c*) that, for $\pi_n(\xi)$ regular, the matrices

$$(-{}^t\bar{X}X)^s, \qquad s = 1, \ldots, q,$$

are linearly independent. This implies the independence of

$$(-{}^t\bar{X}X)^s, \qquad s = 0, \ldots, q-1.$$

If $p = q+1$, we obtain the independence of these matrices from (*b*) and (*d*). Then, for almost all $x \in \mathbb{C}^q$, the vectors

$$x(-{}^t\bar{X}X)^s, \qquad s = 0, \ldots, q-1,$$

are also independent. This is equivalent to the linear independence of

$$w_k, \qquad k = 1, 3, \ldots, 2q-1.$$

The same arguments apply to the subalgebras $\mathfrak{u}(j)$, $j \geq 2q$. Note that the Hamiltonian vector fields arising from polynomials of even degree are purely horizontal

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at ξ, whereas the Hamiltonian vector fields arising from polynomials of odd degree are purely vertical. Moreover, these vertical vectors w_k coming from the subalgebra $\mathfrak{u}(j)$ satisfy $\pi_j(w_k)=0$; however, the

$$\pi_{j+1}(w_k), \qquad k=1,3,\ldots,2q-1,$$

are linearly independent at almost all ξ. With $\mathfrak{u}(2q-1)$ one obtains $q-1$ independent horizontal Hamiltonian vector fields and q independent vertical vector fields as follows from (a) and (d).

To study the subsequent subalgebras we apply the same transformations g_i to $\pi(e)$ in $G_{p,q}(\mathbb{C})$ as with the real Grassmannians. The number of independent horizontal Hamiltonian vector fields then is reduced by one, at every second step. As to the Hamiltonian vector fields arising from polynomials of odd degree, we merely consider $\mathfrak{u}(2q-2)$. All further cases are treated in the same way.

$$\pi_{2q-2}(\mathrm{Ad}\,(g_1)\mathfrak{m})=\mathrm{Ad}\,(g_1)\mathfrak{m}\cap\mathfrak{u}(2q-2)$$

is the Lie triple system of a $G_{q-1,q-1}(\mathbb{C})$. Let

$$\zeta=\pi_{2q-2}(\mathrm{Ad}\,(g_1)\xi)$$

be regular in this space. The (purely vertical) Hamiltonian vector fields induced by $h_1, h_3, \ldots, h_{2q-3}$ are given at $g_1\xi$ by

$$\begin{aligned} w_k &= \mathrm{Ad}\,(g_1^{-1})[i\zeta^{k-1}, \mathrm{Ad}\,(g_1)\xi] \\ &= \mathrm{Ad}\,(g_1^{-1})[i\zeta^{k-1}, \mathrm{Ad}\,(g_1)\xi-\zeta], \qquad k=1,3,\ldots,2q-3. \end{aligned}$$

The independence of these vectors for almost all ξ follows from (b) and (d). Note that their orthogonal projections to the subspace

$$\mathrm{Ad}\,(g_1^{-1})\pi_{2q-2}(\mathrm{Ad}\,(g_1)\mathfrak{m})$$

of $\mathfrak{m}$ vanish. However, for almost all ξ, these vectors project in $\pi_{2q-1}(\mathfrak{m})$ to a complex basis for its orthogonal complement.

One now proceeds in the same way up until $\mathfrak{u}(1)$ which generates one first integral. The Hamiltonian vector field of this first integral is purely vertical over $\pi(g)$, where g is constructed just as in the last section. At each step one observes the following. For almost all ξ the vertical Hamiltonian vector fields w_k at $g\xi$ defined by $\mathfrak{u}(j)$ project in

$$\mathrm{Ad}\,(g^{-1})\pi_{j+1}(\mathrm{Ad}\,(g)\mathfrak{m})$$

to a complex basis for the subspace orthogonal to

$$\mathrm{Ad}\,(g^{-1})\pi_j(\mathrm{Ad}\,(g)\mathfrak{m}).$$

The total number of first integrals that we obtain in this way is:

$$q+2q(p-q)+((q-1)+q)+2(q-1)+((q-2)+(q-1))+\cdots+(1+2)+(1+1)+1 = 2qp.$$

It remains to show that the union of all their Hamiltonian vector fields is independent at $g\xi$ for an appropriate $\xi\in\mathfrak{m}$. To prove the independence of the horizontal vector fields, one can construct such a ξ in the same way as with the real Grassmannians.

For $2q-1 \leq j \leq n$ one supplements $\xi_j \in \pi_j(\mathfrak{m})$ by an $x \in \mathbb{C}^q$ to a regular $\xi_{j+1} \in \pi_{j+1}(\mathfrak{m})$ so that

$$\mathrm{Ca}\,(\xi_{j+1}) \cap \pi_j(\mathfrak{m}) = \{0\}.$$

Similarly, in the cases $j = 1, 3, \ldots, 2q-3$.

For $j = 2, 4, \ldots, 2q-2$, one supplements

$$\mathrm{Ad}\,(g_i^{-1})\zeta_j, \qquad \zeta_j \in \pi_j(\mathrm{Ad}\,(g_{i+1})\mathfrak{m}),$$

by a complex vector x to a regular

$$\zeta_{j+1} \in \pi_{j+1}(\mathrm{Ad}\,(g_{i-1})\mathfrak{m})$$

so that

$$\mathrm{Ca}\,(\zeta_{j+1}) \cap \mathrm{Ad}\,(g_i^{-1})\pi_j(\mathrm{Ad}\,(g_{i+1})\mathfrak{m}) = \{0\}.$$

At the same time, one can choose x in such a way that the vertical Hamiltonian vector fields defined by $\mathfrak{u}(j)$ are independent at $g\xi$. Doing this it is clear that, together, all vertical Hamiltonian vector fields are independent at $g\xi$.

(6.1) THEOREM. *Every G-invariant Hamiltonian on $T(G_{p,q}(\mathbb{C}))$ defines a completely integrable Hamiltonian system. In particular, the geodesic flow of $G_{p,q}(\mathbb{C})$ is integrable.* □

For $q = 1$ the independence of the integrals is shown without using transformations of the base point. Thus, for the dual symmetric space

$$\mathbb{C}H^n = \mathrm{U}\,(n, 1)/\mathrm{U}\,(n) \times \mathrm{U}\,(1)$$

of $\mathbb{C}P^n$, we have the corollary:

(6.2) COROLLARY. *Every* $\mathrm{U}\,(n, 1)$*-invariant Hamiltonian on $T(\mathbb{C}H^n)$ defines a completely integrable Hamiltonian system. In particular, the geodesic flow of $\mathbb{C}H^n$ is integrable.* □

7. *Further examples*

In this section we shall consider homogeneous spaces satisfying (3.1) whose isometry group contains $\mathrm{SU}\,(n+1)$ as a subgroup. The restriction of B to the subalgebra $\mathfrak{su}(n+1)$ of $\mathfrak{g}$ will coincide with the canonical scalar product

$$-\tfrac{1}{2}\,\mathrm{Re}\,\mathrm{tr}\,(\xi \cdot \eta).$$

In order to construct sufficiently large sets of independent first integrals in involution we consider, together with the non-degenerate subalgebra $\mathfrak{su}(n+1)$, a chain

$$\mathfrak{u}(1) \hookrightarrow \mathfrak{u}(2) \hookrightarrow \cdots \hookrightarrow \mathfrak{u}(n)$$

and the 'orthogonal projections'

$$\pi_j : \mathfrak{g} \to \mathfrak{su}(n+1) \to \mathfrak{u}(j)$$

defined by

$$\mathfrak{su}(n+1) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D \in \mathfrak{u}(j).$$

Any U (j)-invariant function $h \in C^{\infty}(\mathfrak{u}(j))$ generates a first integral f for G-invariant Hamiltonian systems by setting $f = h \circ \pi_j \circ P$. The Hamiltonian vector field H_f of f is then given by proposition (3.6) with

$$\zeta = (\operatorname{grad} h(\pi_j \operatorname{Ad}(g)\xi))_{\mathfrak{su}(n+1)} = \operatorname{grad} h(\pi_j \operatorname{Ad}(g)\xi)$$
$$-\frac{1}{n+1} \operatorname{tr}(\operatorname{grad} h(\pi_j \operatorname{Ad}(g)\xi)) \cdot \mathrm{Id}.$$

It is clear that proposition (4.1) extends to such a chain of 'subalgebras'.

(a) Our first example in this section is the symmetric space

$$M = \mathrm{SU}(n+1)/\mathrm{SO}(n+1).$$

(7.1) Theorem. *Every* SU $(n+1)$*-invariant Hamiltonian on*

$$\mathrm{T}(\mathrm{SU}(n+1)/\mathrm{SO}(n+1))$$

defines a completely integrable Hamiltonian system. In particular, the geodesic flow of SU $(n+1)/$SO $(n+1)$ *is integrable.*

Proof. The complement of $\mathfrak{k} = \mathfrak{so}(n+1)$ is

$$\mathfrak{m} = \{\xi\colon i\xi \in \mathfrak{gl}(n+1; \mathbb{R}), {}^t\xi = \xi, \operatorname{tr} \xi = 0\}.$$

Since $\mathfrak{m}$ contains a Cartan algebra of $\mathfrak{su}(n+1)$, there are n invariant polynomials on $\mathfrak{g}$ whose gradients are linearly independent at every regular $\xi \in \mathfrak{m}$. For $1 \le j \le n$, $\pi_j(\mathfrak{m})$ contains a maximal abelian subalgebra of $\mathfrak{u}(j)$. For almost all $\xi \in \mathfrak{m}$, $\pi_j(\xi)$ is regular in $\pi_j(\mathfrak{m})$ and the gradients of exactly j invariant polynomials on $\mathfrak{u}(j)$ are independent at $\pi_j(\xi)$. The total number of first integrals we obtain is:

$$1 + 2 + \cdots + n + n = \dim M.$$

By the same arguments used in the preceding sections, one shows that the gradients of all these polynomials are independent almost everywhere on $\mathfrak{m}$. □

It is easy to see that theorem (7.1) also holds for the compact symmetric space U $(n+1)/$O $(n+1)$, which can be interpreted as the manifold of Lagrangian subspaces of a symplectic vector space. Further, its proof carries over immediately to the dual symmetric space SL $(n+1, \mathbb{R})/$SO $(n+1)$ of SU $(n+1)/$SO $(n+1)$. Here on uses the algebras

$$\mathfrak{gl}(1, \mathbb{R}) \hookrightarrow \mathfrak{gl}(2, \mathbb{R}) \hookrightarrow \cdots \hookrightarrow \mathfrak{gl}(n, \mathbb{R}), \mathfrak{sl}(n+1, \mathbb{R}).$$

(7.2) Corollary. *Every* SL $(n+1, \mathbb{R})$*-invariant Hamiltonian on*

$$\mathrm{T}(\mathrm{SL}(n+1, \mathbb{R})/\mathrm{SO}(n+1))$$

defines a completely integrable Hamiltonian system. In particular, the geodesic flow is integrable. □

(b) We consider the compact Lie groups SO $(n+1)$ and SU $(n+1)$.
The results of § 5 and of (a) above directly give:

(7.3) Proposition. *Let G be one of the Lie groups* SO $(n+1)$ *or* SU $(n+1)$. *Then every Hamiltonian on TG which is invariant under the G-action on TG induced by* *left translations admits k independent first integrals in involution, where*

$$k=\tfrac{1}{4}(n+1)^2,\quad \textit{if } G=\mathrm{SO}\,(n+1) \textit{ and } n+1 \textit{ is even.}$$
$$k=\tfrac{1}{4}n(n+2),\quad \textit{if } G=\mathrm{SO}\,(n+1) \textit{ and } n+1 \textit{ is odd.}$$
$$k=\tfrac{1}{2}n(n+3),\quad \textit{if } G=\mathrm{SU}\,(n+1). \qquad \square$$

For $G=\mathrm{SO}\,(3)$ it follows that every G-left invariant Hamiltonian on TG admits 2 first integrals in involution. This immediately gives the classical result that the geodesic flow of every left invariant metric on SO (3) is completely integrable.

Finally, let us remark that, for Hamiltonian systems on TG which are not only left invariant but have further invariance properties, one can define larger classes of first integrals which are in involution. Let $H\subset G$ be a subgroup and consider the action of H on G from the right: $(h, g)\mapsto gh^{-1}$. Suppose $f\in C^\infty(TG)$ is a G-left invariant Hamiltonian which is also invariant under the induced H-right action on TG.

Represent G as $(G\times H)/H^*$, where $H^*=\{(g, g): g\in H\}$, and consider two chains of non-degenerate subalgebras in $\mathfrak{g}\times\mathfrak{h}$, one in $\mathfrak{g}\times\{0\}$ and the other in $\{0\}\times\mathfrak{h}$. The first integrals of H_f arising from these two chains are all in involution.

(c) As a last example, we consider the distance spheres in complex projective space. The distance spheres in $\mathbb{C}P^{n+1}$ are diffeomorphic to S^{2n+1} but the Riemannian metrics induced by the Fubini–Study metric of $\mathbb{C}P^{n+1}$ are different from the metric of the standard sphere. They can be described as homogeneous spaces as follows, cf. [**11**]. Let

$$\bar{G}=\mathrm{SU}\,(n+1)\times\mathbb{R}$$

and define on

$$\bar{\mathfrak{g}}=\mathfrak{su}(n+1)\oplus\mathbb{R}$$

the Ad $(\bar{G})$-invariant, non-degenerate, symmetric bilinear form

$$\bar{B}=B\oplus c\cdot\langle\ ,\ \rangle,$$

where B is the standard metric of $\mathfrak{su}(n+1)$, $\langle\ ,\ \rangle$ is the ordinary scalar product of $\mathbb{R}$, and $c\in(-\infty, -1)\cup(0, \infty)$ is a constant number. Let

$$A'=i\begin{pmatrix} E_n & 0\\ 0 & -n\end{pmatrix},\qquad a=\|A'\|,\qquad A=A'/a\in\mathfrak{su}(n+1),$$

and D be the canonical base vector of $T_0\mathbb{R}=\mathbb{R}$. In $\bar{G}$ one embeds $\bar{K}=\mathrm{SU}\,(n)\times\mathbb{R}$ in the following way as a closed subgroup:

$$(h, t)\mapsto\left(\begin{pmatrix} h\exp(it/a) & 0\\ 0 & \exp(-int/a)\end{pmatrix}, t\right).$$

Then

$$\bar{\mathfrak{k}}=\left\{\begin{pmatrix} B & 0\\ 0 & 0\end{pmatrix}+t(A+D): B\in\mathfrak{su}(n), t\in\mathbb{R}\right\}$$

and as $\bar{B}$-complement of $\bar{\mathfrak{k}}$ in $\bar{\mathfrak{g}}$ one finds:

$$\bar{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & x \\ -{}^t\bar{x} & 0 \end{pmatrix} + t(s^2A + (s^2-1)D) : x \in \mathbb{C}^n, t \in \mathbb{R} \right\},$$

where $s^2 = c/(c+1)$. Note that $\bar{B}_{|\bar{\mathfrak{m}}\times\bar{\mathfrak{m}}}$ is positive definite. $\bar{M} = \bar{G}/\bar{K}$ is isometric to a distance sphere in $\mathbb{C}P^{n+1}$. If $c > 0$, $\bar{M}$ is normal homogeneous and is also called a Berger sphere.

(7.4) THEOREM. *Every $\bar{G}$-invariant Hamiltonian on $T\bar{M}$ defines a completely integrable Hamiltonian system. In particular, the geodesic flow of $\bar{M}$ is integrable.*

Proof. One can define first integrals in involution by using the algebras

$$\mathfrak{su}(n+1), \mathfrak{u}(1) \hookrightarrow \mathfrak{u}(2) \hookrightarrow \cdots \hookrightarrow \mathfrak{u}(n), \mathbb{R}.$$

Let

$$\pi_{n+1} : \bar{\mathfrak{g}} \to \mathfrak{su}(n+1), \qquad \pi_j : \bar{\mathfrak{g}} \to \mathfrak{su}(n+1) \to \mathfrak{u}(j), \qquad j = 1, \ldots, n$$

be the orthogonal projections. As system of first integrals in involution one obtains:

$$f_0(g\xi) = \bar{B}(D, \operatorname{Ad}(g)\xi);$$

$$f_j(g\xi) = -\tfrac{1}{2} i \operatorname{tr}(\pi_j \operatorname{Ad}(g)\xi), \qquad j = 1, \ldots, n;$$

$$f_{n+j-1}(g\xi) = -\tfrac{1}{4} \operatorname{tr}(\pi_j \operatorname{Ad}(g)\xi)^2, \qquad j = 2, \ldots, n+1, \xi \in \bar{\mathfrak{m}}, g \in \bar{G}.$$

For their Hamiltonian vector fields at ξ $(g = e)$ we have (see proposition (3.6)):

$$v_0 = D_{\bar{\mathfrak{m}}} = -(s^2A + (s^2-1)D)$$

$$w_0 = 0 - \tfrac{1}{2}[v_0, \xi];$$

$$\operatorname{grad} h_j(\pi_j\xi)_{\mathfrak{su}(n+1)} = i\left(\begin{pmatrix} 0 & 0 \\ 0 & E_j \end{pmatrix} - \frac{j}{n+1} E_{n+1}\right), \qquad j = 1, \ldots, n.$$

Thus v_j is a multiple of $s^2A + (s^2-1)D$ and

$$w_j = \left[i\begin{pmatrix} 0 & 0 \\ 0 & E_j \end{pmatrix}, \xi \right] - \tfrac{1}{2}[v_j, \xi], \qquad j = 1, \ldots, n;$$

$$v_{n+j-1} = ((\pi_j\xi)_{\mathfrak{su}(n+1)})_{\bar{\mathfrak{m}}}, \qquad j = 2, \ldots, n+1.$$

If one chooses $\xi \in \bar{\mathfrak{m}}$ in such a way that all coefficients of its x-component are different from 0, then one easily concludes from these expressions that the $2n+1$ Hamiltonian vector fields are independent at ξ. □

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