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On a Singular Integral of Christ–Journé Type with Homogeneous Kernel

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Abstract. In this paper, we prove that the singular integral defined by

$$T_{\Omega,a}f(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\cdot m_{x,y}a\cdot f(y)dy$$

is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ and is of weak type (1,1), where $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$ and $m_{x,y}a =: \int_0^1 a(sx+(1-s)y)ds$, with $a \in L^\infty(\mathbb{R}^d)$ satisfying some restricted conditions.

1 Introduction

In 1965, A. P. Calderón [2] introduced the *commutator* $[A, S]$ on $\mathbb{R}$, defined by

$$[A,S]f(x) = A(x)Sf(x) - S(Af)(x),$$

where $A \in \text{Lip}(\mathbb{R})$, $S := \frac{d}{dx}\circ H$, and H denotes the Hilbert transform, defined by

$$Hf(x) = \text{p.v.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{x-y}\,dy.$$

Note that the commutator $[A, S]$ can be rewritten as $[A, \sqrt{-\Delta}]$, where $\Delta = \frac{d^2}{dx^2}$ is the Laplacian operator on $\mathbb{R}$. Therefore, the study of the commutator $[A, S]$ plays an important role in the theory of linear partial differential equations, Cauchy integrals along Lipschitz curves in $\mathbb{C}$, and the Kato square root problem on $\mathbb{R}$ (see [3, 4, 6, 7, 16, 21–23] for details).

By a formal computation, we see that

$$[A,S]f(x) = (-1)\,\text{p.v.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A(x)-A(y)}{x-y}\,\frac{f(y)}{x-y}\,dy.$$

The operator $[A, S]$ is the so-called *Calderón commutator*. In [2], Calderón proved that if $A \in \text{Lip}(\mathbb{R})$, then the Calderón commutator $[A, S]$ is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$.

In 1987, Christ and Journé [9] introduced a variant singular integral of the Calderón commutator in higher dimensions as follows:

$$T_a f(x) = \text{p.v.}\int_{\mathbb{R}^d} K(x-y)\cdot m_{x,y}a\cdot f(y)dy, \tag{1.1}$$

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where K is the *standard Calderón-Zygmund convolution kernel*, which means that K satisfies the following conditions:

(k1) $|K(x)| \le C|x|^{-d}$;
(k2) $\int_{R<|x|<2R} K(x)dx = 0$, for all $R > 0$;
(k3) $|K(x-h) - K(x)| \le C|h|^{\nu}|x|^{-d-\nu}$ if $|x| > 2|h|$, where $0 < \nu \le 1$.

Here and in the sequel, for $a \in L^{\infty}(\mathbb{R}^d)$,

$$m_{x,y}a = \int_0^1 a\big(sx + (1-s)y\big)\,ds.$$

When the dimension $d = 1$, we have

$$m_{x,y}a = \frac{\int_0^x a(z)dz - \int_0^y a(z)dz}{x-y} =: \frac{A(x) - A(y)}{x-y}.$$

Obviously, $A'(x) = a(x) \in L^{\infty}(\mathbb{R})$. So, if taking $K(x) = -\frac{1}{\pi x}$, we see that

$$T_a f(x) = (-1)\,\text{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{A(x)-A(y)}{x-y}\frac{f(y)}{x-y}dy.$$

Hence, when $d = 1$, the operator T_a is just the *Calderón commutator* $[A, S]$. In [9], Christ and Journé showed that T_a is bounded on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$.

In 1995, taking $K(x) = \Omega(x)|x|^{-d}$ $(x \ne 0)$, S. Hofmann [20] discussed the singular integral of Christ-Journé type with homogeneous kernel defined by

$$T_{\Omega,a}f(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\cdot m_{x,y}a \cdot f(y)dy, \tag{1.2}$$

where

$$\Omega(rx') = \Omega(x'), \text{ for any } r > 0 \text{ and } x' \in \mathbb{S}^{d-1} \tag{1.3}$$

and where Ω satisfies

$$\int_{\mathbb{S}^{d-1}} \Omega(x')d\sigma(x') = 0. \tag{1.4}$$

In [20], S. Hofmann proved the weighted L^p boundedness of $T_{\Omega,a}$ if $\Omega \in L^{\infty}(\mathbb{S}^{d-1})$ satisfies (1.3), (1.4), and $a \in L^{\infty}(\mathbb{R}^d)$. Recently, weak type estimates for the singular integral T_a defined by (1.1) have also been discussed. In 2012, Grafakos and Honzík [18] proved that T_a is of weak type (1,1) in dimension $d = 2$. Further, Seeger [25] showed that T_a is of weak type (1,1) for all dimension $d \ge 2$. In 2015, the authors [11] established a weighted weak (1,1) boundedness of T_a for dimension $d = 2$ with power weight $\omega(x) = |x|^{\alpha}$ for $-2 < \alpha < 0$, later extended to more general $A_1(\mathbb{R}^d)$ weight for dimension $d \ge 2$ in [12].

It is well known that if $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$ and satisfies (1.3) and (1.4), the singular integral operator with rough kernel defined by

$$T_{\Omega}(f)(x) = \text{p.v.}\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}f(y)dy \tag{1.5}$$

is bounded from $L^p(\mathbb{R}^d)$ to itself for $1 < p < \infty$ (see [5]) and is of weak type (1,1) (see [24]). Now a natural question is whether similar results hold for $T_{\Omega,a}$ defined in (1.2)

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if $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$. In this paper, we give a partial answer to this question. Our main result is as follows.

Theorem 1.1 *Suppose $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$ and satisfies (1.3) and (1.4). Let $a \in L^1(\mathbb{R}^d)$ and satisfy $\widehat{a} \in L^1(\mathbb{R}^d)$.*

(i) *For $1 < p < \infty$, we have*

$$\|T_{\Omega,a}f\|_p \le C\|\widehat{a}\|_1\|\Omega\|_{L\log^+ L}\|f\|_p.$$

(ii) *For $p = 1$, we have*

$$m\big(\{x \in \mathbb{R}^d : |T_{\Omega,a}f(x)| > \lambda\}\big) \le \frac{C}{\lambda}\|\widehat{a}\|_1\|f\|_1.$$

The constant C above depends only on the dimension d and Ω.

Remark 1.2 It is clear that the conditions $a \in L^1(\mathbb{R}^d)$ and $\widehat{a} \in L^1(\mathbb{R}^d)$ imply $a \in L^\infty(\mathbb{R}^d)$. It seems difficult to get the L^p and weak (1,1) boundedness of $T_{\Omega,a}$ with $a \in L^\infty(\mathbb{R}^d)$ only by the method presented in this paper. So it is still an open question whether the commutator $T_{\Omega,a}$ is L^p bounded for $1 < p < \infty$ and is of weak type (1,1) for $a \in L^\infty(\mathbb{R}^d)$ and $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$ with (1.3) and (1.4).

The proof of part (i) is quite simple. We use the Fourier inversion formula for a, and then the problem can be reduced to the L^p boundedness of T_Ω. The main content of this paper is the proof of Theorem 1.1(ii). The proof is based on a variant Calderón–Zygmund decomposition. More precisely, we make a Calderón–Zygmund type decomposition of an L^1 function with some parameters, where the constants that appear in the estimate are independent of these parameters. For the rest of the proof, we use some nice ideas from Seeger's works [24, 25]. Recall that when the dimension $d = 1$, $m_{x,y}a$ can be rewritten as $(A(x) - A(y))/(x - y)$, which has some smoothness in variables x, y. For dimension $d \ge 2$, $m_{x,y}a$ has no smoothness in x and y, since $a \in L^\infty(\mathbb{R}^d)$. Note that the kernel K satisfying (k1)–(k3) has some smoothness and the commutator T_a defined in (1.1) has only one rough factor $m_{x,y}a$. However, for the commutator $T_{\Omega,a}$, it is much harder to establish the weak (1,1) boundedness, since it involves two rough factors: Ω and $m_{x,y}a$.

Besides the higher dimensional variant form of the Calderón commutator defined in (1.2), there are some other types of Calderón commutators in higher dimensions. For example, in [2], Calderón considered the following commutator

$$\mathfrak{T}_{\Omega,A}f(x) = \text{p.v.}\int_{\mathbb{R}^d}\frac{\Omega(x-y)}{|x-y|^d}\cdot\frac{A(x)-A(y)}{|x-y|}\cdot f(y)\,dy,$$

where $A \in \mathrm{Lip}(\mathbb{R}^d)$ and Ω satisfies (1.3) and

$$\int_{\mathbb{S}^{d-1}}\Omega(x')x'^\alpha\,d\sigma(x') = 0, \quad \text{for all } \alpha \in \mathbb{Z}_+^d \text{ with } |\alpha| = 1.$$

Calderón showed that $\mathfrak{T}_{\Omega,A}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ if $\nabla A \in L^\infty(\mathbb{R}^d)$ and $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$. Recently the authors of this paper established a weak type-(1, 1) criterion for singular integral with rough kernel in [13] and used this criteria to show $\mathfrak{T}_{\Omega,A}$ is weak type-(1, 1) bounded if $\Omega \in L\log^+ L(\mathbb{S}^{d-1})$. However, this criterion is not

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efficient for the operator $T_{\Omega,a}$ discussed in this paper if $a \in L^\infty$. For more discussion about singular integral with rough kernel, we refer the reader to [1,5,8,10,14,15,19,26, 27].

This paper is organized as follows. In Section 2, we complete the proof of part (i) of Theorem 1.1 and part (ii) based on some lemmas; their proofs are given in Section 3 and 4, respectively. Throughout this paper, the letter C stands for a positive constant that is independent of the essential variables and not necessarily the same one in each occurrence. For a Lebesgue measurable set $E \subset \mathbb{R}^d$, we denote its measure by $|E|$ or $m(E)$. Here, $\mathcal{F}f$ and $\widehat{f}$ denote the Fourier transform of f defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx.$$

We let $\mathbb{Z}_+^d$ denote the space of nonnegative multi-indices and let $\mathbb{Z}_+$ denote the set of all nonnegative integers. Moreover, set

$$\|\Omega\|_q := \Big(\int_{\mathbb{S}^{d-1}} |\Omega(x')|^q d\sigma(x') \Big)^{\frac{1}{q}},$$
$$\|\Omega\|_{L\log^+ L} := \int_{\mathbb{S}^{d-1}} |\Omega(x')| \log\big(2 + |\Omega(x')|\big)\, d\sigma(x').$$

2 Proof of Theorem 1.1

Proof of Theorem 1.1(i) Using the inversion Fourier formula, we write

$$m_{x,y}a = \frac{1}{(2\pi)^d} \int_0^1 \int_{\mathbb{R}^d} \widehat{a}(\eta) e^{is\langle \eta, x\rangle} e^{i(1-s)\langle y,\eta\rangle} d\eta ds.$$

Therefore by Fubini's theorem, we have

$$\begin{aligned} T_{\Omega,a}(f)(x) &= \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} \\ &\quad \times \Big(\frac{1}{(2\pi)^d} \iint_{[0,1]\times\mathbb{R}^d} \widehat{a}(\eta) e^{is\langle x,\eta\rangle} e^{i(1-s)\langle y,\eta\rangle} ds d\eta \Big) f(y) dy \\ &= \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) T_\Omega(W^{\eta,s} f)(x)\, d\eta ds, \end{aligned} \tag{2.1}$$

where $a^{x,s}(\eta) = \frac{1}{(2\pi)^d} \widehat{a}(\eta) e^{is\langle x,\eta\rangle}$, $W^{\eta,s}(y) = e^{i(1-s)\langle y,\eta\rangle}$ and T_Ω is defined by (1.5). Now, applying Minkowski's inequality, the above inequality and that T_Ω is bounded on $L^p(\mathbb{R}^d)$, we have

$$\|T_{\Omega,a}(f)\|_p \le \iint_{[0,1]\times\mathbb{R}^d} |\widehat{a}| \|T_\Omega(W^{\eta,s} f)\|_p \, d\eta ds \le C \|\widehat{a}\|_1 \|\Omega\|_{L\log^+ L} \|f\|_p. \qquad \blacksquare$$

Proof Theorem 1.1(ii) We will finish the proof of part (ii) based on some lemmas, whose proofs are given in Sections 3 and 4. We focus only on dimension $d \ge 2$. By using scaling arguments, we can assume $\|\Omega\|_{L\log^+ L(\mathbb{S}^{d-1})} = \|\widehat{a}\|_{L^1(\mathbb{R}^d)} = 1$. Write $T_{\Omega,a}$ in the form (2.1). In the sequel, we try to make a Calderón-Zygmund decomposition of $W^{\eta,s} f$ with the underlying cubes independent of η, s.

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Lemma 2.1 *Fix η, s. Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Set*

$$\Omega_\lambda = \{x \in \mathbb{R}^d : M(f)(x) > \lambda\},$$

where M is the Hardy–Littlewood maximal operator. Then we have the following conclusions:

(i) $\Omega_\lambda = \bigcup Q$, *where the Q's are disjoint dyadic cubes. Let $\mathcal{Q}$ be the collection of all these cubes.*
(ii) $m(\Omega_\lambda) \le C\lambda^{-1}\|f\|_1$.
(iii) $f W^{\eta,s} = g^{\eta,s} + b^{\eta,s}$.
(iv) $b^{\eta,s} = \sum_{Q\in\mathcal{Q}} b_Q^{\eta,s}$, $\operatorname{supp} b_Q^{\eta,s} \subset Q$, $\int b_Q^{\eta,s} = 0$, $\|b_Q^{\eta,s}\|_1 \le C\lambda|Q|$, $\|b^{\eta,s}\|_1 \le C\|f\|_1$.
(v) $\|g^{\eta,s}\|_2^2 \le C\lambda\|f\|_1$.

Here, all the constants C in (i)–(v) *are independent of η, s.*

Proof We first make a Whitney decomposition of the set Ω_λ. Then there exists a family of dyadic closed cubes $\{Q_j\}_j$ (see [17]) such that

(a) $\bigcup Q_j = \Omega_\lambda$ and the Q_j's have disjoint interior.
(b) $\sqrt{d}\cdot l(Q_j) \le \operatorname{dist}(Q_j, \Omega_\lambda^c) \le 4\sqrt{d}\cdot l(Q_j)$, where $l(Q_j)$ denotes the side length of Q_j.

By the weak type-(1,1) bound of M, we have

$$m(\Omega_\lambda) \le \frac{C}{\lambda}\|f\|_1. \tag{2.2}$$

We write $f W^{\eta,s} = g^{\eta,s} + b^{\eta,s}$, where

$$g^{\eta,s} = f W^{\eta,s}\chi_{\Omega_\lambda^c} + \sum_Q \frac{1}{|Q|}\int_Q f(x)W^{\eta,s}(x)\,dx\,\chi_Q,$$
$$b^{\eta,s} = \sum_Q \Big\{ f W^{\eta,s} - \frac{1}{|Q|}\int_Q f(x)W^{\eta,s}(x)\,dx \Big\}\chi_Q =: \sum_Q b_Q^{\eta,s}.$$

So, $b_Q^{\eta,s}$ is supported in Q and $\int b_Q^{\eta,s} = 0$. Let tQ denote the cube with t times the side length of Q and the same center. We first claim that

$$\frac{1}{|Q|}\int_Q |f(x)|\,dx \le C\lambda, \tag{2.3}$$

where C is only dependent on the dimension d. In fact, by the Whitney decomposition property (b), we have $9\sqrt{d}Q \cap \Omega_\lambda^c \ne \varnothing$. Thus, by the definition of Ω_λ^c, there exists $x_0 \in 9\sqrt{d}Q$ such that $Mf(x_0) \le \lambda$. Using the maximal function property, we have $\frac{1}{|9\sqrt{d}Q|}\int_{9\sqrt{d}Q}|f(x)|\,dx \le C'\lambda$, where C' is only dependent on the dimension d. Hence, we have the estimate

$$\frac{1}{|Q|}\int_Q |f(x)|\,dx \le \frac{(9\sqrt{d})^d}{|9\sqrt{d}Q|}\int_{9\sqrt{d}Q}|f(x)|\,dx \le C\lambda.$$

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For $b_Q^{\eta,s}$ and $b^{\eta,s}$, by (2.2) and (2.3) we have

$$\|b_Q^{\eta,s}\|_1 \le 2\int_Q |f(x)|dx \le C\lambda|Q|,$$
$$\|b^{\eta,s}\|_1 \le C\|f\|_1 + \lambda m(\Omega_\lambda) \le C\|f\|_1.$$

Note that $|f(x)| \le \lambda$ almost everywhere in $(\Omega_\lambda)^c$; by (2.2) and (2.3), we have

$$\|g^{\eta,s}\|_2^2 \le C\lambda\|f\|_1 + C\lambda^2 m(\Omega_\lambda) \le C\lambda\|f\|_1. \qquad \blacksquare$$

By Lemma 2.1(iii) and (2.1), we have

$$\begin{aligned} &m\big(\{x : |T_{\Omega,a}(f)(x)| > \lambda\}\big)\\ &\quad \le m\Big(\Big\{x : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(g^{\eta,s})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big)\\ &\qquad + m\Big(\Big\{x : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big). \end{aligned}$$

Notice that T_Ω is bounded from $L^p(\mathbb{R}^d)$ to itself with bound $\|\Omega\|_{L\log^+ L}$. Hence, combining this with Chebyshev's inequality, Minkowski's inequality, and Lemma 2.1(v),

$$\begin{aligned} &m\Big(\Big\{x : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(g^{\eta,s})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big)\\ &\quad \le \frac{4}{\lambda^2}\Big(\iint_{[0,1]\times\mathbb{R}^d} |\widehat{a}(\eta)|\cdot\|T_\Omega(g^{\eta,s})\|_2 d\eta ds\Big)^2\\ &\quad \le \frac{C}{\lambda}\|f\|_1. \end{aligned}$$

For $Q \in \mathcal{Q}$, denote by $l(Q)$ the side length of cube Q. Set $E^* = \bigcup_{Q\in\mathcal{Q}} 2^{200}Q$. Then we have

$$\begin{aligned} &m\Big(\Big\{x : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big) \le\\ &\quad m(E^*) + m\Big(\Big\{x \in (E^*)^c : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big). \end{aligned}$$

By Lemma 2.1(ii), the set E^* satisfies

$$m(E^*) \le Cm(\Omega_\lambda) \le \frac{C}{\lambda}\|f\|_1.$$

Thus, to complete the proof of Theorem 1.1(ii), it remains to show that

$$m\Big(\Big\{x \in (E^*)^c : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big) \le \frac{C}{\lambda}\|f\|_1,$$

where C is only dependent on the dimension d.

Denote $\mathfrak{Q}_k = \{Q \in \mathcal{Q} : l(Q) = 2^k\}$ and let $B_k^{\eta,s} = \sum_{Q\in\mathfrak{Q}_k} b_Q^{\eta,s}$. Then $b^{\eta,s}$ can be rewritten as $b^{\eta,s} = \sum_{j\in\mathbb{Z}} B_j^{\eta,s}$. Take a smooth radial function ϕ on $\mathbb{R}^d$ such that

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$\operatorname{supp}\phi \subset \{x : \frac{1}{4} \le |x| \le 1\}$ and $\sum_j \phi_j(x) = 1$ for all $x \in \mathbb{R}^d\backslash\{0\}$, where $\phi_j(x) = \phi(2^{-j}x)$. Now we define the operator T_j as

$$T_j(f)(x) = \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\phi_j(x-y)f(y)dy.$$

Then we have $T_\Omega = \sum_j T_j$. We write

$$T_\Omega(b^{\eta,s})(x) = \sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} T_j(B^{\eta,s}_{j-n})(x).$$

Note that $T_j(B^{\eta,s}_{j-n})(x) = 0$ for $x \in (E^*)^c$ and $n < 100$. Therefore,

$$m\Big(\Big\{x \in (E^*)^c : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)T_\Omega(b^{\eta,s})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big) = m\Big(\Big\{x \in (E^*)^c : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j\sum_{n\ge100} T_j(B^{\eta,s}_{j-n})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big).$$

Hence, to finish the proof of part (ii), it suffices to verify the estimate

$$(2.4)\quad m\Big(\Big\{x \in (E^*)^c : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j\sum_{n\ge100} T_j(B^{\eta,s}_{j-n})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big) \le \frac{C}{\lambda}\|f\|_1.$$

2.1 Some Key Estimates

In the sequel we will show that (2.4) holds if Ω is restricted in some subset of $\mathbb{S}^{d-1}$. More precisely, for a fixed $n \ge 100$, denote $D^\iota = \{\theta \in \mathbb{S}^{d-1} : |\Omega(\theta)| \ge 2^{\iota n}\|\Omega\|_1\}$, where $0 < \iota < \frac{\gamma}{2}$ will be chosen later. The operator $T^n_{j,\iota}$ is defined by

$$T^n_{j,\iota}(f)(x) = \text{p.v.} \int_{\mathbb{R}^d} \Omega\chi_{D^\iota}\Big(\frac{x-y}{|x-y|}\Big)\frac{\phi_j(x-y)}{|x-y|^d}\cdot f(y)dy.$$

We have the following result, which will be proved in next section.

Lemma 2.2 *Under the conditions of Theorem* 1.1 *with* $0 < \iota < \gamma/2$, *we have*

$$m\Big(\Big\{x \in (E^*)^c : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j\sum_{n\ge100} T^n_{j,\iota}(B^{\eta,s}_{j-n})(x)\,d\eta ds\Big| > \frac{\lambda}{2}\Big\}\Big) \le C\frac{\|f\|_1}{\lambda}.$$

Thus, to finish the proof of Theorem 1.1, by Lemma 2.2 it suffices to verify (2.4) for the kernel function Ω, which satisfies $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$ in each $T_j(B^{\eta,s}_{j-n})$.

In the following, we need to give a partition of unity on the unit surface $\mathbb{S}^{d-1}$. Fix $n \ge 100$. Let $\Theta_n = \{e^n_\nu\}_\nu$ be a collection of unit vectors on $\mathbb{S}^{d-1}$ which satisfies the following two conditions:

(a) $|e^n_\nu - e^n_{\nu'}| \ge 2^{-n\gamma-4}$, if $\nu \ne \nu'$.

(b) If $\theta \in \mathbb{S}^{d-1}$, there exists a e^n_ν such that $|e^n_\nu - \theta| \le 2^{-n\gamma-4}$.

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The constant $0 < \gamma < 1$ in (a) and (b) will be chosen later. To do this, we can simply take a maximal collection $\{e_\nu^n\}_\nu$ for which (a) holds. Notice that there are $C2^{n\gamma(d-1)}$ elements in the collection $\{e_\nu^n\}_\nu$. For every $\theta \in \mathbb{S}^{d-1}$, there only exists finite e_ν^n such that $|e_\nu^n - \theta| \le 2^{-n\gamma-4}$. Now we can construct an associated partition of unity on the unit surface $\mathbb{S}^{d-1}$. Let ζ be a smooth, nonnegative, radial function with $\zeta(u) = 1$ for $|u| \le \frac{1}{2}$ and $\zeta = 0$ for $|u| > 1$. Set

$$\widetilde{\Gamma}_\nu^n(\xi) = \zeta\Big(2^{n\gamma}\Big(\frac{\xi}{|\xi|} - e_\nu^n\Big)\Big)$$

and define

$$\Gamma_\nu^n(\xi) = \widetilde{\Gamma}_\nu^n(\xi)\Big(\sum_\nu \widetilde{\Gamma}_\nu^n(\xi)\Big)^{-1}.$$

Then it is easy to see that Γ_ν^n is homogeneous of degree 0 with

$$\sum_\nu \Gamma_\nu^n(\xi) = 1, \text{ for all } \xi \neq 0 \text{ and all } n.$$

Now we define operator $T_j^{n,\nu}$ by

$$T_j^{n,\nu}(h)(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d}\phi_j(x-y)\Gamma_\nu^n(x-y)\cdot h(y)dy.$$

For convenience, define the kernel of $T_j^{n,\nu}$ as $K_j^{n,\nu}(x) = \frac{\Omega(x)}{|x|^d}\phi_j(x)\Gamma_\nu^n(x)$. Therefore, for fixed $n \ge 100$ we have

$$T_j = \sum_\nu T_j^{n,\nu}.$$

In the sequel, we need to separate the phase of the kernel into different directions. Hence we define a multiple operator by

$$\widehat{G_{n,\nu}h}(\xi) = \Phi(2^{n\gamma}\langle e_\nu^n, \xi/|\xi|\rangle)\widehat{h}(\xi),$$

where h is a Schwartz function and Φ is a smooth, nonnegative, radial function such that $0 \le \Phi(x) \le 1$ and $\Phi(x) = 1$ on $|x| \le 2$, $\Phi(x) = 0$ on $|x| > 4$. Now we can split $T_j^{n,\nu}$ into two parts:

$$T_j^{n,\nu} = G_{n,\nu}T_j^{n,\nu} + (I - G_{n,\nu})T_j^{n,\nu}.$$

The following lemma gives the L^2 estimate involving $G_{n,\nu}T_j^{n,\nu}$, which will be proved in the next section.

Lemma 2.3 *For $n \ge 100$, $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$ with $0 < \iota < \gamma/2$, there exists a constant C such that*

$$\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_\nu\sum_j G_{n,\nu}T_j^{n,\nu}(B_{j-n}^{\eta,s})(x)d\eta ds\Big\|_2^2 \le C2^{-n\gamma+2n\iota}\lambda\|f\|_1,$$

where constant C is independent of n, λ, and f.

The terms involving $(I - G_{n,\nu})T_j^{n,\nu}$ are more complicated. In Section 4, we will prove the following lemma.

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Lemma 2.4 *For $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$ in $T_j^{n,\nu}$, then*

$$\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) \sum_{n\ge 100}\sum_{\nu}\sum_{j}(I-G_{n,\nu})T_j^{n,\nu}(B_{j-n}^{\eta,s})(x)\,d\eta\,ds\Big\|_1 \le C\|f\|_1$$

where C is independent of λ and f.

Proof Theorem 1.1(ii) We now complete the proof of (2.4) with $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$ in each T_j. By Chebyshev's inequality, we have

$$\begin{aligned} m\Big(\Big\{x\in(E^*)^c : \Big|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_j\sum_{n\ge 100} T_j^n(B_{j-n}^{\eta,s})(x)\,d\eta\,ds\Big| > \frac{\lambda}{2}\Big\}\Big) \\ \le \frac{16}{\lambda^2}\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_{n\ge100}\sum_\nu\sum_j G_{n,\nu}T_j^{n,\nu}(B_{j-n}^{\eta,s})(x)\,d\eta\,ds\Big\|_2^2 \\ + \frac{4}{\lambda}\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_{n\ge100}\sum_\nu\sum_j (I-G_{n,\nu})T_j^{n,\nu}(B_{j-n}^{\eta,s})(x)\,d\eta\,ds\Big\|_1 \\ =: \mathrm{I}+\mathrm{II}. \end{aligned}$$

Using Lemma 2.4, we can get the desired estimate of II. Notice that we choose $0<\iota<\frac{\gamma}{2}$. For I, by Minkowski's inequality and Lemma 2.3, we have

$$\begin{aligned} \mathrm{I} &\le C\lambda^{-2}\Big(\sum_{n\ge100}\Big\|\iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta)\sum_\nu\sum_j G_{n,\nu}T_j^{n,\nu}(B_{j-n}^{\eta,s})\,d\eta\,ds\Big\|_2\Big)^2 \\ &\le C\lambda^{-2}\Big(\sum_{n\ge100}(2^{-n\gamma+2n\iota}\lambda\|f\|_1)^{\frac12}\Big)^2 \le C\lambda^{-1}\|f\|_1. \end{aligned}$$

Combining this with Lemma 2.2, we complete the proof of Theorem 1.1(ii), once Lemmas 2.2–2.4 hold. ∎

3 Proofs of Lemmas 2.2 and 2.3

Proof of Lemma 2.2 Denote the kernel of operator $T_{j,\iota}^n$ by

$$K_{j,\iota}^n(y) := \Omega\chi_{D^\iota}\Big(\frac{y}{|y|}\Big)\frac{\phi_j(y)}{|y|^d}.$$

It is easy to see that

$$\Big|\int_{\mathbb{R}^d} K_{j,\iota}^n(y)\,dy\Big| \le C\int_{D^\iota}\int_{2^{j-2}}^{2^j}|\Omega(\theta)|r^{-1}\,dr\,d\sigma(\theta) \le C\int_{D^\iota}|\Omega(\theta)|\,d\sigma(\theta).$$

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Therefore, by Chebyshev's inequality, Minkowski's inequality and Lemma 2.1(iv), we get

$$\begin{aligned}
&m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{[0,1]\times\mathbb{R}^d}a^{x,s}(\eta)\sum_{n\geq100}\sum_{j\in\mathbb{Z}}T^n_{j,\iota}(B^{\eta,s}_{j-n})(x)d\eta ds\Big|>\frac{\lambda}{2}\Big\}\Big)\\
&\quad\leq\frac{C}{\lambda}\Big\|\iint_{[0,1]\times\mathbb{R}^d}a^{x,s}(\eta)\sum_{n\geq100}\sum_{j\in\mathbb{Z}}T^n_{j,\iota}(B^{\eta,s}_{j-n})(x)d\eta ds\Big\|_1\\
&\quad\leq\frac{C}{\lambda}\sum_{n\geq100}\iint_{[0,1]\times\mathbb{R}^d}|\widehat{a}(\eta)|\sum_j\|B^{\eta,s}_{j-n}\|_1d\eta ds\int_{D^\iota}|\Omega(\theta)|d\sigma(\theta)\\
&\quad\leq\frac{C}{\lambda}\|\widehat{a}\|_1\|f\|_1\int_{\mathbb{S}^{d-1}}\operatorname{card}\Big\{n\in\mathbb{N}:n\geq100,2^{\iota n}\leq|\Omega(\theta)|/\|\Omega\|_1\Big\}|\Omega(\theta)|d\sigma(\theta)\\
&\quad\leq\frac{C}{\lambda}\|\widehat{a}\|_1\|f\|_1. \qquad\blacksquare
\end{aligned}$$

Proof of Lemma 2.3 We will use some ideas from [24] in the proof of Lemma 2.3. As usual, we adopt the TT^* method in the L^2 estimate. Moreover, we also use an orthogonality argument based on the following observation of the support of $\mathcal{F}(G_{n,\nu}T^{n,\nu}_j)$. For a fixed $n\geq100$, one has

$$\sup_{\xi\neq0}\sum_\nu|\Phi^2(2^{n\gamma}\langle e^n_\nu,\xi/|\xi|\rangle)|\leq C2^{n\gamma(d-2)}. \tag{3.1}$$

In fact, by the homogeneity of Φ, it suffices to take the supremum over the surface $\mathbb{S}^{d-1}$. For $|\xi|=1$ and $\xi\in\operatorname{supp}\Phi(2^{n\gamma}\langle e^n_\nu,\xi/|\xi|\rangle)$, denote by $\xi^\perp$ the hyperplane perpendicular to ξ. Thus,

$$\operatorname{dist}(e^n_\nu,\xi^\perp)\leq C2^{-n\gamma}. \tag{3.2}$$

Since the mutual distance of e^n_ν's is bounded by $2^{-n\gamma-4}$, there are at most $C2^{n\gamma(d-2)}$ vectors satisfy (3.2). We hence get (3.1).

By applying Minkowski's inequality, Plancherel's theorem, and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\Big\|\iint_{[0,1]\times\mathbb{R}^d}a^{x,s}(\eta)\sum_\nu\sum_jG_{n,\nu}T^{n,\nu}_j(B^{\eta,s}_{j-n})(x)d\eta ds\Big\|_2^2\\
&\quad\leq\Big(\iint_{[0,1]\times\mathbb{R}^d}|\widehat{a}(\eta)|\Big\|\sum_\nu\Phi(2^{n\gamma}\langle e^n_\nu,\xi/|\xi|\rangle)\mathcal{F}\Big(\sum_jT^{n,\nu}_j(B^{\eta,s}_{j-n})\Big)(\xi)\Big\|_2d\eta ds\Big)^2\\
&\quad\leq C2^{n\gamma(d-2)}\Big(\iint_{[0,1]\times\mathbb{R}^d}|\widehat{a}(\eta)|\Big\|\sum_\nu\Big|\mathcal{F}\Big(\sum_jT^{n,\nu}_j(B^{\eta,s}_{j-n})\Big)\Big|^2\Big\|_1^{\frac12}d\eta ds\Big)^2\\
&\quad\leq C2^{n\gamma(d-2)}\Big(\iint_{[0,1]\times\mathbb{R}^d}|\widehat{a}(\eta)|\Big(\sum_\nu\Big\|\sum_jT^{n,\nu}_j(B^{\eta,s}_{j-n})\Big\|_2^2\Big)^{\frac12}d\eta ds\Big)^2.
\end{aligned} \tag{3.3}$$

Next we will show that for a fixed e^n_ν,η,s,

$$\Big\|\sum_jT^{n,\nu}_j(B^{\eta,s}_{j-n})\Big\|_2^2\leq C2^{-2n\gamma(d-1)+2n\iota}\lambda\|f\|_1. \tag{3.4}$$

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Then, using $\mathrm{card}(\Theta_n) \le C2^{n\gamma(d-1)}$ and applying (3.3) and (3.4), we get

$$\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) \sum_\nu \sum_j G_{n,\nu} T_j^{n,\nu}(B_{j-n}^{\eta,s}) d\eta ds \Big\|_2^2 \le C2^{-n\gamma+2n\iota} \lambda \|f\|_1,$$

which is just the desired bound of Lemma 2.3. Thus, to finish the proof of Lemma 2.3, it is enough to prove (3.4). By applying $\|\Omega\|_\infty \le 2^{\iota n}\|\Omega\|_1$, we have

$$\begin{aligned}|T_j^{n,\nu}(B_{j-n}^{\eta,s})(x)| &\le C2^{-jd}2^{\iota n}\|\Omega\|_1 \int_{\mathbb{R}^d} \phi_j(x-y)\Gamma_\nu^n(x-y)|B_{j-n}^{\eta,s}(y)|dy\\ &\le C2^{\iota n} H_j^{n,\nu} * |B_{j-n}^{\eta,s}|(x),\end{aligned}$$

where $H_j^{n,\nu}(x) := 2^{-jd}\chi_{E_j^{n,\nu}}(x)$ and $\chi_{E_j^{n,\nu}}(x)$ is a characteristic function of the set

$$E_j^{n,\nu} := \left\{ x \in \mathbb{R}^d : |\langle x, e_\nu^n\rangle| \le 2^j, |x - \langle x, e_\nu^n\rangle e_\nu^n| \le 2^{j-n\gamma} \right\}.$$

For a fixed e_ν^n, we write

$$\begin{aligned}(3.5)\qquad \Big\| \sum_j T_j^{n,\nu}(B_{j-n}^{\eta,s}) \Big\|_2^2 &\le C2^{2\iota n} \sum_j \int_{\mathbb{R}^d} H_j^{n,\nu} * H_j^{n,\nu} * |B_{j-n}^{\eta,s}|(x) \cdot |B_{j-n}^{\eta,s}(x)| dx\\ &\quad + C2^{2\iota n} \sum_j \sum_{i=-\infty}^{j-1} \int_{\mathbb{R}^d} H_j^{n,\nu} * H_i^{n,\nu} * |B_{i-n}^{\eta,s}|(x) \cdot |B_{j-n}^{\eta,s}(x)| dx.\end{aligned}$$

Observe that $\|H_i^{n,\nu}\|_1 \le C2^{-id} m(E_i^{n,\nu}) \le C2^{-n\gamma(d-1)}$; therefore, for any $i \le j$,

$$H_j^{n,\nu} * H_i^{n,\nu}(x) \le 2^{-n\gamma(d-1)} 2^{-jd} \chi_{\widetilde{E}_j^{n,\nu}},$$

where $\widetilde{E}_j^{n,\nu} = E_j^{n,\nu} + E_j^{n,\nu}$. Hence for a fixed j, n, e_ν^n, and x, we have

$$\begin{aligned}(3.6)\qquad & H_j^{n,\nu} * H_j^{n,\nu} * |B_{j-n}^{\eta,s}|(x) + \sum_{i=-\infty}^{j-1} H_j^{n,\nu} * H_i^{n,\nu} * |B_{i-n}^{\eta,s}|(x)\\ &\le C2^{-n\gamma(d-1)} 2^{-jd} \sum_{i\le j} \int_{x+\widetilde{E}_j^{n,\nu}} |B_{i-n}^{\eta,s}(y)| dy\\ &\le C2^{-n\gamma(d-1)} 2^{-jd} \sum_{i\le j} \sum_{\substack{Q\in\mathfrak{Q}_{i-n}\\ Q\cap\{x+\widetilde{E}_j^{n,\nu}\}\ne\varnothing}} \int_{\mathbb{R}^d} |b_Q^{\eta,s}(y)| dy\\ &\le C2^{-n\gamma(d-1)} 2^{-jd} \sum_{i\le j} \sum_{\substack{Q\in\mathfrak{Q}_{i-n}\\ Q\cap\{x+\widetilde{E}_j^{n,\nu}\}\ne\varnothing}} \lambda|Q|\\ &\le C2^{-n\gamma(d-1)} 2^{-jd} 2^{jd-n\gamma(d-1)} \lambda\\ &\le C\lambda 2^{-2n\gamma(d-1)},\end{aligned}$$

where in third inequality above, we use $\int |b_Q^{\eta,s}(y)| dy \le C\lambda|Q|$ (see Lemma 2.1(iv)) and in the fourth inequality we use the fact that the cubes in $\mathfrak{Q}$ are disjoint (see Lemma 2.1(i)). By (3.5), (3.6) and $\sum_j \|B_{j-n}^{\eta,s}\|_1 \le C\|f\|_1$, we obtain

$$\Big\| \sum_j T_j^{n,\nu}(B_{j-n}^{\eta,s}) \Big\|_2^2 \le C\lambda 2^{-2n\gamma(d-1)+2n\iota} \sum_j \|B_{j-n}^{\eta,s}\|_1 \le C\lambda 2^{-2n\gamma(d-1)+2n\iota} \|f\|_1,$$

which is just (3.4), and we complete the proof of Lemma 2.3. ∎

4 Proof of Lemma 2.4

To prove Lemma 2.4, we have to consider some oscillatory integrals that come from the term $(I - G_{n,\nu})T_j^{n,\nu}$.

Before stating the proof of Lemma 2.4, let us give some notations. We introduce a frequency decomposition. Let ψ be a radial C^∞ function such that $\psi(\xi) = 1$ for $|\xi| \le 1$, $\psi(\xi) = 0$ for $|\xi| \ge 2$ and $0 \le \psi(\xi) \le 1$ for all $\xi \in \mathbb{R}^d$. Define $\beta(\xi) = \psi(\xi) - \psi(2\xi)$, $\beta_k(\xi) = \beta(2^k\xi)$; then β_k is supported in $\{\xi : 2^{-k-1} \le |\xi| \le 2^{-k+1}\}$. Define the convolution operators Λ_k with Fourier multipliers β_k. That is, $\widehat{\Lambda_k f}(\xi) = \beta_k(\xi)\widehat{f}(\xi)$. Then by the construction of β_k, we have

$$I = \sum_{k\in\mathbb{Z}} \Lambda_k,$$

where I is the identity. Write $(I - G_{n,\nu})T_j^{n,\nu} = \sum_k (I - G_{n,\nu})\Lambda_k T_j^{n,\nu}$. By using Minkowski's inequality,

$$\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) \sum_{n\ge100}\sum_\nu\sum_j (I - G_{n,\nu})T_j^{n,\nu}(B_{j-n}^{\eta,s})(x)\,d\eta\,ds\Big\|_1 \le \sum_{n\ge100}\sum_\nu\sum_j\sum_k \sum_{l(Q)=2^{j-n}} \iint_{[0,1]\times\mathbb{R}^d} |\widehat{a}(\eta)|\cdot\|(I - G_{n,\nu})\Lambda_k T_j^{n,\nu}(b_Q^{\eta,s})\|_1\,d\eta\,ds. \tag{4.1}$$

Lemma 4.1 *There exists $N > 0$ such that for any $N_1 \in \mathbb{Z}_+$*

$$\|(I - G_{n,\nu})\Lambda_k T_j^{n,\nu}(b_Q^{\eta,s})\|_1 \le C2^{-n\gamma(d-1)+n\iota+(-j+k)N_1+n\gamma(N_1+2N)}\|b_Q^{\eta,s}\|_1, \tag{4.2}$$

where C is a constant only dependent on N_1.

Proof Denote $h_{k,n,\nu}(\xi) = (1 - \Phi(2^{n\gamma}\langle e_\nu^n, \xi/|\xi|\rangle))\beta_k(\xi)$. Then

$$\big\|(I - G_{n,\nu})\Lambda_k T_j^{n,\nu}(b_Q^{\eta,s})\big\|_1 \le \big\|\mathcal{F}^{-1}(h_{k,n,\nu}\widehat{K_j^{n,\nu}})\big\|_1\|b_Q^{\eta,s}\|_1.$$

Write

$$\mathcal{F}^{-1}(h_{k,n,\nu}\widehat{K_j^{n,\nu}})(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}h_{k,n,\nu}(\xi)\int_{\mathbb{R}^d} e^{-i\xi\cdot\omega}K_j^{n,\nu}(\omega)\,d\omega\,d\xi.$$

In order to separate the rough kernel, we change to polar coordinates $\omega = r\theta$; then the integral above can be written as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}} \Omega(\theta)\Gamma_\nu^n(\theta)\Big\{\int_{\mathbb{R}^d}\int_0^\infty e^{i(\langle x-r\theta,\xi\rangle)}h_{k,n,\nu}(\xi)\cdot\frac{\phi_j(r)}{r}\,dr\,d\xi\Big\}\,d\sigma(\theta). \tag{4.3}$$

Since $\theta \in \operatorname{supp}\Gamma_\nu^n$, $|\theta - e_\nu^n| \le 2^{-n\gamma}$. By the support of Φ, we see $|\langle e_\nu^n, \xi/|\xi|\rangle| \ge 2^{1-nr}$. Thus,

$$|\langle\theta, \xi/|\xi|\rangle| \ge |\langle e_\nu^n, \xi/|\xi|\rangle| - |\langle e_\nu^n - \theta, \xi/|\xi|\rangle| \ge 2^{-n\gamma}. \tag{4.4}$$

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Noting that ϕ_j is supported in $[2^{j-2}, 2^j]$, we can integrate by parts N_1 times with r. Hence the integral (4.3) can be rewritten as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma_\nu^n(\theta)\bigg\{\int_{\{2^{-k-1}\le|\xi|\le 2^{-k+1}\}}\int_{2^{j-2}}^{2^j} e^{i(\langle x-r\theta,\xi\rangle)}h_{k,n,\nu}(\xi) \times (i\langle\theta,\xi\rangle)^{-N_1}\cdot\partial_r^{N_1}[\phi_j(r)r^{-1}]drd\xi\bigg\}d\sigma(\theta),$$

since $h_{k,n,\nu}$ is supported in $\{2^{-k-1}\le|\xi|\le 2^{-k+1}\}$. Integrating by parts in ξ, the integral in curly brackets above can be rewritten as

$$\int_{\{2^{-k-1}\le|\xi|\le 2^{-k+1}\}}\int_{2^{j-2}}^{2^j} e^{i\langle x-r\theta,\xi\rangle}\frac{(I-2^{-2k}\Delta_\xi)^N\big[(i\langle\theta,\xi\rangle)^{-N_1}h_{k,n,\nu}(\xi)\big]}{(1+2^{-2k}|x-r\theta|^2)^N} \times\partial_r^{N_1}[\phi_j(r)r^{-1}]drd\xi. \tag{4.5}$$

We first give an estimate of the term in (4.5). Note that $2^{j-2}\le r\le 2^j$, and we get

$$\big|\partial_r^{N_1}[\phi_j(r)r^{-1}]\big|\le C2^{-j(1+N_1)}. \tag{4.6}$$

In the following, we claim that

$$\big|(I-2^{-2k}\Delta_\xi)^N[\langle\theta,\xi\rangle^{-N_1}h_{k,n,\nu}(\xi)]\big|\le C2^{(n\gamma+k)N_1+2n\gamma N}. \tag{4.7}$$

In fact, by (4.4), it is easy to see that

$$|(-i\langle\theta,\xi\rangle)^{-N_1}\cdot h_{k,n,\nu}(\xi)|\le C|\langle\theta,\xi\rangle|^{-N_1}\le C2^{(n\gamma+k)N_1}.$$

Using the product rule, we get

$$\begin{aligned}&|\partial_{\xi_i}h_{k,n,\nu}(\xi)|\\ &\quad=\Big|-\partial_{\xi_i}\big[\Phi(2^{n\gamma}\langle e_\nu^n,\xi/|\xi|\rangle)\big]\cdot\beta_k(\xi)+\partial_{\xi_i}\beta_k(\xi)\cdot\big(1-\Phi(2^{n\gamma}\langle e_\nu^n,\xi/|\xi|\rangle)\big)\Big|\\ &\quad\le C2^{n\gamma+k}.\end{aligned}$$

Therefore by induction, we have $|\partial_\xi^\alpha h_{k,n,\nu}(\xi)|\le C2^{(n\gamma+k)|\alpha|}$ for any multi-indices $\alpha\in\mathbb{Z}_+^d$. By using (4.4) and the product rule again, we have

$$\begin{aligned}&\big|\partial_{\xi_k}^2(\langle\theta,\xi\rangle)^{-N_1}h_{k,n,\nu}(\xi))\big|\\ &\quad=\big|\langle\theta,\xi\rangle^{-N_1-2}\cdot N_1(N_1+1)\theta_k^2\cdot h_{k,n,\nu}\\ &\qquad+2\langle\theta,\xi\rangle^{-N_1-1}\cdot(-N_1)\cdot\theta_k\partial_{\xi_k}h_{k,n,\nu}(\xi)+\langle\theta,\xi\rangle^{-N_1}\partial_{\xi_k}^2h_{k,n,\nu}(\xi)\big|\\ &\quad\le C2^{(n\gamma+k)(N_1+2)}.\end{aligned}$$

Hence, we conclude that

$$2^{-2k}\big|\Delta_\xi[(\langle\theta,\xi\rangle)^{-N_1}h_{k,n,\nu}(\xi)]\big|\le C2^{(n\gamma+k)N_1+2n\gamma}.$$

Proceeding by induction, we get (4.7). Now we choose $N=[d/2]+1$. Since we need to get the L^1 estimate of (4.3), by the support of $h_{k,n,\nu}$,

$$\int_{\{2^{-k-1}\le|\xi|\le 2^{-k+1}\}}\int\big(1+2^{-2k}|x-r\theta|^2\big)^{-N}dxd\xi\le C.$$

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Integrating with r, we get a bound 2^j. Note that we assume that $\|\Omega\|_\infty \le 2^{n\iota}\|\Omega\|_1$. Next, integrating with θ, we get a bound $2^{-n\gamma(d-1)+n\iota}\|\Omega\|_1$. Combining (4.6), (4.7), and the above estimates, (4.2) is bounded by

$$2^{-j(1+N_1)+(n\gamma+k)N_1+2n\gamma N+j-n\gamma(d-1)+n\iota}\|\Omega\|_1 \le C2^{-n\gamma(d-1)+n\iota}2^{(-j+k)N_1+n\gamma(N_1+2N)}.$$

Hence, we complete the proof of Lemma 4.1 with $N = [d/2] + 1$. ∎

Lemma 4.2 *There exists $N > 0$ such that*

$$\|(I - G_{n,v})\Lambda_k T_j^{n,v}(b_Q^{\eta,s})\|_1 \le C2^{-n\gamma(d-1)+n\iota+j-n-k+2n\gamma N}\|b_Q^{\eta,s}\|_1.$$

Proof The proof of this lemma is similar to that of Lemma 4.1. However, we will not integrate by parts with r, but use some cancellation of $b_Q^{\eta,s}$. Denote $h_{k,n,v}(\xi) = (1 - \Phi(2^{n\gamma}\langle e_v^n, \xi/|\xi|\rangle))\beta_k(\xi)$. Then

$$(4.8)\quad (I - G_{n,v})\Lambda_k T_j^{n,v}(b_Q^{\eta,s})(x) = \int_{\mathbb{R}^d}\Big(\mathcal{F}^{-1}(h_{k,n,v}\widehat{K_j^{n,v}})(x-y) - \mathcal{F}^{-1}(h_{k,n,v}\widehat{K_j^{n,v}})(x-y_Q)\Big)b_Q^{\eta,s}(y)dy,$$

where y_Q is the center of Q. Here we use the cancellation of $b_Q^{\eta,s}$ (see Lemma 2.1(iv)). By changing to polar coordinate and integrating by parts with ξ, we can rewrite $\mathcal{F}^{-1}(h_{k,n,v}\widehat{K_j^{n,v}})(x-y)$ as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma_v^n(\theta)\Big\{\int_{\{2^{-k-1}\le|\xi|\le 2^{-k+1}\}}\int_{2^{j-2}}^{2^j} e^{i\langle x-y-r\theta,\xi\rangle} \times \frac{(I-2^{-2k}\Delta_\xi)^N[h_{k,n,v}(\xi)]}{(1+2^{-2k}|x-y-r\theta|^2)^N}\cdot\phi_j(r)r^{-1}drd\xi\Big\}d\sigma(\theta).$$

Here we choose $N = [d/2]+1$. Thus, (4.8) can be rewritten as two parts: $\mathrm{I}(x) + \mathrm{II}(x)$, where

$$I(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma_v^n(\theta)\Big\{\int_\xi\int_r e^{i\langle x-r\theta,\xi\rangle}\Big(e^{-i\langle y,\xi\rangle} - e^{-i\langle y_Q,\xi\rangle}\Big) \times \frac{(I-2^{-2k}\Delta_\xi)^N[h_{k,n,v}(\xi)]}{(1+2^{-2k}|x-y-r\theta|^2)^N}\phi_j(r)r^{-1}drd\xi\Big\}d\sigma(\theta)\cdot b_Q^{\eta,s}(y)dy$$

and

$$II(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\Omega(\theta)\Gamma_v^n(\theta) \times\Big\{\int_\xi\int_r e^{i\langle x-y_Q-r\theta,\xi\rangle}(I-2^{-2k}\Delta_\xi)^N[h_{k,n,v}(\xi)]\phi_j(r)r^{-1} \times\Big((1+2^{-2k}|x-y-r\theta|^2)^{-N} - (1+2^{-2k}|x-y_Q-r\theta|^2)^{-N}\Big)drd\xi\Big\} \times d\sigma(\theta)\cdot b_Q^{\eta,s}(y)dy.$$

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Note that $y \in Q$ and y_Q is the center of Q, then $|y - y_Q| \le C2^{j-n}$. By applying (4.7) with $N_1 = 0$, we get

$$\left|(I - 2^{-2k}\Delta_\xi)^N\left(h_{k,n,\nu}(\xi)\right)\right| \le C2^{2n\gamma N}$$

Notice that $|e^{-i\langle y,\xi\rangle} - e^{-i\langle y_Q,\xi\rangle}| \le C2^{j-n-k}$. Now, integrating with the variables in the order as we did in proving Lemma 4.1, we can obtain that the L^1 norm of $I(x)$ is bounded by $2^{-n\gamma(d-1)+n\iota+j-n-k+2n\gamma N}\|b_Q^{\eta,s}\|_1$.

For $II(x)$, using the observation

$$\begin{aligned}\left|\Psi(y) - \Psi(y_Q)\right| &= \left|\int_0^1 \left\langle y - y_Q, \nabla\Psi(ty + (1-t)y_0)\right\rangle dt\right|\\ &\le C|y - y_Q| \int_0^1 \frac{N2^{-2k}|x - (ty + (1-t)y_Q) - r\theta|}{(1 + 2^{-2k}|x - (ty + (1-t)y_Q) - r\theta|^2)^{N+1}} dt,\end{aligned}$$

where $\Psi(y) = (1 + 2^{-2k}|x - y - r\theta|^2)^{-N}$, we can also get that the L^1 norm of $II(x)$ is bounded by $2^{-n\gamma(d-1)+n\iota+j-n-k+2n\gamma N}\|b_Q^{\eta,s}\|_1$. Thus, we finish the proof of Lemma 4.2 ∎

Proof of Lemma 2.4 Let us come back to the proof of Lemma 2.4. Denote by $[x]$ the integral part of x. Let ε_0 satisfy $0 < \varepsilon_0 < 1$ and will be chosen later. By (4.1),

$$\begin{aligned}&\Big\| \iint_{[0,1]\times\mathbb{R}^d} a^{x,s}(\eta) \sum_{n\ge100}\sum_\nu\sum_j (I - G_{n,\nu})T_j^{n,\nu}(B_{j-n}^{\eta,s})(x)d\eta ds\Big\|_1\\ &\le \sum_{n\ge100}\sum_\nu\sum_j \sum_{k<j-[n\varepsilon_0]} \sum_{l(Q)=2^{j-n}} \iint_{[0,1]\times\mathbb{R}^d} |\widehat{a}(\eta)|\\ &\quad\times \|(I - G_{n,\nu})\Lambda_k T_j^{n,\nu}(b_Q^{\eta,s})\|_1 d\eta ds\\ &\quad+ \sum_{n\ge100}\sum_\nu\sum_j \sum_{k\ge j-[n\varepsilon_0]} \sum_{l(Q)=2^{j-n}} \iint_{[0,1]\times\mathbb{R}^d} |\widehat{a}(\eta)|\\ &\quad\times \|(I - G_{n,\nu})\Lambda_k T_j^{n,\nu}(b_Q^{\eta,s})\|_1 d\eta ds\end{aligned}$$

Now, using Lemma 4.1 with $N = [d/2] + 1$ for the first term, Lemma 4.2 with $N = [d/2] + 1$ for the second term, the fact $[n\varepsilon_0] \le n\varepsilon_0 < [n\varepsilon_0] + 1$, Lemma 2.1(iv) and $\mathrm{card}(\Theta_n) \le C2^{n\gamma(d-1)}$, the above sum is bounded by

$$\sum_{n\ge100} (2^{n\tau_1} + 2^{n\tau_2})\|\widehat{a}\|_1\|f\|_1,$$

where

$$\tau_1 = -\varepsilon_0 N_1 + \iota + \gamma\left(N_1 + 2\left([d/2] + 1\right)\right), \qquad \tau_2 = 2\gamma\left([d/2] + 1\right) + \varepsilon_0 + \iota - 1.$$

Choose $0 < \iota \ll \gamma \ll \varepsilon_0 \ll 1$ and N_1 large enough such that $\max\{\tau_1, \tau_2\} < 0$. Therefore, the sum is convergent for $n \ge 100$, and we finish the proof of Lemma 2.4, thus proving Theorem 1.1(ii). ∎

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