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VALUES OF ZETA FUNCTIONS OF ARITHMETIC SURFACES AT $s=1$

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Abstract We show that the conjecture of [27] for the special value at $s=1$ of the zeta function of an arithmetic surface is equivalent to the Birch–Swinnerton–Dyer conjecture for the Jacobian of the generic fibre.

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1. Introduction and statement of results

Let V be a connected regular scheme with $V \to \text{Spec } \mathbb{Z}$ flat projective and consider its scheme zeta function

$$\zeta(V,s) = \prod_{x \in V} \frac{1}{(1-N(x)^{-s})};$$

the product is over all closed points x of V and $N(x)$ is the size of the finite residue field of x. The first author [27, Conjecture 3.1] has recently conjectured a formula for the special value $\zeta^*(V,r)$ as a generalised Euler characteristic (up to powers of 2):

$$\zeta^*(V,r) = \pm\chi(V,r). \tag{1}$$

The basic example is the case when $V = \text{Spec } \mathcal{O}_F$ where $\mathcal{O}_F$ is the ring of integers of a number field F; then (1) recovers [27, §7] the well-known class number formula (15) for $r=0$ and $r=1$ (see Subsection 2.7) and, for $r \neq 0,1$, the formula (1) recovers the conjecture [25] of the first-named author relating $\zeta^*(V,r)$ and algebraic K-theory. The next case of interest would be that of arithmetic surfaces and $r=1$ (considered in this article). The compatibility of (1) with the functional equation, proved in [27, §6] using the deep results of Bloch–Kato–Saito [22], is a strong point in support of the conjecture. Our results for arithmetic surfaces provide additional evidence for (1).

1.1. History

It is a fascinating and important problem to give a description of these special values in terms of cohomological invariants of V. The first conjectural attempt at such a description was given in 1991 by Fontaine and Perrin-Riou [16, 15], building on previous work by Deligne, Beilinson, Bloch and Kato. Roughly speaking, Bloch and Kato [2] considered values at positive integers, while Deligne and Beilinson considered values at negative integers. Fontaine and Perrin-Riou were the first to give a uniform approach valid for all integers.

The reader should be warned that the conjectures of Fontaine and Perrin-Riou [16, 15] do not in fact concern the scheme zeta function $\zeta(V,s)$ above but rather the Hasse–Weil zeta function $\zeta_{HW}(V_0,s)$, which is the alternating product of Hasse–Weil L-functions $L_{HW}(h^j(V_0),s)$ [35]. These L-functions $L_{HW}(h^j(V_0),s)$ only depend on the generic fibre V_0 of $V \to \mathrm{Spec}\ \mathbb{Z}$, while $\zeta(V,s)$ depends on the special fibres as well. In fact, Fontaine and Perrin-Riou gave conjectures for the special values of the L-functions $L_{HW}(h^j(V_0),s)$, whose alternating product yields conjectures for the Hasse–Weil zeta function $\zeta_{HW}(V_0,s) = \prod_j (L_{HW}(h^j(V_0),s))^{(-1)^{j+1}}$. These two zeta functions agree if V is smooth over Spec $\mathbb{Z}$ but not in general.

Fontaine and Perrin-Riou first introduced various (conjecturally) finite-dimensional vector spaces and made a conjecture (which we will call Conjecture FPR) giving the special value of the (Hasse–Weil) zeta function $\zeta_{HW}(V_0,s)$ up to a rational number in terms of determinants of maps between these vector spaces tensored with $\mathbb{R}$. They then used spaces taken from p-adic Hodge theory to refine these conjectures to obtain a conjecture for the special value of $\zeta_{HW}(V_0,s)$ up to sign.

A crude description of the main conjecture made in 2017 by the first author [27] (which we will call Conjecture L) would be to say that it refines Conjecture FPR by using canonical integral models for the vector spaces used in that conjecture, avoiding the necessity for p-adic Hodge theory. In fact, the difference between the scheme zeta function and the Hasse–Weil zeta function forces minor changes in the vector spaces to be considered. These changes are similar to those made in [12] for the same reason. In 2016, Flach and Morin [12] also made a general special values conjecture, which we will refer to as Conjecture FM1. Conjecture FM1 does make use of p-adic Hodge theory and is more closely related to Conjecture FPR than is Conjecture L. In fact, Conjecture FM1 can be shown to be equivalent to Conjecture FPR if V is smooth over Spec $\mathbb{Z}$; see [12].

In 2019, Flach and Morin [13] made a new conjecture (referred to here as Conjecture FM2) which avoids p-adic Hodge theory and so is more closely modeled on Conjecture L and less related to Conjecture FPR. In fact, it seems plausible that it is not too difficult to show that Conjecture L is equivalent to Conjecture FM2, except that Conjecture FM2 eliminates the 2-power indeterminacy. Both Conjecture L and Conjecture FM2 have been shown to be compatible [27, 13] with a form of the functional equation for the scheme zeta function. This has not been shown for either Conjecture FM1 or Conjecture FPR.

Conjecture L is a bit more ad hoc than Conjecture FM2 but has the advantage that it involves much less elaborate machinery. Of course, as previously mentioned, Conjecture

FM2 is more precise, but we hope that is possible to remedy this by modifying Conjecture L using the Artin–Verdier topology.

Conjecture FPR is *local*: for each prime p, a conjecture is formulated involving sophisticated p-adic Hodge theory and the corresponding local variety $V \times \operatorname{Spec} \mathbb{Q}_p$, and it is their combination (for all primes p) that determines the special value up to sign. In contrast, Conjecture L (and FM2) is truly *global* as it produces a description of $\zeta^*(V,r)$ intrinsically in terms of invariants of V. For instance, in the case of an arithmetic surface X, where Conjecture FPR calculates the order of the Tate–Shafarevich group $Ш(J)$ of the Jacobian J of the generic fibre X_0 of X, Conjecture L for $\zeta^*(X,1)$ concerns its intrinsic counterpart: the Brauer group $\operatorname{Br}(X)$.

1.2. The conjecture for arithmetic surfaces

Let $S = \operatorname{Spec} \mathcal{O}_F$ be the spectrum of the ring of integers $\mathcal{O}_F$ in a number field F. In this article, *an arithmetic surface* X over S will mean

- a regular scheme X (of dimension 2) together with a projective flat morphism $\pi : X \to S$;
- the generic fibre $X_0 \to \operatorname{Spec} F$ is a geometrically connected smooth projective curve of genus g.

We recall the conjecture of [27, Conjecture 3.1] for the case at hand: $V = X$ and $r = 1$.

Conjecture 1. *For an arithmetic surface X as above, the special value of $\zeta^*(X,1)$ at $s = 1$ is a generalised Euler characteristic $\chi(X,1)$ (up to powers of 2),*

$$\zeta^*(X,1) = \pm\chi(X,1) = \pm\frac{\chi_{A,C}(X,1)}{\chi_B(X,1)}.$$

As we shall see in Subsection 7.5, this provides a description of $\zeta^*(X,1)$ in terms of (periods computed using) the finitely generated $\mathcal{O}_F$-modules $H^*(X,\mathcal{O})$, the Betti cohomology $H^*_B(X_{\mathbb{C}},\mathbb{Z}(1))$, the conjecturally finite group $\operatorname{Br}(X)$, the finitely generated abelian group $\operatorname{Pic}(X)$ and the Arakelov intersection pairing on $\operatorname{Pic}^0(X)$.

Our main result is the following theorem.

Theorem 2. *Conjecture 1 is equivalent to the Birch–Swinnerton–Dyer conjecture (Conjecture 20) for the Jacobian J of X_0 over F.*

Remark 3.

(i) This theorem is the analogue (Subsection 7.5) for arithmetic surfaces of Conjecture (d) [36, p. 427]; see [26, 18, 30, 32]. Such an arithmetic analogue had long been expected. Flach–Siebel [14] proved a similar relation between the conjectures of Flach–Morin [12] for X and Conjecture 20.

(ii) In a certain sense, Conjecture 1 is dual to Conjecture 20. When X_0 is an elliptic curve, the former uses the proper map $\pi : X \to S$ that may not be smooth, whereas the latter uses the Néron model $\mathcal{N}$ of X_0 (and the smooth map $\mathcal{N} \to S$ may not

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be proper). This suggests that one should start with the case when π is smooth and admits a section; this case is considered in Section 5.

(iii) If $g = 0$, then J is trivial. Since Conjecture 20 is obviously true in this case, Theorem 2 gives a proof of Conjecture 1 for $g = 0$. See Remark 43 for details.

Since Tate [36] has proved that Conjecture 20 is invariant under isogeny, one obtains the following.

Corollary 4. *Suppose X and X' are arithmetic surfaces over S such that the Jacobians of X_0' and X_0 are isogenous over F. Then Conjecture 1 for X is equivalent to Conjecture 1 for X'.*

In particular, Conjecture 1 for (the regular integral models of) an elliptic curve E and a torsor T over E are equivalent.

1.3. Plan of the article

We start in Section 2 with a presentation of the main conjecture from [27] and provide a detailed description of the basic examples ($V = S$); in particular, we show that the conjecture recovers the analytic class number formula (15). This is followed in Section 3 by the conjecture (Conjecture 20) of Birch–Swinnerton–Dyer (and Tate) for the global L-function of an abelian variety over a number field. Here we rely on [19] because it contains the formulation of Conjecture 20 in terms of Néron models and provides the definition of a period even when the (projective) module of invariant differentials on the Néron model is not free as a $\mathcal{O}_F$-module.

In Section 4, we prove the key relation (Theorem 23) between the period in Conjecture 20 and the period in Conjecture 1.

Section 5 contains the proof of Theorem 2 in the special but illustrative and important case where $\pi : X \to S$ is smooth and $X_0(F)$ is nonempty. The rich intersection theory on a regular surface figures prominently in Section 6. The beautiful results of Raynaud–Liu–Lorenzini [30, 3, 4] on the Néron model, the connected components of the special fibre and the relation to intersection theory play a key role in our article. Though the analysis of the role of torsion in derived de Rham cohomology could complicate the study of $\zeta^*(V,r)$ in general, these results help us bypass the torsion in $H^1(X,\mathcal{O}_X)$. An equally important role is played by the result of Geisser [17]. Finally, the proof of Theorem 2 is completed in Section 7.

The concept of integral structures and derived differentials is resonant with the methods used in the study of ϵ-constants, Arakelov theory [5, 6] and Bloch's conjectural formula for the conductor [22].

Notations

As usual, we write d_F for the discriminant of F over $\mathbb{Q}$, h for the class number of F and w the number of roots of unity in F. Using the notation A_{tor} for the torsion subgroup of an abelian group A and $[B]$ for the cardinality of a finite group B, one has $w = [F^*_{tor}]$.

We shall use S to indicate both Spec $\mathcal{O}_F$ and the set of nonzero prime ideals of $\mathcal{O}_F$. Thus, $v \in S$ means a closed point of S or a nonzero prime ideal in $\mathcal{O}_F$. We write q_v for the size of the finite residue field $k(v)$.

Let $\tilde{S}$ denote the set of embeddings of F into $\mathbb{C}$; complex conjugation c on $\mathbb{C}$ induces an involution $\sigma \mapsto c\sigma = \bar{\sigma}$ on $\tilde{S}$ via

$$\sigma : F \to \mathbb{C} \qquad \mapsto \qquad \bar{\sigma} : F \xrightarrow{\sigma} \mathbb{C} \xrightarrow{c} \mathbb{C}.$$

The set S_∞ of infinite places of S is the set of orbits of the involution; the set of fixed points of the involution is exactly the subset $S_\mathbb{R}$ of embeddings of F into $\mathbb{R}$. As usual, r_1 is the size of $S_\mathbb{R}$ and $r_1 + r_2$ is the size of S_∞.

For any scheme Y over S, we put $Y_\mathbb{C} = Y \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{C}$ and Y_σ for the base change along $\sigma : F \to \mathbb{C}$:

$$\begin{array}{ccc} Y_\mathbb{C} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{C} & \longrightarrow & \text{Spec } \mathbb{Z} \end{array} \qquad\qquad \begin{array}{ccc} Y_\sigma & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{C} & \longrightarrow & S \end{array} \tag{2}$$

We write $H^*_{et}(Y,\mathbb{Z}(r))$ for the étale motivic cohomology groups of Y; these are the hypercohomology groups of Bloch's higher Chow group complex in the étale topology of Y.

2. Statement of the main conjecture from [27]

Fix a connected regular scheme V of dimension d together with a projective flat morphism $V \to S$; we assume that the generic fibre $V_0 \to \text{Spec } F$ is a geometrically connected smooth variety of dimension $d-1$ over Spec F. The aim of this section is to present the conjectural formula from [27] for the special value $\zeta^*(V,r)$.

2.1. The Bloch–Kato–Fontaine–Perrin-Riou conjecture [15, 16]

The motive $M = h^j(V_0)(r)$ has a Betti realisation $H^j_B(V_\mathbb{C},\mathbb{Z}(r))$, a de Rham realisation $H^j_{dR}(V_\mathbb{C},\mathbb{C}(r))$, an action of complex conjugation c on $H^j_B(V_\mathbb{C},\mathbb{Z}(0)) = H^j_B(V_\mathbb{C},\mathbb{Z})$, a Hodge filtration F on $H^j_{dR}(V_\mathbb{C},\mathbb{C}(r))$ and a period map

$$\theta_{j,r} : H^j(V_\mathbb{C},\mathbb{C}(r)) \xrightarrow{\sim} H^j_{dR}(V_\mathbb{C},\mathbb{C}(r)).$$

As abelian groups, $H^j_B(V_\mathbb{C},\mathbb{Z}(r)) = H^j_B(V_\mathbb{C},\mathbb{Z})$ and $H^j_B(V_\mathbb{C},\mathbb{C}(r)) = H^j_B(V_\mathbb{C},\mathbb{C})$; also, $H^j_B(V_\mathbb{C},\mathbb{C}(r)) = H^j_B(V_\mathbb{C},\mathbb{Z}(r)) \otimes_\mathbb{Z} \mathbb{C}$. The map $\theta_{j,0}$ is the classical period map from $H^j_B(V_\mathbb{C},\mathbb{C})$ to $H^j_{dR}(V_\mathbb{C},\mathbb{C})$. We define

$$H^j(V_\mathbb{C},\mathbb{Z}(r))^+ = \{x \in H^j(V_\mathbb{C},\mathbb{Z}) \mid c(x) = (-1)^r \cdot x\},$$
$$H^j(V_\mathbb{C},\mathbb{C}(r))^+ = \{x \in H^j(V_\mathbb{C},\mathbb{C}) \mid c(x) = (-1)^r \cdot x\}.$$

Finally, the map $\theta_{j,r}$ defined as $(2\pi i)^r \cdot \theta_{j,0}$ induces a map

$$\alpha_M : H^j(V_\mathbb{C},\mathbb{C}(r))^+ \to M_B \xrightarrow{\sim} M_{dR} \to t_M = \frac{M_{dR}}{F^0}. \tag{3}$$

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Fontaine–Perrin-Riou [16, 15] introduced certain vector spaces $H^0_f(M)$ and $H^1_f(M)$ that are conjecturally finite-dimensional $\mathbb{Q}$-vector spaces. Further, they conjectured the existence of *the fundamental exact sequence*

$$0 \to H^0_f(M)\otimes\mathbb{R} \xrightarrow{c} \mathrm{Ker}(\alpha_M) \to H^1_f(M^*(1))^*\otimes\mathbb{R} \xrightarrow{h} H^1_f(M)\otimes\mathbb{R} \xrightarrow{r} \mathrm{Coker}(\alpha_M) \to H^0_f(M^*(1))^*\otimes\mathbb{R} \to 0, \tag{4}$$

where c is a cycle class map, h is a (Arakelov) height pairing and r is the Beilinson regulator. They used it to reformulate the Deligne–Beilinson conjecture $C_{DB}(M)$ [15] which predicts the special value $L^*(M,0) \in \mathbb{R}^\times/\mathbb{Q}^\times$ at $s=0$ of the L-function $L(M,s)$.

To predict $L^*(M,0) \in \mathbb{R}^\times$, they introduced similar exact sequences for each prime p; this is the conjecture $C_{BK}(M)$ [15] for p. One obtains a prediction for the special value of $\zeta_{HW}(V_0,s)$ at $s=r$ by combining $C_{BK}(M)$ for the motives $h^j(V_0)(r)$ for all j. For more details, see [2, 16, 23, 24, 15, 10, 11, 12].

But our approach, following [27], to $\zeta^*(V,r)$ is different: we endow the vector spaces in (4) with integral structures using Weil-étale motivic cohomology groups and the de Rham cohomology groups of V. The determinants of the maps in (4) with respect to these integral structures will be used for a description of $\zeta^*(V,r)$.

2.2. Weil-étale motivic cohomology groups

These groups $H^i_W(V,\mathbb{Z}(r))$ [27, §2.1] are defined as étale motivic cohomology $H^i_{et}(V,\mathbb{Z}(r))$ for $i \le 2r$ and $r \ge 0$ and then, for $i > 2r$, as the dual of étale motivic cohomology

$$H^i(\mathrm{RHom}(R\Gamma_{et}(V,\mathbb{Z}(d-r)),\mathbb{Z}[-2d-1]));$$

thereby, for $i > 2r$, they sit in an exact sequence

$$0 \to \mathrm{Ext}^1(H^{2d+2-i}_{et}(V,\mathbb{Z}(d-r)),\mathbb{Z}) \to H^i_W(V,\mathbb{Z}(r)) \to \mathrm{Hom}(H^{2d+1-i}_{et}(V,\mathbb{Z}(d-r)),\mathbb{Z}) \to 0.$$

We define $H^i_W(X,\mathbb{Z}(r))$ to be zero when $r<0$ and $i \le 2r$.

The groups $H^i_W(V,\mathbb{Z}(r))$ are conjectured to be finitely generated.

2.3. Integral structures and complexes

We recall the well-known theory of determinants [24, Lecture 1, §5].

A lattice L in a finite-dimensional vector space C is the $\mathbb{Z}$-submodule generated by a basis of C. Write $\Lambda L \cong \mathbb{Z}$ (noncanonically) for the highest exterior power of L and $\Lambda C \cong \mathbb{C}$ (noncanonically) for the highest exterior power of C and $(\Lambda C)^{-1}$ for its dual. Given a finite complex of finite-dimensional complex vector spaces C_i and lattices $L_i \subset C_i$

$$C_\bullet: \qquad 0 \to \cdots C_i \xrightarrow{f_i} C_{i+1} \xrightarrow{f_{i+1}} C_{i+2} \to \cdots,$$

one has a canonical map $g : \Lambda(L_\bullet) \to \Lambda(C_\bullet)$ where

$$\Lambda(L_\bullet) = \otimes_j(\Lambda L_j)^{(-1)^j} \cong \mathbb{Z}, \qquad \Lambda(C_\bullet) = \otimes_i(\Lambda C_i)^{(-1)^i}.$$

If $C_\bullet$ is exact, then one has a canonical isomorphism

$$h : \Lambda(C_\bullet) \xrightarrow{\sim} \mathbb{C}.$$

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We define

$$det(C_\bullet,L_\bullet) = h(g(e))$$

as the image of a generator e of $\Lambda(L_\bullet) \cong \mathbb{Z}$ under the composite map

$$\Lambda(L_\bullet) \xrightarrow{g} \Lambda(C_\bullet) \xrightarrow{h} \mathbb{C};$$

$det(C_\bullet,L_\bullet)$ is well-defined as an element of

$$\mathbb{C}^\times/\{\pm 1\}.$$

2.3.1. Integral structures.

Definition 5.

(i) An integral structure on a complex vector space V is a pair (M,m) consisting of a lattice $M \subset V$ and a positive rational number $m \in \mathbb{Q}_{>0}$.

(ii) The Euler characteristic of a map from (M,m) on V to (N,n) on W is the determinant of the map $f : V \to W$ (with respect to the lattices M and N) multiplied by n/m.

(iii) Two integral structures (M,m) and (M',m') on V are Euler-equivalent if the Euler characteristic (relative to M and M') of the identity map on V is one.

(iv) The $\mathbb{Z}$-dual of (M,m) is the integral structure $(\mathrm{Hom}_\mathbb{Z}(M,\mathbb{Z}), \frac{1}{m})$ on $\mathrm{Hom}_\mathbb{C}(V,\mathbb{C})$.

The Euler characteristic of f in (ii) does not change if (M,m) is replaced with a Euler-equivalent integral structure (M',m') on V. If M' is a sublattice of M with index e, then (M',em) and (M,m) are Euler-equivalent, as are $(\mathrm{Hom}_\mathbb{Z}(M,\mathbb{Z}),1)$ and $(\mathrm{Hom}_\mathbb{Z}(M',\mathbb{Z}),1/e)$: Euler equivalence is compatible with duality.

If $\phi : A \to V$ is a homomorphism, A is a finitely generated abelian group, $\mathrm{Ker}(\phi)$ is finite and $\phi(A)$ is a lattice in V, then $(\phi(A),[\mathrm{Ker}(\phi)])$ is an integral structure. If A is torsion-free, $(\phi(A),1)$ essentially determines ϕ.

2.3.2. Integral structures on complexes. Given a finite complex $C_\bullet$ of finite-dimensional complex vector spaces, an *integral structure* L on $C_\bullet$ is a collection $L_\bullet = (L_i,t_i)$ where L_i is a lattice in C_i and t_i is a positive rational number. If $C_\bullet$ is exact, we define

$$\chi(C_\bullet,L_\bullet) = \frac{det(C_\bullet,L_\bullet)}{T} \in \mathbb{C}^\times/\{\pm 1\}, \qquad T = \frac{t_0 \cdot \ \cdot t_2 \cdot \cdots}{t_1 \ \cdot t_3 \ \cdot \ \cdots} \in \mathbb{Q}_{>0}.$$

Here, T is a multiplicative Euler characteristic.

Example 6. A collection ϕ of homomorphisms $\phi_i : A_i \to C_i$ where, for all i, A_i is a finitely generated abelian group, $\mathrm{Ker}(\phi_i)$ is finite and $\mathrm{Im}(\phi_i)$ is a lattice in C_i gives an integral structure $L_\bullet = (L_i,t_i)$ where $L_i = \phi_i(A_i)$ and $t_i = [\mathrm{Ker}(\phi_i)]$; in this case, we write $T = \chi(A_{tor})$. If $C_\bullet$ is exact, then we have

$$\chi(C_\bullet,A_\bullet) = \frac{det(C_\bullet,L_\bullet)}{\chi(A_{tor})} \in \mathbb{C}^\times/\{\pm 1\}.$$

We often refer to $A_\bullet$ as an integral structure on $C_\bullet$.

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If there is a canonical integral structure $L_\bullet$ on $C_\bullet$, then we write $det(C_\bullet)$ and $\chi(C_\bullet)$ for $det(C_\bullet, L_\bullet)$ and $\chi(C_\bullet, L_\bullet)$.

2.3.3. Integral structures on $\mathcal{O}_F$-modules. Let M be a finitely generated $\mathcal{O}_F$-module of rank d; there are two natural integral structures i_M and j_M on $V = M \otimes_{\mathbb{Z}} \mathbb{C}$. Let $\phi : M \to V$ be the canonical map. The integral structure i_M is obtained by entirely forgetting the $\mathcal{O}_F$-structure:

$$i_M = (\phi(M), [M_{tor}]). \tag{5}$$

While i_M appears in the conjectural formula (1), another[1] integral structure j_M appears in Conjecture 20; j_M uses the $\mathcal{O}_F$-structure on M. Recall the standard isomorphism

$$\psi : \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}^{r_1+2r_2} = \mathbb{Z}^{r_1+2r_2} \otimes \mathbb{C} \tag{6}$$

that sends $x \otimes 1 \in \mathcal{O}_F \otimes \mathbb{C}$ to the element of $\mathbb{C}^{r_1+2r_2}$ given by the collection of $\sigma(x)$ as $\sigma \in \tilde{S}$ runs through the embeddings of F in $\mathbb{C}$. Define $M_\sigma = M \otimes_{\mathcal{O}_F} \mathbb{C}$ using $\sigma : F \to \mathbb{C}$.

The isomorphism (6) shows

$$V = M \otimes_{\mathbb{Z}} \mathbb{C} \cong M \otimes_{\mathcal{O}_F} (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{C}) \cong M \otimes_{\mathcal{O}_F} \mathbb{C}^{r_1+2r_2} \cong \prod_{\sigma \in \tilde{S}} M_\sigma.$$

Suppose $B = \{x_1, \cdots, x_d\} \subset M$ is linearly independent over $\mathcal{O}_F$. Let $\{\epsilon_j\}$ be the standard basis of $\mathbb{C}^{r_1+2r_2}$. Define $\tilde{B} =< x_i \otimes \epsilon_j >$ to be the $\mathbb{Z}$-span of $x_i \otimes \epsilon_j$ inside V. Consider the integral structure $i(B) = (\tilde{B}, b)$ where b is the order of the quotient of M by the $\mathcal{O}_F$-submodule $< x_1, \cdots, x_d >$ generated by B.

Proposition 7. *For any two such subsets B and B' of M, $i(B)$ and $i(B')$ are Euler-equivalent.*

Proof. Let $B' = \{c_1x_1, c_2x_2, \cdots, c_dx_d\}$ and $C = c_1.c_2 \cdots c_d$ and consider $i(B')$. As b and b' are related by $b' = C.b$ and the determinant of the identity map of V with respect to $i(B)$ and $i(B')$ is C, $i(B)$ and $i(B')$ are Euler-equivalent. Given B and B', one can find B'' such that the previous argument applies to the pairs B, B'' and B', B''. □

Definition 8. For any finitely generated $\mathcal{O}_F$-module M, the $\mathcal{O}_F$-integral structure j_M on $V = M \otimes_{\mathbb{Z}} \mathbb{C}$ is the Euler equivalence class in Proposition 7.

2.3.4. Results on integral structures. Let M be a finitely generated $\mathcal{O}_F$-module of rank d.

Proposition 9.

(i) The Euler characteristic of the identity map on V from i_M to j_M is $(\sqrt{d_F})^d$.

(ii) Let M be torsion-free and $M^\vee = Hom_{\mathcal{O}_F}(M, \mathcal{O}_F)$. The $\mathbb{Z}$-dual of j_M is the $\mathcal{O}_F$-integral structure $j_{M^\vee}$.

[1]Recall that two lattices L and L' in V are commensurable if $L \cap L'$ is finite index in L and L'. The integral structures i_M and j_M for $M = \mathcal{O}_F$ are not commensurable in general.

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Proof. (i) Write κ_M for the Euler characteristic. For $M = \mathcal{O}_F$, this says $\sqrt{d_F}$ is the determinant of the standard map $\psi : \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{Z}^{r_1+2r_2} \otimes \mathbb{C}$ in (6). So $\kappa_M = \sqrt{d_F}^d$ if M is free of finite rank $d > 0$. Even if M is not free, it contains a free submodule M' of finite index. As $\kappa_M = \kappa_{M'}$, the claim for M' implies the one for M.

(ii) Define the index $(M : N)$ of any two torsion-free $\mathcal{O}_F$-modules M and N of rank d inside V by $(M : N) = (M : P)/(N : P)$ for any submodule P of finite index contained in M and N. Let $\{x_1, \cdots, x_d\}$ be a basis of $V = M \otimes_{\mathcal{O}_F} F$. Let $\{\epsilon_j\}$ be the standard basis of $\mathbb{C}^{r_1+2r_2}$. Let $\{y_1, \cdots, y_d\}$ be an F-basis of $M^\vee \otimes_{\mathcal{O}_F} F$ dual to $\{x_1, \cdots, x_d\}$. Then $(< x_i \otimes \epsilon_j >, (M :< x_1, \cdots, x_d >))$ is a representative for the $\mathcal{O}_F$-integral structure j_M. Its $\mathbb{Z}$-dual is $(< y \otimes \epsilon_j >, (< x_1, \cdots, x_d >: M))$. But $(< y \otimes \epsilon_j >, (M^\vee :< y \otimes \epsilon_j >))$ is a representative for $j_{M^\vee}$, and since $(M : N)^{-1} = (M^\vee, N^\vee)$, we are done. □

2.3.5. Pairings and determinants. Let N be a finitely generated abelian group and $\psi : N/N_{tor} \times N/N_{tor} \to \mathbb{R}$ be a nondegenerate symmetric bilinear form on N. One defines $\Delta(N) \in \mathbb{C}^\times/\{\pm 1\}$ as the determinant of the matrix $\psi(b_i, b_j)$ divided by $(N : N_0)^2$ where N_0 is the subgroup of finite index generated by a maximal linearly independent subset b_i of N; therefore, $\Delta(N)$ is independent of the choice of b_is and incorporates the order of the torsion subgroup of N. We can rewrite ψ as the integral structure $(C_\bullet, L_\bullet)$ with $L_0 = N, L_1 = \mathrm{Hom}(N, \mathbb{Z})$:

$$C_0 = N \otimes \mathbb{C} \xrightarrow{\psi_*} \mathrm{Hom}(N, \mathbb{Z}) \otimes \mathbb{C} = C_1, \qquad \Delta(N) = \frac{det(\psi_*)}{[N_{tor}]^2};$$

here $det(C_\bullet, L_\bullet)$ is the determinant $\det(\psi_*)$ of ψ computed with the bases L_0 and L_1. Given a short exact sequence of finitely generated groups $0 \to N' \to N \to N'' \to 0$ which splits over $\mathbb{Q}$ as an orthogonal direct sum

$$N_{\mathbb{Q}} \cong N'_{\mathbb{Q}} \oplus N''_{\mathbb{Q}}$$

with respect to a definite pairing ψ on N, one has the following standard relation:

$$\Delta(N) = \Delta(N') \cdot \Delta(N''). \tag{7}$$

2.4. Motives

We start with the integral structures underlying the Betti and de Rham realisations of the motives $M = h^j(V_0)(r)$ for $0 \le j \le 2(d-1)$.

Definition 10. The integral structure $H^j_B(V_{\mathbb{C}}, \mathbb{Z}(r))^+$ is defined as the complex vector space $H^j_B(V_{\mathbb{C}}, \mathbb{C}(r))^+$ together with the homomorphism

$$H^j_B(V_{\mathbb{C}}, \mathbb{Z})^+ \to H^j_B(V_{\mathbb{C}}, \mathbb{C})^+ \xrightarrow{\times (2\pi i)^r} H^j_B(V_{\mathbb{C}}, \mathbb{C}(r))^+;$$

the first map is the natural map and the second map is multiplication by $(2\pi i)^r$.

Example 11. When $V = S$, one has $H^0(S_{\mathbb{C}}, \mathbb{Z}(r))^+$ is isomorphic to $\mathbb{Z}^{r_2}$ when r is odd and $\mathbb{Z}^{r_1+r_2}$ when r is even.

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2.5. de Rham realisation of $M = h^j(V_0)(r)$

The de Rham realisation $M_{dR} = H^j_{dR}(V_{\mathbb{C}}, \mathbb{C}(r))$ is $H^j_{dR}(V_{\mathbb{C}}, \mathbb{C})$ with the Hodge filtration shifted by r; that is, the Tate twist in de Rham realisation shifts the Hodge filtration but does not change the vector space. The tangent space $(t_M)_{\mathbb{C}}$ of M is defined as follows:

$$(t_M)_{\mathbb{C}} = \frac{H^j_{dR}(V_{\mathbb{C}}, \mathbb{C}(r))}{F^0} = \prod_{k<r} H^{j-k}(V_{\mathbb{C}}, \Omega^k_{V_{\mathbb{C}}}) = \prod_{\sigma \in \tilde{S}} \prod_{k<r} H^{j-k}(V_\sigma, \Omega^k_{V_\sigma}).$$

Let $\lambda^k \Omega_{V/S}$ denote the kth derived exterior power of the sheaf $\Omega_{V/S}$ on V; see the appendix of [27].

Definition 12. The integral structure on the complex vector space $(t_M)_{\mathbb{C}}$ is given by the map

$$t_M := \prod_{k<r} H^{j-k}(V, \lambda^k \Omega_{V/S}) \to \prod_{\sigma \in \tilde{S}} \prod_{k<r} H^{j-k}(V_\sigma, \Omega^k_{V_\sigma}) = (t_M)_{\mathbb{C}}.$$

As in (5), one views $H^{j-k}(V, \lambda^k \Omega_{V/S})$ as an abelian group; its structure as a module over $\mathcal{O}_F$ is not used.

Remark 13. We can use the Hodge decomposition of $(t_M)_{\mathbb{C}}$ to decompose the period map α_M of (3) as a sum of α_k where α_k is the composite map

$$H^j(V_{\mathbb{C}}, \mathbb{C}(r))^+ \xrightarrow{\alpha} (t_M)_{\mathbb{C}} = \prod_{k<r} H^{j-k}(V_{\mathbb{C}}, \Omega^k_{V_{\mathbb{C}}}) \to H^{j-k}(V_{\mathbb{C}}, \Omega^k_{V_{\mathbb{C}}}).$$

The enhanced period map γ_M of [27, §2.3],

$$\gamma_M : H^j(V_{\mathbb{C}}, \mathbb{C}(r))^+ \to (t_M)_{\mathbb{C}},$$

likewise decomposes as a sum of $\gamma_k := \Gamma^*(r-k) \cdot \alpha_k$. Here Γ is the classical gamma function and $\Gamma^*(m)$ denotes the special value at $s = m$.

2.6. Integral structures on certain motivic exact sequences

The conjectural formula (1) from [27, Conjecture 3.1] predicts that the special value $\zeta^*(V,r)$ of $\zeta(V,s)$ at $s = r$ is equal to $\pm\chi(V,r)$ (up to powers of 2) defined as follows:

$$\chi(V,r) = \frac{\chi_{A,C}(V,r)}{\chi_B(V,r)} = \frac{\chi(C(r)) \cdot \chi_A(V,r) \cdot \chi_{A'}(V,r)}{\chi_B(V,r)}.$$

In more detail,

- M_j is the motive $h^j(V_0, \mathbb{Z}(r))$ with $0 \le j \le 2d-2$ with the enhanced period map

  $$\gamma_{M_j} : H^j_B(V_{\mathbb{C}}, \mathbb{Z}(r))^+_{\mathbb{C}} \to t_{M_j}.$$

- We fix an integral structure on $\mathrm{Ker}(\gamma_{M_j})$ and $\mathrm{Coker}(\gamma_{M_j})$ for all j and use them implicitly in all that follows. Under this condition, the final conjecture is actually independent of the choice of these integral structures.

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- The exact sequences $B(j,r)$ with integral structures

$$0 \to \mathrm{Ker}(\gamma_{M_j}) \to H^j_B(V_{\mathbb{C}},\mathbb{Z}(r))^+_{\mathbb{C}} \to t_{M_j} \to \mathrm{Coker}(\gamma_{M_j}) \to 0. \tag{8}$$

We write

$$\chi_B(V,r) = \prod_j (\chi(B(j,r)))^{(-1)^j} = \frac{\chi(B(0,r)) \cdot \chi(B(2,r)) \cdots}{\chi(B(1,r)) \cdot \chi(B(3,r)) \cdots}.$$

- The integral structures $A(j,r)$ for $0 \le j \le \min(2d-1,2r-3)$ and $A'(j,r)$ for $2d+1 \ge j \ge \max(0,2r+1)$ are defined as

$$A(j,r) : H^{j+1}_W(V,\mathbb{Z}(r))_{\mathbb{C}} \to \mathrm{Coker}(\gamma_{M_j}), \qquad A'(j,r) : \mathrm{Ker}(\gamma_{M_j}) \to H^{j+2}_W(V,\mathbb{Z}(r))_{\mathbb{C}}. \tag{9}$$

We write

$$\chi_A(V,r) = \prod_{j=0}^{\min(2d-1,2r-3)} (\chi(A(j,r)))^{(-1)^j},$$
$$\chi_{A'}(V,r) = \prod_{j=\max(0,2r+1)}^{j=2d-1} (\chi(A'(j,r)))^{(-1)^j}. \tag{10}$$

The map in $A(j,r)$ is Beilinson's regulator from (4), usually stated as a map from algebraic K-theory of V_0 to the Deligne cohomology of V_0; it is conjectured to be an isomorphism. The group $\mathrm{Ker}(M_{2d-2-j,d-r})$ can be identified with the dual of $\mathrm{Coker}(M_{j,r})$ - see [27, §2] or [15, §6.9]. The integral structures $A'(j,r)$ are defined as the dual of $A(2d-2-j,d-r)$ in [27].

- The exact sequence $C(r)_{\mathbb{C}}$ with integral structure (for $0 \le r \le d$)

$$0 \to H^{2r-1}_W(V,\mathbb{Z}(r))_{\mathbb{C}} \to \mathrm{Coker}(\gamma_{M_{2r-2}}) \to H^{2r+1}_W(V,\mathbb{Z}(r))_{\mathbb{C}} \xrightarrow{h}$$
$$\to H^{2r}_W(V,\mathbb{Z}(r))_{\mathbb{C}} \to \mathrm{Ker}(\gamma_{M_{2r}}) \to H^{2r+2}_W(V,\mathbb{Z}(r))_{\mathbb{C}} \to 0. \tag{11}$$

The maps here are integral versions of the maps from (4): the first is Beilinson's regulator, the second is the dual of the cycle map, the third is the Arakelov intersection/height pairing map, the fourth map is the cycle class map and the fifth is the dual of Beilinson's regulator. Beilinson's conjectures [27, Conjectures 2.3.4-2.3.7] imply that $C(r)_{\mathbb{C}}$ is exact; we refer to [27, §2] for more details. Integral versions of (4) have also been introduced by Flach–Morin [12, Conjecture 2.9 and (28), (29)].

Remark 14. (see Table 1) (i) By [15, §7.1] or [34, §2.1 and (2.2.1)], if the weight of M is positive (so $j > 2r$), then one has $\mathrm{Coker}(\gamma_M) = 0$. Dually, if $j-2r < 0$, then one has $\mathrm{Ker}(\gamma_M) = 0$. It remains to consider the case $M = h^{2r}(V_0,\mathbb{Z}(r))$. Since $N = h^{2r}(V_0,\mathbb{Z}(r+1))$ has negative weight, it follows that $\mathrm{Ker}(\gamma_N) = 0$. Hence, $\mathrm{Coker}(\gamma_{M_{2r}}) = 0$ as it is dual to $\mathrm{Ker}(\gamma_N)$.

(ii) Note that each group $H^i_W(V,\mathbb{Z}(r))$ appears exactly once.

All of the exact sequences above are variants of motivic exact sequences (4) but equipped with an integral structure. Our convention is that the terms $\mathrm{Ker}(\gamma_M)$ have even degree and

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Table 1. Tabulation of invariants in the various integral structures for the motives $M_j = h^j(V_0)(r)$

Groups	Type B	Type A or C
$H^j_W(V,\mathbb{Z}(r))$ and $\max(1,2r+2) < j < 2d+2$	–	$A'(j-2,r)$
$H^j_W(V,\mathbb{Z}(r))$ and $0 < j < \min(2d+1,2r-2)$	–	$A(j-1,r)$
$H^j_W(V,\mathbb{Z}(r))$ and $2r-2 < j < 2r+3$	–	$C(r)$
$\mathrm{Ker}(\gamma_{M_j})$ and $j \geq 2r+1$	$B(j,r)$	$A'(j,r)$
$\mathrm{Ker}(\gamma_{M_j}) = 0$ for $j \leq 2r-1$	$B(j,r)$	–
$\mathrm{Coker}(\gamma_{M_j}) = 0$ for $j > 2r$	$B(j,r)$	–
$\mathrm{Coker}(\gamma_{M_j}) = 0$ for $j = 2r$	$B(j,r)$	–
$\mathrm{Coker}(\gamma_{M_j})$ and $j \leq 2r-3$	$B(j,r)$	$A(j,r)$
$\mathrm{Ker}(\gamma_{M_{2r}})$	$B(2r,r)$	$C(r)$
$\mathrm{Coker}(\gamma_{M_{2r-2}})$	$B(2r-2,r)$	$C(r)$
$\mathrm{Coker}(\gamma_{M_{2r-1}}) = 0 = \mathrm{Ker}(\gamma_{M_{2r-1}})$		

$\mathrm{Coker}(\gamma_M)$ have odd degree. If $j = 2r-1$, then $\mathrm{Ker}(\gamma_{M_j}) = 0 = \mathrm{Coker}(\gamma_{M_j})$; if $j \neq 2r-1$, these terms occur exactly twice in the conjecture, once in the B-complexes and once in A, A' and C-complexes. Since χ_B is the denominator and the others (χ_A, $\chi_{A'}$ and $\chi(C)$) are in the numerator of $\chi(V,r)$, the conjecture is independent of the choice of integral structures on the terms $\mathrm{Ker}(\gamma_M)$ and $\mathrm{Coker}(\gamma_M)$.

(iii) The conjectural formula (1) from [27, Conjecture 3.1] tacitly assumes the generalised Beilinson–Soulé conjecture from [27] which predicts

$$\text{if } r < 0, \text{ then } H^i_W(X,\mathbb{Z}(r)) = \text{finite 2-group for } i \leq 1.$$

If this is not assumed, then (10) has to be replaced with

$$\chi_A(V,r) = \prod_{j=0}^{2r-3} (\chi(A(j,r)))^{(-1)^j}, \qquad \chi_{A'}(V,r) = \prod_{j=2r+1}^{j=2d-1} (\chi(A'(j,r)))^{(-1)^j}$$

and the limits on j in the first two lines of Table 1 have to be replaced with $2d+2 > j > 2r+2$ (in the first line) and $0 < j < 2r-1$ (in the second line). We refer to [27] for more details.

(iv) We refer to [27] for the conjectures such as the finite generation of $H^*_W(V,\mathbb{Z}(r))$ which are implicit in the formulation of (1).

It is instructive to consider the conjecture [27, Conjecture 3.1] in the case of number fields before delving into the case of arithmetic surfaces.

2.7. The case of number fields: $V = S$, $d = 1$ and $r = 0$ and $r = 1$

This example is worked out in [27, §7].

2.7.1. The case $r = 0$.

The motives are $M_j = h^j(S)(0)$.

We know $H^j_{et}(S,\mathbb{Z}(1)) = 0$ if $j < 1$ and that $H^1_{et}(S,\mathbb{Z}(1)) = \mathcal{O}_F^\times$, $H^2_{et}(S,\mathbb{Z}(1)) = \mathrm{Pic}(S)$, $H^3_{et}(S,\mathbb{Z}(1)) = 0$ (up to a finite 2-group) and $H^0_{et}(S,\mathbb{Z}(0)) = \mathbb{Z}$.

It immediately follows from the definitions that $H^0_W(S,\mathbb{Z}(0))=\mathbb{Z}$, $H^1_W(S,\mathbb{Z}(0))=0$ (up to a finite 2-group) and $H^3_W(S,\mathbb{Z}(0))=\mu_F^\vee$, the dual of the roots of unity in F.

There is also an exact sequence

$$0\to \mathrm{Pic}(S)^\vee \to H^2_W(S,\mathbb{Z}(0))\to \mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{Z})\to 0.$$

Note that $t_{M_j}=0$ for all j.

$\chi_B(S,0)=1$: The sequence $B(j,0)$ is zero for all $j\neq 0$ and we can arrange $\chi(B(0,0))=1$ by using the isomorphism

$$B(0,0):\qquad \mathrm{Ker}(\gamma_{M_0})\xrightarrow{\sim} H^0_B(S_\mathbb{C},\mathbb{Z}(0))^+_\mathbb{C}$$

to transfer the integral structure $H^0_B(S_\mathbb{C},\mathbb{Z}(0))^+$ to $\mathrm{Ker}(\gamma_{M_0})$.

Since $A(j,r)$ contributes only for $j\leq 2r-3=-3$, it follows that there is no contribution from any of the $A(j,0)$-terms. We see that $A'(j,0)=0$ unless $j=1$, in which case $A'(1,0)=\mu_F^\vee$ in degree 2. Thus, we obtain $\chi(A'(1,0))=w$ and

$$\chi_{A'}(S,0)=\frac{1}{w}.$$

The sequence $C(0)_\mathbb{C}$ for $V=S$

$$0\to H^0_W(S,\mathbb{Z}(0))_\mathbb{C}\to \ker(\gamma_{M_0})\to H^2_W(S,\mathbb{Z}(0))_\mathbb{C}\to 0$$

becomes the sequence (our convention is that $\mathrm{Ker}(\gamma)$ has even degree, say, degree 2)

$$0\to \underset{\text{degree }1}{\mathbb{C}}\to \underset{\text{degree }2}{\mathbb{C}^{r_1+r_2}}\to \underset{\text{degree }3}{\mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{C})}\to 0, \tag{12}$$

with $H^2_W(S,\mathbb{Z}(0))$ providing the standard integral structure $\mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{Z})$ on the last term and the second map being the dual of the classical regulator. So $\chi(C(0))=hR$ and so

$$\chi(S,0)=\frac{\chi_{A,C}(S,0)}{\chi_B(S,0)}=\frac{\chi(C(0))\ \cdot\ \chi_{A'}(S,0)}{1}=\frac{hR}{w}.$$

As is well-known,

$$\zeta^*(S,0)=-hR/w. \tag{13}$$

So the conjectural formula (1) from [27, Conjecture 3.1] is valid in this case.

2.7.2. The case $r=1$. The motives are $M_j=h^j(S)(1)$.

As $A(j,1)$ contribute only for $j<-1$, we have $\chi_A(S,1)=1$. The complex $C(1)_\mathbb{C}$ (our convention is that $\mathrm{Coker}(\gamma)$ has odd degree, say 1)

$$0\to H^1_W(S,\mathbb{Z}(1))_\mathbb{C}\to \mathrm{Coker}(\gamma_{M_0})\to H^3_W(S,\mathbb{Z}(1))_\mathbb{C}\to 0\to 0\cdots$$

becomes

$$0\to \underset{\text{degree }0}{\mathcal{O}_F^\times\otimes\mathbb{C}}\to \underset{\text{degree }1}{\mathbb{C}^{r_1+r_2}}\to \underset{\text{degree }2}{\mathbb{C}}\to 0, \tag{14}$$

with the last two terms getting the standard basis and the first term a basis from $\mathcal{O}_F^\times$. This is exactly as in [27, §7]. Thus, $det(C(1))$ is the classical regulator R. One

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checks that $H^i_W(S,\mathbb{Z}(1))=0$ for $i>3$ and that $H^3_W(S,\mathbb{Z}(1))=\operatorname{Hom}(H^0_{et}(S,\mathbb{Z}(0)),\mathbb{Z})$. As $H^i_W(S,\mathbb{Z}(1))=H^i_{et}(S,\mathbb{Z}(1))$ for $i\le 2$, this completes the computation of $H^*_W(S,\mathbb{Z}(1))$. From this, we see that

$$\chi(C(1))=\frac{hR}{w}.$$

We need to consider $A'(j,1)$ only when $j\ge 3$. Since $t_{M_j}=0$ for $j>0$ and $H^{j+2}_W(S,\mathbb{Z}(1))=0$ for $j\ge 3$, we see $A'(j,1)=0$ for $j\ge 3$. As a result, $\chi_{A'}(S,1)=1$.

Finally, $B(j,1)=0$ if $j\ne 0$ and $B(0,1)$ given by

$$0\to H^0_B(S_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}}\to t_{M_0}\to \operatorname{Coker}(\gamma_{M_0})\to 0$$

becomes (our convention is that $\operatorname{Coker}(\gamma)$ has odd degree)

$$0\to \underset{\text{degree }1}{\mathbb{C}^{r_2}}\to \underset{\text{degree }2}{\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{C}}\to \underset{\text{degree }3}{\mathbb{C}^{r_1+r_2}}\to 0.$$

The first map is the composition of the natural inclusion of $\mathbb{C}^{r_2}$ in $\mathbb{C}^{r_1+2r_2}$ (multiplied by $2\pi i$) with the inverse of ψ of (6). The integral structure on $\operatorname{Coker}(\gamma_{M_0})\cong\mathbb{C}^{r_1+r_2}$ is defined by $H^0_B(S,\mathbb{Z}(1))^-\cong\mathbb{Z}^{r_1+r_2}$. Since the determinant of ψ with respect to the usual bases is $\sqrt{d_F}$ and the integral structures are torsion-free, we obtain

$$det(B(0,1))=\frac{\sqrt{d_F}}{(2\pi i)^{r_2}}=\chi(B(0,1)).$$

Hence, we obtain

$$\chi(S,1)=\frac{\chi(C(1))}{\chi(B(0,1))}=\frac{hR(2\pi i)^{r_2}}{w\sqrt{d_F}};$$

as the sign of d_F is $(-1)^{r_2}$, this is equal to the usual formula (up to a power of 2)

$$\zeta^*(S,1)=\frac{2^{r_1}hR(2\pi)^{r_2}}{w\sqrt{|d_F|}}. \tag{15}$$

So the conjectural formula (1) from [27, Conjecture 3.1] is valid in this case, too.

2.8. The case of arithmetic surfaces: $V=X$ and $r=1$

We now turn to an explicit description of the terms that enter into the description of $\zeta^*(X,1)$; this uses the motives $M_j=h^j(X_0)(1)$ for $j=0,1,2$ of the algebraic curve X_0 over F and $t_{M_j}=H^j(X_0,\mathcal{O}_X)$. We begin with the Weil-étale motivic cohomology groups of X.

2.8.1. The groups $H^*_W(X,\mathbb{Z}(1))$.

Fix an arithmetic surface $\pi: X\to S$.

Since the motivic complex $\mathbb{Z}(1)$ is $\mathbb{G}_m[-1]$, one has the identifications

$$H^j_{et}(X,\mathbb{Z}(1))=H^{j-1}_{et}(X,\mathbb{G}_m),$$

the Picard group $\operatorname{Pic}(X)=H^2_{et}(X,\mathbb{Z}(1))$ and the Brauer group $\operatorname{Br}(X)=H^3_{et}(X,\mathbb{Z}(1))$.

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Recall that $H^i_W(X,\mathbb{Z}(1))$ [27, §2.1] are defined as étale motivic cohomology $H^i_{et}(X,\mathbb{Z}(1))$ for $i \le 2$ and, for $i > 2$, as the dual of étale motivic cohomology

$$H^i(\mathrm{RHom}(R\Gamma_{et}(X,\mathbb{Z}(1)),\mathbb{Z}[-3])),$$

thereby sitting in an exact sequence (for $i > 2$)

$$0 \to \mathrm{Ext}^1(H^{6-i}_{et}(X,\mathbb{Z}(1)),\mathbb{Z}) \to H^i_W(X,\mathbb{Z}(1)) \to \mathrm{Hom}(H^{5-i}_{et}(X,\mathbb{Z}(1)),\mathbb{Z}) \to 0.$$

As $H^i_W(X,\mathbb{Z}(1)) = 0$ for $i = 0$ and $i \ge 6$, the following is a complete description of $H^*_W(X,\mathbb{Z}(1))$:

$$H^1_W(X,\mathbb{Z}(1)) = \mathcal{O}_F{}^\times, \quad H^2_W(X,\mathbb{Z}(1)) = \mathrm{Pic}(X), \quad H^5_W(X,\mathbb{Z}(1)) = \mathrm{Ext}^1(\mathcal{O}_F{}^\times,\mathbb{Z}), \tag{16}$$

$$0 \to \mathrm{Ext}^1(\mathrm{Br}(X),\mathbb{Z}) \to H^3_W(X,\mathbb{Z}(1)) \to \mathrm{Hom}(\mathrm{Pic}(X),\mathbb{Z}) \to 0, \tag{17}$$

$$0 \to \mathrm{Ext}^1(\mathrm{Pic}(X),\mathbb{Z}) \to H^4_W(X,\mathbb{Z}(1)) \to \mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{Z}) \to 0. \tag{18}$$

Remark. As the finite generation of $\mathrm{Pic}(X)$ is well-known (theorem of Mordell–Weil–Roquette), the finite generation of $H^*_W(X,\mathbb{Z}(1))$ reduces to the finiteness of $\mathrm{Br}(X)$.

2.8.2. The complexes $A(j,1)$ and $A'(j,1)$. In our case, only $A'(3,1)$ could be nonzero.

In our case, $r = 1$. For $A(j,1)$ to intervene, one needs $j \le 2r-3 = -1$ as $2d-1 = 3$. Since $H^0_W(X,\mathbb{Z}(1)) = 0$, we obtain $\chi(A(j,1) = 1)$ for all j.

For $A'(j,1)$ to intervene, one needs $j \ge 3$. For $j = 3$, since $H^5_W(X,\mathbb{Z}(1))$ is a finite group of order w, we have $det(A'(3,1)) = 1$ and $\chi(A'(3,1)) = w$. If $j > 3$, then $H^{j+2}_W(X,\mathbb{Z}(1)) = 0$ and $M_j = 0$; so $\chi(A'(j,1)) = 1$ for $j > 3$. Thus,

$$\chi_A(X,1) = 1, \qquad \chi_{A'}(X,1) = \frac{1}{w}. \tag{19}$$

2.8.3. The groups $H^i(X,\mathcal{O}_X)$. Since $\pi : X \to S$ is proper, $H^i(X,\mathcal{O}_X)$ are finitely generated $\mathcal{O}_F$-modules.

Lemma 15. *$H^i(X,\mathcal{O}_X)$ is zero for $i \ge 2$.*

Proof. Since $X \to S$ is a relative curve, $H^i(X_s,\mathcal{O}_{X_s}) = 0$ (for $i > 1$) at all points $s \in S$. Since S is a reduced noetherian scheme and $\pi : X \to S$ is proper and $\mathcal{O}_X$ is a coherent sheaf on X flat over S, by Grothendieck's theorem on formal functions (see L. Illusie's article [9, Corollary 8.3.4]), $R^i\pi_*\mathcal{O}_X = 0$ for $i > 1$. □

Using Subsection 2.8.2 and Lemma 15, we find that

$$\chi(X,1) = \frac{\chi(C(1))}{\chi(A'(3,1))} \cdot \frac{\chi(B(1,1))}{\chi(B(0,1))} = \frac{\chi(C(1))}{w} \cdot \frac{\chi(B(1,1))}{\chi(B(0,1))}, \tag{20}$$

where the exact sequence $B(j,1)$ is

$$0 \to \mathrm{Ker}(\gamma_{M_j}) \to H^j_B(X_\mathbb{C},\mathbb{Z}(1))^+_\mathbb{C} \to t_{M_j} \to \mathrm{Coker}(\gamma_{M_j}) \to 0 \tag{21}$$

and $C(1)_{\mathbb{C}}$ is

$$0 \to H^1_W(X,\mathbb{Z}(1))_{\mathbb{C}} \to \mathrm{Coker}(\gamma_{M_0}) \to H^3_W(X,\mathbb{Z}(1))_{\mathbb{C}} \to H^2_W(X,\mathbb{Z}(1))_{\mathbb{C}} \to \mathrm{Ker}(\gamma_{M_2}) \to H^4_W(X,\mathbb{Z}(1))_{\mathbb{C}} \to 0. \tag{22}$$

So Conjecture 1 provides an intrinsic formula for $\zeta^*(X,1)$ using only $B(0,1)$, $B(1,1)$, $C(1)$ and $H^*_W(X,\mathbb{Z}(1))$. We compute $\chi(B(0,1))$ and explain how $\chi(B(2,1))$ can be assumed to be one.

2.8.4. $\chi(B(0,1))$. This has been computed ([27, §7]) in Subsection 2.7. Here

$$\mathrm{Ker}(\gamma_{M_0}) = 0, \qquad \mathrm{Coker}(\gamma_{M_0}) \cong \mathbb{C}^{r_1+r_2}, \qquad H^0(X,\mathcal{O}) = \mathcal{O}_F. \tag{23}$$

Our convention for $B(0,1)$ is that $\mathrm{Coker}(\gamma_M)$ is in odd degree, say, degree 3. As the integral structure

$$B(0,1): \qquad 0 \to \underset{\text{degree }1}{H^0_B(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}}} \to \underset{\text{degree }2}{(\mathcal{O}_F)_{\mathbb{C}}} \to \underset{\text{degree }3}{\mathrm{Coker}(\gamma_{M_0})} \to 0$$

is torsion-free, we have (again using that the sign of d_F is $(-1)^{r_2}$)

$$\chi(B(0,1)) = det(B(0,1)) = \frac{\sqrt{d_F}}{(2\pi i)^{r_2}} = \frac{\sqrt{|d_F|}}{(2\pi)^{r_2}}. \tag{24}$$

2.8.5. The sequence $B(2,1)$. By Lemma 15, $t_{M_2} = H^2(X_0,\mathcal{O}) = 0$; so, in $B(2,1)$,

$$\mathrm{Ker}(\gamma_{M_2}) \xrightarrow{\sim} H^2_B(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}},$$

we use the isomorphism to transfer the integral structure from $H^2_B(X_{\mathbb{C}},\mathbb{Z}(1))^+$ to $\mathrm{Ker}(\gamma_{M_2})$, thereby obtaining

$$\chi(B(2,1)) = 1. \tag{25}$$

3. The conjecture of Birch–Swinnerton–Dyer–Tate

3.1. Statement of the conjecture [36, 19]

Our presentation of this conjecture follows (almost verbatim) [19, § 2] where it is formulated in terms of Néron models.

Let A be an abelian variety of dimension g over Spec F. Let $L(A,s)$ be its L-series over F, which is defined – see (28) by a Euler product convergent in the half-plane Re $(s) > \frac{3}{2}$. We assume that $L(A,s)$ has an analytic continuation to the entire complex plane. We write $L^*(A,1)$ for its special value at $s=1$. Write

$$L(A,s) \ \sim \ c \cdot (s-1)^{r(A)} \qquad s \to 1,$$

where $r(A)$ is a nonnegative integer and c is nonzero (the special value $L^*(A,1)$). Birch and Swinnerton–Dyer and Tate have conjectured that $r(A)$ is equal to the rank of the finitely generated group $A(F)$ of points of A over F; they have also given a conjectural formula for $L^*(A,1)$ in terms of certain arithmetic invariants of A which we now recall.

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Let $\mathcal{N}$ be the Néron model of A over S. Let $\mathcal{N}^0$ be the largest open subgroup scheme of $\mathcal{N}$ in which all fibres are connected. For any nonzero prime v of $\mathcal{O}_F$ (i.e., $v \in S$), we write

$$\mathcal{N}_v, \qquad \mathcal{N}_v^0$$

for the fibres over v. Thus, $\mathcal{N}_v$ is a commutative group scheme over $k(v)$ and $\mathcal{N}_v^0$ is the connected component of the identity of $\mathcal{N}_v$. We have an exact sequence of group schemes over $k(v)$:

$$0 \to H_v \to \mathcal{N}_v^0 \to B_v \to 0$$

where H_v is a commutative linear group scheme and B_v is an abelian variety. For almost all v, we have

$$\mathcal{N}_v = \mathcal{N}_v^0 = B_v;$$

these are the primes where A has good reduction. Let $Y_v = \mathrm{Hom}(H_v, \mathbb{G}_m)$ be the character group of H_v and π_v be the Frobenius endomorphism of B_v.

We write $\Phi_v = \pi_0(\mathcal{N}_v)$ for the finite group scheme of connected components of the fibre $\mathcal{N}_v$ over v. The Tamagawa number of A at v, denoted c_v, is the size of the group $\Phi_v(k(v))$ of the group of rational points of Φ_v. The Tamagawa number c_v is one for almost all $v \in S$; we may define the product

$$P_{A,fin} = \prod_{v \in S} c_v. \tag{26}$$

Let Γ_v be a decomposition group for v in $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ and let I_v be the inertia subgroup of Γ_v and σ_v be an arithmetic Frobenius, a topological generator of the quotient Γ_v/I_v. For any prime ℓ distinct from the characteristic of $k(v)$, consider the ℓ-adic Tate module $T_\ell A$ of A; this is a free $\mathbb{Z}_\ell$-module of rank $2g$ which admits a continuous $\mathbb{Z}_\ell$-linear action of Γ_F. We define the local L-factor of A at v by the formula

$$L_v(A,t) = \det(1 - \sigma_v^{-1} t \mid \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A, \mathbb{Z}_\ell)^{I_v}). \tag{27}$$

The characteristic polynomial $L_v(A,t)$ has integral coefficients which are independent of ℓ: this is evident from the formula

$$L_v(A,t) = \det(1 - \sigma_v t \mid Y_v) \cdot \det(1 - \pi_v t \mid T_\ell B_v)$$

where the first determinant is clearly independent of ℓ and the second is the characteristic polynomial of an endomorphism of the abelian variety B_v which has integral coefficients independent of ℓ by a theorem of A. Weil.

The roots of the first factor have complex absolute value 1, whereas those of the second factor have complex absolute value $q_v^{-1/2}$. Therefore, the global L-series $L(A,s)$ defined by

$$L(A,s) = L(A/F,s) = \prod_{v \in S} \frac{1}{L_v(A, q_v^{-s})} \tag{28}$$

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converges for Re $(s) > \frac{3}{2}$. We shall assume that it has an analytic continuation to the entire complex plane.

3.2. The archimedean period of A

Let $\omega_{\mathcal{N}}$ denote the *projective* $\mathcal{O}_F$-module of invariant differentials of the Néron model $\mathcal{N}$. Since g is the rank of $\omega_{\mathcal{N}}$, the module $\Lambda^g\omega_{\mathcal{N}}$ is a rank 1 $\mathcal{O}_F$-submodule of $H^0(A,\Omega^g)$, a vector space over F of dimension 1. Let $\{w_1,\cdots,w_g\}$ be an F-basis of $H^0(A,\Omega^1)$ and put $\eta = w_1\wedge\cdots\wedge w_g$. We have

$$\Lambda^g\omega_{\mathcal{N}} = \eta\cdot\mathfrak{a}_\eta \quad \subset H^0(A,\Omega^g) \tag{29}$$

where $\mathfrak{a}_\eta$ is a fractional ideal of F and

$$\mathfrak{a}_\eta = \mathfrak{a}_{c\eta}\cdot(c) \qquad c\in F^\times. \tag{30}$$

For any $\sigma\in S_{\mathbb{R}}$ (corresponds to a real embedding $\sigma: F\to\mathbb{R}$), let H^+ denote the submodule of $H_1(A_\sigma(\mathbb{C}),\mathbb{Z})$ which is fixed by complex conjugation; it is a free $\mathbb{Z}$-module of rank g and if $\{\gamma_1,\cdots,\gamma_g\}$ denotes a basis, we put

$$p_\sigma(A,\eta) = [(\pi_0(A_\sigma(\mathbb{R})))]\cdot\left|\det\left(\left(\int_{\gamma_i}\omega_j\right)\right)\right|. \tag{31}$$

For any complex place v of S (corresponding to $\sigma: F\to\mathbb{C}$ and its conjugate), let $\{\gamma_1,\cdots,\gamma_{2g}\}$ be a basis of the free module $H_1(A_\sigma(\mathbb{C}),\mathbb{Z})$ of rank $2g$ and define the period

$$p_\sigma(A,\eta) = \left|\det(M_v(A,\eta))\right|, \qquad M_v(A,\eta) = \left(\left(\int_{\gamma_i}\omega_j, \overline{\int_{\gamma_i}\omega_j}\right)\right); \tag{32}$$

this period is nonzero and depends only on the differential form η and the place v. The period of A (relative to η) is the real number

$$P_{A,\infty}(\eta) = \prod_{\sigma\in S_\infty} p_\sigma(A,\eta). \tag{33}$$

Remark 16. We recall the well-known fact that the determinant of $M_v(A,\eta)$ is $(\sqrt{-1})^g$ times a real number. In the basis $\{\kappa_1,\cdots,\kappa_g\}$ of $H^0(A_\sigma(\mathbb{C}),\Omega^1)$ defined by $\int_{\gamma_j}\kappa_i = \delta_{ij}$ for $1\le i,j\le g$, the $2g\times 2g$-period matrix $(\int_{\gamma_i}\kappa_j,\overline{\int_{\gamma_i}\kappa_j})$ is a block matrix of the form

$$M' = \begin{bmatrix} I_g & I_g \\ \Omega & \overline{\Omega}\end{bmatrix}, \qquad \Omega = \left(\int_{\gamma_{g+i}}\kappa_j\right)_{1\le i,j\le g}$$

whose determinant is equal to the determinant of the $g\times g$-matrix $\overline{\Omega}-\Omega$ (all of its entries have zero real part) and hence $\det(M')$ is the product of a real number with $\sqrt{-1}^g$. If K is the change of basis matrix from the basis $\{w_1,\cdots,w_g\}$ to the basis $\{\kappa_1,\cdots,\kappa_g\}$, then

$$M_v(A,\eta) = M'\cdot\begin{bmatrix} K & 0\\ 0 & \overline{K}\end{bmatrix} \implies \det(M_v(A,\eta)) = \det(M')\cdot\det(K)\cdot\overline{\det(K)}.$$

Lemma 17. *The number*

$$P_{A,\infty}(\eta)\cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)=\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)\cdot \prod_{v\in S_\infty} p_v(A,\eta)$$

is independent of η. *It will be denoted* $P_{A,\infty}$.

Proof. For any $0\neq k\in F$ and any $v\in S_\infty$, one has

$$p_v(A,k\eta)=\begin{cases} p_v(A,\eta)\cdot \sigma(k) & \text{if } v\in S_{\mathbb{R}} \\ p_v(A,\eta)\cdot \overline{\sigma(k)}\cdot \sigma(k) & \text{if } v\in S_{\mathbb{C}} \end{cases}.$$

On the other hand, we see from (30) that $\mathfrak{a}_{k\eta}=(k)^{-1}\mathfrak{a}_\eta$. Since

$$\prod_{\sigma\in\tilde{S}}\sigma(k)=\mathbb{N}_{F/\mathbb{Q}}(k), \tag{34}$$

one has

$$\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_{k\eta})\cdot \prod_{\sigma\in S_\infty} p_\sigma(A,k\eta)=\mathbb{N}_{F/\mathbb{Q}}(k)^{-1}\cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)\cdot \prod_{\sigma\in S_\infty} p_\sigma(A,\eta)$$
$$\cdot\Big(\prod_{\sigma\in\tilde{S}}\sigma(k)\Big)=\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)\cdot \prod_{\sigma\in S_\infty} p_\sigma(A,\eta).$$

□

Definition 18. The global volume P_A of A is defined by

$$P_A=\frac{P_{A,fin}\cdot P_{A,\infty}}{|d_F|^{g/2}}=\frac{P_{A,fin}\cdot P_{A,\infty}(\eta)\cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}{|d_F|^{g/2}}.$$

Remark 19. It is known that P_A is the volume $m_A(A(\mathbb{A}_F))$ of the adelic points $A(\mathbb{A}_F)$ with respect to the (canonical) Tamagawa measure m_A. If A' is the Weil restriction of A to $F'\subset F$, then $L(A,s)=L(A',s)$. One can compare the terms in Conjecture 20 for A and A':

$$Ш(A/F)\cong Ш(A'/F'),\quad \Theta_{NT}(A)=\Theta_{NT}(A'),\quad A'(F')=A(F),\quad (A')^t(F')=A^t(F);$$

see Remark 23 for the the comparison of P_A and $P_{A'}$. Conjecture 20 is compatible with restriction of scalars [33] [7, §6]: Conjecture 20 for A over F is equivalent to A' over F'; see [7, §6] for a beautiful exposition.

Let A^t be the dual abelian variety. We write $\Theta_{NT}(A)$ for the determinant of the Néron–Tate height pairing

$$\frac{A(F)}{A(F)_{tor}}\times\frac{A^t(F)}{A^t(F)_{tor}}\to\mathbb{R}. \tag{35}$$

Since this height pairing corresponds to the Poincaré divisor on $A\times A^t$ which is symmetric, it follows that $\Theta_{NT}(A)=\Theta_{NT}(A^t)$. Following Subsection 2.3.5, one has

$$\Delta_{NT}(A)=\frac{\Theta_{NT}(A)}{([A(F)_{tor}]\cdot[A^t(F)_{tor}])}.$$

Finally, let $Ш(A/F)$ denote the Tate–Shafarevich group of A over F.

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The BSD conjecture states that the order of vanishing of $L(A,s)$ at $s=1$ is equal to the rank of the Mordell–Weil group $A(F)$ of A. The strong BSD conjecture [36] (we follow Gross's formulation [19, Conjecture 2.10] which is shown by him to be equivalent to the one in [36]) concerns the special value $L^*(A,1)$ of $L(A,s)$ at $s=1$.

Conjecture 20. *$Ш(A/F)$ is finite and $L^*(A,1)$ satisfies*

$$L^*(A,1) = \frac{P_A \cdot \Theta_{NT}(A) \cdot [Ш(A/F)]}{([A(F)_{tor}] \cdot [A^t(F)_{tor}])} = P_A \cdot \Delta_{NT}(A) \cdot [Ш(A/F)]. \tag{36}$$

3.3. The case of everywhere good reduction

If A has good reduction everywhere, then $c_v = 1$ for all $v \in S$ and $P_{A,fin} = 1$; also, $A_v = \mathcal{N}_v = \mathcal{N}_v^0 = B_v$. Conjecture 20 simplifies to the following.

Conjecture 21. *If A has good reduction everywhere, then*

$$L^*(A,1) = \frac{P_{A,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta) \cdot \Theta_{NT}(A) \cdot [Ш(A/F)]}{|d_F|^{g/2}([A(F)_{tor}] \cdot [A^t(F)_{tor}])}.$$

4. Comparison of periods

In order to relate Conjectures 1 and 20, it is necessary to compare the period $P_{J,\infty}$ of Lemma 17 with the determinant of $B(1,1)$ in (21). Let $\omega_{\mathcal{J}}$ denote the projective $\mathcal{O}_F$-module of invariant differentials on the Néron model $\mathcal{J}$ of the Jacobian J of X_0. Recall that the integral structure $i_{\omega_{\mathcal{J}}}$ (Definition 5) uses the abelian group underlying $\omega_{\mathcal{J}}$ and forgets the $\mathcal{O}_F$-structure.

Definition 22. (i) $det(\gamma_{\mathcal{J}}^*)$ is the determinant of the period isomorphism

$$\gamma_{\mathcal{J}}^* : H_B^1(J_{\mathbb{C}},\mathbb{Z})_{\mathbb{C}}^+ \xleftarrow{\sim} \omega_{\mathcal{J}} \otimes_{\mathbb{Z}} \mathbb{C} \tag{37}$$

calculated with respect to $H_B^1(J_{\mathbb{C}},\mathbb{Z})^+$ and the integral structure $i_{\omega_{\mathcal{J}}}$.

(ii) $det_{\mathcal{O}_F}(\gamma_{\mathcal{J}}^*)$ is the determinant of (37) with respect to $H_B^1(J_{\mathbb{C}},\mathbb{Z})^+$ and the $\mathcal{O}_F$-integral structure $j_{\omega_{\mathcal{J}}}$ (Definition 8).

The main result of this section is the following.

Theorem 23. *One has (up to a power of 2)*

$$det(\gamma_{\mathcal{J}}^*) = \pm\frac{P_{J,\infty}}{|d_F|^{g/2}} = \pm(\sqrt{-1})^{g \cdot r_2} \cdot \frac{P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}{\sqrt{d_F}^g}.$$

Remark 24. (i) As we shall see below, the factor $\sqrt{d_F}^g$ arises from the change of integral structures and the factor $\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)$ appears if $\omega_{\mathcal{J}}$ is not free and if F is totally real; then $P_{J,\infty} = det_{\mathcal{O}_F}(\gamma_{\mathcal{J}}^*)$ up to a power of 2 (because (31) – but not (37) – uses $\pi_0(A_\sigma(\mathbb{R}))$, a finite 2-group). Finally, there is a factor of $\sqrt{-1}^{r_2 \cdot g}$ because the definition of $P_{J,\infty}$ uses a different Betti lattice at complex places.

(ii) As $det(\gamma_{\mathcal{J}}^*)$ uses only the underlying abelian group of $\omega_{\mathcal{J}}$, one observes that the actual $\mathcal{O}_F$-structure on $\omega_{\mathcal{J}}$ is irrelevant for Conjecture 20. This observation has the following implication. As in Remark 19, let A be an abelian variety over F and let A' be its restriction to $F' \subset F$; let $\mathcal{N}$ and $\mathcal{N}'$ be their Néron models. Conjecture 20 for an abelian variety A over F is equivalent to Conjecture 20 for A' over F'. A key step in the proof of the equivalence is showing that [7, §6]

$$\frac{P_{A,\infty}}{(|d_F|)^{g/2}} = \frac{P_{A',\infty}}{(|d_{F'}|)^{\frac{g[F:F']}{2}}}.$$

This is clear from Theorem 23 as $\omega_{\mathcal{N}}$ is isomorphic to $\omega_{\mathcal{N}'}$ as abelian groups and the period isomorphism $\gamma_{\mathcal{N}}$ is the same as $\gamma_{\mathcal{N}'}$.

Proof. (of Theorem 23) It is difficult to calculate or write down a formula directly for $det(\gamma^*_{\mathcal{J}})$ as the integral structure $i_{\omega_{\mathcal{J}}}$ is not easy to describe explicitly. So one has to use $j_{\omega_{\mathcal{J}}}$. By Proposition 9, one has

$$det_{\mathcal{O}_F}(\gamma^*_{\mathcal{J}}) = det(\gamma^*_{\mathcal{J}}) \cdot \sqrt{d_F}^{\,g}.$$

We shall now see how to compute $det_{\mathcal{O}_F}(\gamma_{\mathcal{J}})$ explicitly in the style of Subsection 3.2.

4.0.1. The case that $N = \omega_{\mathcal{J}}$ is free

If N is free as an $\mathcal{O}_F$-module (its rank is g), then pick an $\mathcal{O}_F$-basis $\mathcal{B} = \{v_1, \cdots, v_g\}$ for it. If $\mathcal{B}'$ is a different basis for N, then the determinant $d(\mathcal{B},\mathcal{B}')$ of the change of basis matrix is a unit in $\mathcal{O}_F$. Note that $\mathcal{B}$ is also a basis for the F-vector space $\omega_{\mathcal{J}} \otimes_{\mathcal{O}_F} F = H^0(J,\Omega^1)$. For any embedding $\sigma : F \to \mathbb{C}$, let

$$\sigma(\mathcal{B}) = \{\sigma(v_1), \cdots, \sigma(v_g)\}$$

be the image of $\mathcal{B}$ under the isomorphism (via σ)

$$H^0(J,\Omega^1) \otimes_F \mathbb{C} \cong H^0(J_\sigma(\mathbb{C}),\Omega^1).$$

So $\sigma(\mathcal{B})$ is a basis for the complex vector space $H^0(J_\sigma(\mathbb{C}),\Omega^1)$. Just as a basis for V and W produces a basis for $V \times W$, we can combine the bases $\sigma(\mathcal{B})$ for $\sigma \in \tilde{S}$ to get a basis $\tilde{\mathcal{B}}$ for the complex vector space

$$H^0(J_{\mathbb{C}},\Omega^1) = H^0(J,\Omega^1) \otimes_{\mathbb{Z}} \mathbb{C} = \prod_{\sigma \in \tilde{S}} H^0(J_\sigma(\mathbb{C}),\Omega^1).$$

The lattice ($\mathbb{Z}$-module) spanned by $\tilde{\mathcal{B}}$ represents the $\mathcal{O}_F$-integral structure j_N on $V = N \otimes_{\mathbb{Z}} \mathbb{C}$ (see Definition 8). The determinant $D(\mathcal{B})$ of (37) computed with respect to the integral structure provided by $\tilde{\mathcal{B}}$ is equal to $det_{\mathcal{O}_F}(\gamma^*_{\mathcal{J}})$. Let us show that $D(\mathcal{B})$ is independent of the basis $\tilde{\mathcal{B}}$.

Given another basis $\mathcal{B}'$ for N, we have

$$D(\mathcal{B}') = D(\mathcal{B}) \prod_{\sigma \in \tilde{S}} \sigma(d_{\mathcal{B},\mathcal{B}'}).$$

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The determinant of the change of basis matrix from $\tilde{\mathcal{B}}$ to $\tilde{\mathcal{B}}'$ is

$$\prod_{\sigma\in\tilde{S}}\sigma(d_{\mathcal{B},\mathcal{B}'}) = \mathbb{N}_{F/\mathbb{Q}}d_{\mathcal{B},\mathcal{B}'} = \pm 1$$

as $d_{\mathcal{B},\mathcal{B}'}$ is a unit in $\mathcal{O}_F$. Though the integral structure on $N\otimes_{\mathbb{Z}}\mathbb{C}$ provided by $\tilde{\mathcal{B}}$ depends on the choice of $\mathcal{B}$, the number $D(\mathcal{B})$ is independent of the basis $\mathcal{B}$: Any two $\mathcal{O}_F$-bases of N provide Euler-equivalent integral structures representing j_N on $V = N\otimes_{\mathbb{Z}}\mathbb{C}$. Thus, in this case, for any basis $\mathcal{B}$, one has

$$det(\gamma^*_{\mathcal{J}}) = \frac{D(\mathcal{B})}{\sqrt{d_F}^g}.$$

4.0.2. The case that $N = \omega_{\mathcal{J}}$ is not free as a $\mathcal{O}_F$-module

In this case, we proceed as in Subsection 3.2 using the top exterior power $\Lambda^g N$ of the projective $\mathcal{O}_F$-module N. Let $\mathcal{B} = \{v_1,\cdots,v_g\}$ be a F-basis of $H^0(J,\Omega^1)$ and put $\rho = v_1\wedge\cdots\wedge v_g$. We have

$$\Lambda^g N = \mathfrak{b}_\rho \cdot \rho \subset \Lambda^g H^0(J,\Omega^1) \tag{38}$$

where $\mathfrak{b}_\rho$ is a fractional ideal of $\mathcal{O}_F$. For any $k \neq 0 \in F$, one has an equality of fractional ideals

$$\mathfrak{b}_\rho = \mathfrak{b}_{k\rho} \cdot (k). \tag{39}$$

As before, let $\tilde{\mathcal{B}}$ denote the basis of $V = N\otimes_{\mathbb{Z}}\mathbb{C}$ obtained from $\mathcal{B}$. Let $D(\mathcal{B})$ denote the determinant of (37) with respect to the basis $\tilde{\mathcal{B}}$ and $H^1_B(J_{\mathbb{C}},\mathbb{Z})^+$. Using (38), one has

$$det_{\mathcal{O}_F}(\gamma^*_{\mathcal{J}}) = D(\mathcal{B}) \cdot \mathbb{N}_{F/\mathbb{Q}}\mathfrak{b}_\rho.$$

Let us directly show that the right-hand side is independent of the basis. Given any basis $\mathcal{B}' = \{v'_1,\cdots,v'_g\}$,

$$D(\mathcal{B}') = D(\mathcal{B})\prod_{\sigma\in\tilde{S}}\sigma(k) \qquad k := d(\mathcal{B},\mathcal{B}').$$

Note that $k = d(\mathcal{B},\mathcal{B}') \in F^\times$ need not be a unit.

If $\rho' = v'_1\wedge\cdots\wedge v'_g$, then

$$\rho' = \rho \cdot k, \qquad k = d(\mathcal{B},\mathcal{B}'),$$

which gives

$$b_{\rho'} = b_\rho \cdot (k)^{-1}.$$

Thus, using (34) as in Lemma 17, we find

$$D(\mathcal{B}')\cdot \mathbb{N}_{F/\mathbb{Q}}\mathfrak{b}_{\rho'} = D(\mathcal{B}) \cdot (\prod_{\sigma\in\tilde{S}}\sigma(k)) \cdot \mathbb{N}_{F/\mathbb{Q}}\mathfrak{b}_{\rho'} = D(\mathcal{B})$$
$$\cdot (\prod_{\sigma\in\tilde{S}}\sigma(k)) \cdot \mathbb{N}_{F/\mathbb{Q}}\mathfrak{b}_\rho \cdot (\mathbb{N}_{F/\mathbb{Q}}(k))^{-1} = D(\mathcal{B}) \cdot \mathbb{N}_{F/\mathbb{Q}}\mathfrak{b}_\rho.$$

Thus, $D(\mathcal{B}) \cdot \mathbb{N}_{F/\mathbb{Q}}\mathfrak{b}_\rho$ is independent of $\mathcal{B}$.

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4.0.3. Betti lattices

It should be clear that the above computation of $det_{\mathcal{O}_F}(\gamma^*_{\mathcal{J}})$ is exactly the computation of $P_{J,\infty}$ in Subsection 3.2. In fact, for any g-dimensional abelian variety A over F, one has (up to a power of 2)

$$det_{\mathcal{O}_F}(\gamma^*_{\mathcal{N}}) = \pm(\sqrt{-1})^{g\,\cdot\,r_2}\,\cdot\,P_{A,\infty}. \tag{40}$$

Here, as in Subsection 3.2, $\mathcal{N}$ is the Néron model of A and $\gamma^*_{\mathcal{N}}$ is the period isomorphism

$$\gamma^*_{\mathcal{N}}:\ H^1_B(A_{\mathbb{C}},\mathbb{Z})^+_{\mathbb{C}} \xleftarrow{\sim} \omega_{\mathcal{N}} \otimes_{\mathbb{Z}} \mathbb{C}.$$

Let $B=\{w_1,\cdots,w_g\}$ be an F-basis of $H^0(A,\Omega^1)$ and put $\eta = w_1\wedge\cdots\wedge w_g$. The discrepancy in (40) arises from the fact that the Betti lattices are different.

For a real place of S corresponding to $\sigma: F\to\mathbb{R}$, consider the period isomorphism

$$\gamma_\sigma = H^1_B(A_\sigma(\mathbb{C}),\mathbb{Z})^+_{\mathbb{C}} \xleftarrow{\sim} H^0(A_\sigma,\Omega^1).$$

For a complex place v of S (corresponding to $\sigma: F\to\mathbb{C}$ and its conjugate $c\sigma$), consider the complex abelian variety

$$A_v = A_\sigma \times A_{c\sigma}$$

and the period isomorphism

$$\gamma_v : H^1_B(A_v,\mathbb{Z})^+_{\mathbb{C}} \xleftarrow{\sim} H^0(A_v,\Omega^1).$$

For real places, the groups in γ_σ are the same as the one used in (31); in this case, $p_\sigma(A,\eta)$ is the determinant[2] of γ_σ relative to the basis $H^1_B(A_\sigma(\mathbb{C}),\mathbb{Z})^+$ and $B=\{w_1,\cdots,w_g\}$.

For complex places v, (32) uses $H^1(A_\sigma,\mathbb{Z})$ instead of $H^1_B(A_v,\mathbb{Z})^+$. In this case, $(\sqrt{-1})^g\,\cdot\,p_v(A,\eta)$ is the determinant of γ_v relative to the basis $H^1_B(A_v,\mathbb{Z})^+$ and B. This is what leads to the discrepancy between the two invariants. What follows is presumably well-known, but we include it for sake of completeness.

Complex conjugation

$$c: A_\sigma \to A_{c\sigma}$$

simply permutes the factors of A_v. One has the decomposition

$$H_1(A_v(\mathbb{C}),\mathbb{Z}) = H_1(A_\sigma(\mathbb{C}),\mathbb{Z}) \oplus H_1(A_{c\sigma}(\mathbb{C}),\mathbb{Z}),$$
$$H^0(A_v,\Omega^1) = H^0(A_\sigma,\Omega^1) \oplus H^0(A_{c\sigma},\Omega^1).$$

Concretely, a basis for $H_1(A_v,\mathbb{Z})$ is given by

$$\{(\gamma_1,0),(\gamma_2,0),\cdots,(\gamma_{2g},0),(0,c\gamma_1),(0,c\gamma_2),\cdots,(0,c\gamma_{2g})\};$$

a basis for $H^0(A_v,\Omega^1)$ is given by (here the image of w_j is denoted as w_j on A_σ and as cw_j on $A_{c\sigma}$)

$$\{(w_1,0),(w_2,0),\cdots,(w_g,0),(0,cw_1),(0,cw_2),\cdots,(0,cw_g)\}.$$

[2] Up to a power of 2.

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So a basis for $H_1(A_v,\mathbb{Z})^+$ is given by

$$\{(\gamma_1,c\gamma_1),\cdots,(\gamma_{2g},c\gamma_{2g})\}.$$

For the complex place v and η, the determinant of γ_v relative to the basis $H^1_B(A_v,\mathbb{Z})^+$ and B is the determinant of the $2g\times 2g$-matrix M

$M =$

$$\begin{bmatrix} \int_{(\gamma_1,c\gamma_1)}(w_1,0) & \int_{(\gamma_1,c\gamma_1)}(w_2,0) & \cdots & \int_{(\gamma_1,c\gamma_1)}(w_g,0) & \int_{(\gamma_1,c\gamma_1)}(0,cw_1) & \cdots & \int_{(\gamma_1,c\gamma_1)}(0,cw_g) \\ \int_{(\gamma_2,c\gamma_2)}(w_1,0) & \int_{(\gamma_2,c\gamma_2)}(w_2,0) & \cdots & \int_{(\gamma_2,c\gamma_2)}(w_g,0) & \int_{(\gamma_2,c\gamma_2)}(0,cw_1) & \cdots & \int_{(\gamma_2,c\gamma_2)}(0,cw_g) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \int_{(\gamma_{2g},c\gamma_{2g})}(w_1,0) & \int_{(\gamma_{2g},c\gamma_{2g})}(w_2,0) & \cdots & \int_{(\gamma_{2g},c\gamma_{2g})}(w_g,0) & \int_{(\gamma_{2g},c\gamma_{2g})}(0,cw_1) & \cdots & \int_{(\gamma_{2g},c\gamma_{2g})}(0,cw_g) \end{bmatrix}$$

Observe that (in a product $V\times W$ of manifolds, a differential form from W does not pair with a cycle from V)

$$\int_{(\gamma_i,c\gamma_i)}(w_j,0)=\int_{\gamma_i} w_j, \qquad \int_{(\gamma_i,c\gamma_i)}(0,cw_j)=\int_{c\gamma_i} cw_j.$$

So the matrix M can be rewritten as

$$M=\begin{bmatrix} \int_{\gamma_1} w_1 & \int_{\gamma_1} w_2 & \cdots & \int_{\gamma_1} w_g & \int_{c\gamma_1} cw_1 & \cdots & \int_{c\gamma_1} cw_g \\ \int_{\gamma_2} w_1 & \int_{\gamma_2} w_2 & \cdots & \int_{\gamma_2} w_g & \int_{c\gamma_2} cw_1 & \cdots & \int_{c\gamma_2} cw_g \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \int_{\gamma_{2g}} w_1 & \int_{\gamma_{2g}} w_2 & \cdots & \int_{\gamma_{2g}} w_g & \int_{c\gamma_{2g}} cw_1 & \cdots & \int_{c\gamma_{2g}} cw_g \end{bmatrix}.$$

As A_σ and $A_{c\sigma}$ are complex conjugate abelian varieties, one has

$$\int_{c\gamma_i} cw_j=\overline{\int_{\gamma_i} w_j};$$

it follows that M and $M_v(A,\eta)$ of (32) satisfy

$$\det(M)=\det(M_v(A,\eta)).$$

Remark 16 shows that

$$\prod_{v\in S_\infty}\det(M_v(A,\eta))$$

is a real number multiplied by $(\sqrt{-1})^{g\,\cdot\,r_2}$. This proves the identity (40); combining it with Proposition 9 provides (up to a power of 2)

$$det(\gamma^*_{\mathcal{J}})=\frac{det_{\mathcal{O}_F}(\gamma^*_{\mathcal{J}})}{\sqrt{d_F}^{\,g}}=\pm(\sqrt{-1})^{g\,\cdot\,r_2}\frac{P_{J,\infty}}{\sqrt{d_F}^{\,g}}.$$

Theorem 23 follows using the well-known result that the sign of d_F is $(-1)^{r_2}$.

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Proposition 25. *(i)* $det_{\mathcal{O}_F}(\gamma^*_{\mathcal{J}})$ *is equal to the determinant* $det_{\mathcal{O}_F}(\gamma_{\mathcal{J}})$ *of*

$$\gamma_{\mathcal{J}}:\ H^1_B(J_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\sim} Lie\ (\mathcal{J})\otimes_{\mathbb{Z}}\mathbb{C}$$

calculated using the $\mathcal{O}_F$*-integral structure* $j_{Lie\ (\mathcal{J})}$ *(Definition 8) and the lattice* $H^1_B(J_{\mathbb{C}},\mathbb{Z}(1))^+$.

(ii) $det(\gamma^*_{\mathcal{J}}) = det(\gamma_{\mathcal{J}})$.

Proof. (i) For any complex abelian variety A, the dual abelian variety A^t satisfies

$$H_1(A^t,\mathbb{Z}) = H^1_B(A,\mathbb{Z}(1)) \qquad \text{Lie } A = H^1(A^t,\mathcal{O})$$
$$\text{Hom}(H^1_B(A^t,\mathbb{Z}),\mathbb{Z}) = H^1_B(A,\mathbb{Z}(1)) \qquad \text{Hom}_{\mathbb{C}}(H^1(A,\mathcal{O}),\mathbb{C}) = H^0(A^t,\Omega^1).$$

This shows that the period isomorphism

$$\gamma_A : H^1_B(A,\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\sim} H^1(A,\mathcal{O})$$

is dual to the map

$$\gamma^*_{A^t} : H^1_B(A^t,\mathbb{Z})^+_{\mathbb{C}} \xleftarrow{\sim} H^0(A^t,\Omega^1).$$

Applying these to the self-dual Jacobian $J_{\mathbb{C}}$ gives that the lattice $H^1_B(J_{\mathbb{C}},\mathbb{Z}(1))^+$ is dual to the lattice $H^1_B(J_{\mathbb{C}},\mathbb{Z})^+$. The natural duality of the projective $\mathcal{O}_F$-modules Lie $(\mathcal{J})$ and $\omega_{\mathcal{J}}$ shows, using Proposition 9, that the $\mathcal{O}_F$-integral structures $j_{\text{Lie }(\mathcal{J})}$ and $j_{\omega_{\mathcal{J}}}$ are dual. As the determinants of dual maps computed with respect to dual lattices are equal, the result follows.

(ii) Proposition 9 and (i) imply $det(\gamma^*_{\mathcal{J}}) \cdot (\sqrt{d_F})^g = det_{\mathcal{O}_F}(\gamma^*_{\mathcal{J}}) = det_{\mathcal{O}_F}(\gamma_{\mathcal{J}}) = det(\gamma_{\mathcal{J}}) \cdot (\sqrt{d_F})^g$. □

5. Proof of Theorem 2 in a special case

Throughout this section, we assume that $\pi : X \to S$ is smooth and $X_0(F)$ is nonempty. We prove Theorem 2 in this case by computing $H^*_W(X,\mathbb{Z}(1))$ and then use it to compare $\chi(X,1)$ with Conjecture 21 for the Jacobian J of X_0.

5.1. The groups $H^*_W(X,\mathbb{Z}(1))$

By Subsection 2.8.1, we need to understand Pic(X) and Br(X).

Any $x \in X_0(F)$ provides Pic$(X_0) = J(F)\times\mathbb{Z}$.

Proposition 26. *One has (neglecting 2-torsion)*

$$(i)\ \ Pic(X) \cong \mathbb{Z}\times J(F)\times Pic(S), \quad [Pic(X)_{tor}] = h\cdot[J(F)_{tor}],$$

$$(ii)\ \ Br(X) \xrightarrow{\sim} Ш(J/F).$$

Proof. As $\pi : X \to S$ is smooth proper, any $x \in X_0(F)$ provides a splitting of the map Pic$(S) \to$ Pic(X). The identity component $\text{Pic}^0_{X/S}$ of the relative Picard scheme $\text{Pic}_{X/S}$ is the Néron model $\mathcal{J}$ of J by [4, Theorem 1, page 264] and $\text{Pic}_{X/S}(S) = \text{Pic}^0_{X/S}(S)\times\mathbb{Z}$ by [4, Theorem 1, page 252]. As $\pi_*\mathcal{O}_X = \mathcal{O}_S$, [4, Proposition 4, page 204] says $\text{Pic}_{X/S}(S) =$

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$\mathrm{Pic}(X)/\mathrm{Pic}(S)$ and $\mathrm{Pic}_{X/S}(F) = \mathrm{Pic}(X_0)$. This proves (i). For (ii), we note that [36, Theorem 3.1] provides an exact (modulo 2-torsion) sequence

$$0 \to \mathrm{Br}(S) \to \mathrm{Br}(X) \to Ш(J/F) \to 0;$$

by class field theory, $\mathrm{Br}(S)$ is a finite 2-group. □

5.2. The Euler characteristic $\chi_B(X,1)$

We now compute the Euler characteristic of (21),

$$\chi_B(X,1) = \frac{\chi(B(0,1))\chi(B(2,1))}{\chi(B(1,1))}.$$

Remark 27. As $\pi : X \to S$ is smooth, $H^1(X_s,\mathcal{O}_{X_s})$ has dimension g over the residue field $k(s)$ for all points $s \in S$. Since S is a reduced noetherian scheme, $\pi : X \to S$ is proper and $\mathcal{O}_X$ is a coherent sheaf on X flat over S, a result of Grothendieck (see L. Illusie's article [9, Corollary 8.3.4]) states that $R^1\pi_*\mathcal{O}_X$ is locally free and that $R^1\pi_*\mathcal{O}_X \otimes k(s) \cong H^1(X_s,\mathcal{O}_{X_s})$. So the $\mathcal{O}_F$-module $H^1(X,\mathcal{O})$ is projective of rank g.

5.2.1. The Euler characteristic of $B(1,1)$. We write $det(\gamma_X)$ for the determinant of

$$\gamma_X : H^1_B(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\sim} H^1(X,\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C} \tag{41}$$

with respect to the lattices $H^1_B(X_{\mathbb{C}},\mathbb{Z}(1))^+$ and the abelian group underlying $H^1(X,\mathcal{O})$.

Lemma 28. *The projective $\mathcal{O}_F$-modules $H^1(\mathcal{J},\mathcal{O})$ and $\omega_{\mathcal{J}}$ are dual.*

Proof. As $\mathcal{J} \to S$ is an abelian scheme, its relative Picard scheme $\mathrm{Pic}_{\mathcal{J}/S}$ exists [4, Theorem 5, p. 234] and its identity component $\mathrm{Pic}^0_{\mathcal{J}/S}$ is the dual abelian scheme $\mathcal{J}^t \to S$. The S-points Lie $\mathrm{Pic}_{\mathcal{J}}$ of its Lie algebra[3] *Lie* $\mathrm{Pic}_{\mathcal{J}}$ satisfy [4, Theorem 1, p. 231], [30, Proposition 1.1 (d)]

$$\mathrm{Lie}\ (\mathcal{J}^t) = \mathrm{Lie}\ (\mathrm{Pic}_{\mathcal{J}/S}) \xrightarrow{\sim} H^1(\mathcal{J},\mathcal{O}). \tag{42}$$

Since J^t is the generic fibre of $\mathcal{J}^t$, the self-duality $J \cong J^t$ shows $\mathcal{J} \cong \mathcal{J}^t$ and $\mathrm{Lie}\ (\mathcal{J}^t) \cong \mathrm{Lie}\ (\mathcal{J})$. Combining this with the natural duality between $\mathrm{Lie}\ (\mathcal{J})$ and $\omega_{\mathcal{J}}$ [30, Proposition 1.1 (c)] proves the lemma. □

Proposition 29. *One has*

$$\chi(B(1,1)) = \frac{P_{J,fin}}{P_J} = \frac{\sqrt{|d_F|}^g}{P_{J,\infty}(\eta)\ \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}.$$

Proof. Any $x \in X_0(F)$ gives a map

$$\beta_x : X \to \mathcal{J}, \qquad X_0 \to J;$$

[3] For any group scheme G over S, we write Lie G for the S-points *Lie* $G(S)$ of the Lie algebra *Lie* G over S.

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the induced map β on cohomology is independent of the choice of x; for example, it provides the following isomorphism:

$$\beta : H^1(\mathcal{J},\mathcal{O}) \xrightarrow{\sim} H^1(X,\mathcal{O})$$

of $\mathcal{O}_F$-modules, which fits into a commutative diagram (using Lemma 28)

$$\begin{array}{ccc} H^1_B(J_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} & \xrightarrow{\gamma_{\mathcal{J}}} & H^1(\mathcal{J},\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ H^1_B(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} & \xrightarrow[\gamma_X]{} & H^1(X,\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C}; \end{array} \tag{43}$$

as the vertical maps β are isomorphisms, the integral structure corresponding to the map $\gamma_{\mathcal{J}}$ (top row) is isomorphic to the one corresponding to γ_X (bottom row). As they are torsion-free, Theorem 23 and Proposition 25 show that

$$\chi(\gamma_X) = \chi(\gamma_{\mathcal{J}}) = det(\gamma_{\mathcal{J}}) = \frac{P_{J,\infty}}{|d_F|^{g/2}}.$$

Our convention for $B(1,1)$ is that $\mathrm{Ker}(\gamma_M)$ is in even degree, say, degree 0. We have

$$B(1,1): \underbrace{H^1_B(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}}}_{\text{degree } 1} \to \underbrace{H^1(X,\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C}}_{\text{degree } 2},$$

$$det(B(1,1)) = \frac{\sqrt{|d_F|}^g}{P_{J,\infty}(\eta)\ \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)} = \chi(B(1,1)).$$

□

Combining Proposition 29 with (24) and (25) yields the following.

Proposition 30. *We have*

$$\frac{1}{\chi_B(X,1)} = \frac{\chi(B(1,1))}{\chi(B(0,1))\chi(B(2,1))} = \frac{(2\pi i)^{r_2}}{\sqrt{d_F}} \cdot \frac{\sqrt{|d_F|}^g}{P_{J,\infty}(\eta)\ \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}.$$

5.3. The determinant of the sequence $C(1)$

Using (16), (17), (18) and Proposition 26, the torsion in (22) satisfies

$$\chi(C(1)_{tor}) = \frac{w \cdot [Br(X)]}{[\mathrm{Pic}(X)_{tor}]^2} = \frac{w \cdot [Br(X)]}{h \cdot [J(F)_{tor}] \cdot h \cdot [J(F)_{tor}]}.$$

As $\mathrm{Hom}(\mathrm{Pic}(X),\mathbb{Z}) \otimes \mathbb{C} \cong \mathbb{C} \times \mathrm{Hom}(J(F),\mathbb{Z}) \otimes \mathbb{C}$ by Proposition 26, the sequence $C(1)_{\mathbb{C}}$ of (22)

$$0 \to \underbrace{\mathcal{O}_F^\times \otimes \mathbb{C}}_{degree\ 0} \to \underbrace{\mathrm{Coker}(\gamma_{M_0})}_{degree\ 1} \to \underbrace{\mathrm{Hom}(\mathrm{Pic}(X),\mathbb{C})}_{degree\ 2} \xrightarrow{h} \underbrace{\mathrm{Pic}(X) \otimes \mathbb{C}}_{degree\ 3} \to \underbrace{\mathrm{Ker}(\gamma_{M_2})}_{degree\ 4}$$
$$\to \underbrace{\mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{C})}_{degree\ 5} \to 0. \tag{44}$$

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breaks up into exact sequences

$$0 \to \underset{degree\ 0}{\mathcal{O}_F^\times \otimes \mathbb{C}} \to \underset{degree\ 1}{\mathrm{Coker}(\gamma_{M_0})} \to \mathbb{C} \to 0,$$

$$\underset{degree\ 2}{\mathrm{Hom}(\mathrm{Pic}^0(X),\mathbb{C})} \xrightarrow{h} \underset{degree\ 3}{\mathrm{Pic}^0(X)\otimes\mathbb{C}},$$

$$0 \to \mathbb{C} \to \underset{degree\ 4}{\mathrm{Ker}(\gamma_{M_2})} \to \underset{degree\ 5}{\mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{Z})\otimes\mathbb{C}} \to 0.$$

We compute $det(C(1))$ in three steps:

- By (23), the first sequence is (14) and so has determinant R.
- By [27, Conjecture 2.3.7] or (11), the map h is the Arakelov intersection pairing

  $$\mathrm{Hom}(J(F),\mathbb{Z})\otimes\mathbb{C} \to J(F)\otimes\mathbb{C}.$$

  By Faltings–Hriljac [8, 21], $-h$ is the Néron–Tate height pairing (35) and so $det(h) = \pm\Theta_{NT}(J)$.
- As $t_{M_2} = H^2(X_0,\mathcal{O}) = 0$, one has $\mathrm{Ker}(\gamma_{M_2}) = H^2(X_{\mathbb{C}},\mathbb{C}(1))^+ \cong \mathbb{C}^{r_1+r_2}$. Using (18), the third sequence is (12) and so has determinant R.

Proposition 31. *We have*

$$det(C(1)) = \frac{R\times R}{\Theta_{NT}} \qquad \chi(C(1)) = \frac{det(C(1))}{\chi(C(1)_{tor})} = \frac{h\times h[J(F)_{tor}][J(F)_{tor}]}{[Br(X)]w}\frac{R\times R}{\Theta_{NT}}.$$

5.4. Completion of the proof of the main theorem in a special case

In this case, J has good reduction everywhere as $\mathrm{Pic}^0_{X/S}$ is the Néron model of J; the group scheme Φ_v is trivial and the Tamagawa numbers $c_v = 1$ for all v and $P_{J,fin} = 1$. Conjecture 20 for $L(J,s)$ becomes Conjecture 21:

$$L^*(J,1) = \frac{P_{J,\infty}(\eta)\ \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}{(\sqrt{|d_F|})^g}\frac{[\text{Ш}(J/F)]\ \cdot\ \Theta_{NT}}{[J(F)_{tor}][J(F)_{tor}]}.$$

We recall that

$$\zeta(X,s) = \prod_{v\in S}\zeta(X_v,s), \qquad \zeta(X_v,s) = \frac{P_1(X_v,s)}{P_0(X_v,s)P_2(X_v,s)}.$$

As $\pi : X \to S$ is smooth and proper, one has

$$P_0(X_v,s) = (1-q_v^{-s}), \quad P_1(C_v,t) = P_1(J_v,t), \quad P_2(X_v,s) = (1-q_v^{1-s})$$

and

$$\prod_{v\in S}\frac{1}{P_0(X_v,s)} = \zeta(S,s), \qquad \prod_{v\in S}\frac{1}{P_2(X_v,s)} = \zeta(S,s-1), \quad L(J,s) = \prod_{v\in S}\frac{1}{P_1(X_v,s)}. \quad (45)$$

Using (13), (15), we see that Conjecture 21 for J/F is equivalent to the equality

$$\zeta^*(X,1) = \frac{\zeta^*(S,1)\zeta^*(S,0)}{L^*(J,1)} = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d_F|}}\cdot\frac{hR}{w}\cdot\frac{(\sqrt{|d_F|})^g[J(F)_{tor}][J(F)_{tor}]}{[\text{Ш}(J/F)]\Theta_{NT}P_{J,\infty}(\eta)\ \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}.$$

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Conjecture 1 for $\zeta^*(X,1)$ says

$$\zeta^*(X,1)=\chi(X,1)=\frac{\chi_{A,C}(X,1)}{\chi_B(X,1)}=\frac{\chi(C(1))}{\chi_B(X,1)\cdot\chi(A'(3,1))}.$$

Propositions 30 and 31 show

$$\chi(X,1)=\frac{1}{w}\cdot\frac{[\mathrm{Pic}(X)_{tor}]^2}{w\ [\mathrm{Br}(X)]}\,\frac{R\times R}{\Theta_{NT}}\cdot\frac{(2\pi i)^{r_2}}{\sqrt{d_F}}\,\frac{\sqrt{|d_F|}^g}{P_{J,\infty}(\eta)\ \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}.$$

Proposition 26 shows that Conjecture 1 is equivalent to Conjecture 21 for J over F, up to powers of 2. This proves Theorem 2 in the special case $\pi:X\to S$ is smooth and $X_0(F)$ is nonempty.

6. Towards the main theorem in the general case: arbitrary reduction

The assumption $\pi:X\to S$ is smooth and $X_0(F)$ is nonempty leads to several simplifications such as $H^1(X,\mathcal{O})$ is torsion-free, $\mathrm{Br}(X)\cong Ш(J/F)$, the intersection pairing on $\mathrm{Pic}^0(X)$ is the Néron–Tate pairing on $J(F)$ and $\mathrm{Pic}^0(X_0)\cong J(F)$. The failure of these identities complicates the proof of Theorem 2 in the general case. In order to handle these complications, we now recall certain results from [18, 30, 31, 32, 17, 37, 29].

6.1. Index and Period

Let C be a smooth proper curve over a field K and $T=\mathrm{Spec}\ K$.

Definition 32.

(i) The index δ_K of C over K is the least positive degree of a K-rational divisor on C. If

$$d:\mathrm{Pic}(C)\to\mathbb{Z}$$

is the degree map, then $\delta_K=[\mathrm{Coker}(d)]$. The kernel of d is $\mathrm{Pic}^0(C)$.

(ii) The period δ'_K of C over K is the least positive degree of a K-rational divisor class on C.

6.2. Curves over local and global fields [3, 30]

Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field $k(v)=\mathbb{F}_q$. Write $T=\mathrm{Spec}\ \mathcal{O}_K$ and $v:\mathrm{Spec}\ \mathbb{F}_q\to T$ for the closed point of T. Let $f:C\to T$ be a flat projective morphism with C regular, $f_*\mathcal{O}_C=\mathcal{O}_T$ and $C_K\to\mathrm{Spec}\ K$ a geometrically connected smooth curve of genus $g>0$. Let $\mathcal{N}$ be the Néron model of the Jacobian J of C_K. We write $\Phi_v=\pi_0(\mathcal{N}_v)$ for the finite group scheme of connected components of the special fibre $\mathcal{N}_v$ over v and c_v is the order of $\Phi_v(k(v))$.

Consider the map $Lie\ (\phi):R^1f_*\mathcal{O}_C\to Lie\ \mathcal{N}$ of coherent sheaves on T and the induced map on T-points

$$\mathrm{Lie}(\phi):H^1(C,\mathcal{O}_C)\to\mathrm{Lie}\ (\mathcal{N}).$$

We will need the following result in Section 7.

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Theorem 33 ([30, Theorem 3.1].).

(i) *The kernel of Lie* (ϕ) *is the torsion subgroup of* $H^1(C,\mathcal{O}_C)$.

(ii) *The kernel and cokernel of Lie* (ϕ) *are torsion sheaves on* T *of the same length.*

Let Γ_i $(i \in I)$ be the irreducible components of the special fibre C_v.

- d_i is the multiplicity of Γ_i in C_v.
- e_i is the geometric multiplicity of Γ_i in C_v.
- r_i is the number of irreducible components of $\Gamma_i \times \text{Spec}\ \overline{\mathbb{F}_q}$.

There are canonical maps [3, Proposition 1.9] $\alpha_C : \mathbb{Z}^I \to \mathbb{Z}^I$ (defined using intersection on C) and $\beta_C : \mathbb{Z}^I \to \mathbb{Z}$. Let d be the gcd of the set $\{d_i, i \in I\}$ and d' be the gcd of the set $\{r_i d_i, i \in I\}$.

Theorem 34 ([3, Theorem 1.7, Corollary 1.12].).

(i) *There is an exact sequence*

$$0 \to \frac{Ker\ (\beta_C)}{Im\ (\alpha_C)} \xrightarrow{h} \Phi_v(\mathbb{F}_q) \to \frac{cd\mathbb{Z}}{d'\mathbb{Z}} \to 0;$$

here $c = 1$ *if* d' *divides* $g-1$ *and* $c = 2$ *otherwise.*

(ii) *If* $C_K(K)$ *is not empty, then* $d' = d$ *and* h *is an isomorphism.*

It is known that [29, §9.1, Theorem 1.23] that the intersection pairing on R_v

$$R_v := \frac{\mathbb{Z}^I}{\mathbb{Z}} = \text{Coker}\ (\mathbb{Z} \to \mathbb{Z}^I) \qquad 1 \mapsto \sum_{i\in I} d_i \Gamma_i$$

is negative definite. The following corollary of Theorem 34 and [3, Theorem 1.11] on $\Delta(R_v)$ (defined as in Subsection 2.3.5) is due to Flach–Siebel.

Corollary 35 (Flach–Siebel [14, Lemma 17].). *If* δ_v *is the index of* C_K *and* δ'_v *is the period of* C_K, *then*

$$\Delta(R_v) = \frac{c_v}{\delta_v \cdot \delta'_v} \prod_{i\in I} r_i.$$

In terms of the Arakelov intersection pairing on R_v *(see [8, p. 390] or [20, (3.7)]), one has*

$$\Delta_{ar}(R_v) = \Delta(R_v)(\log q_v)^{\#G_v - 1}.$$

Lemma 36. *Let* $L_v(J,t)$ *be the local L-factor of* J *as in (27). The zeta function*

$$Z(C_v,t) = \frac{P_1(C_v,t)}{P_0(C_v,t) \cdot P_2(C_v,t)}$$

of C_v *satisfies*

$$Z(C_v,t) = \frac{L_v(J,t)}{(1-t) \cdot \prod_{i\in I}(1-(qt)^{r_i})}.$$

Proof. As C_K is smooth projective and geometrically connected, C_v is geometrically connected. So $P_0(C_v,t) = (1-t)$.

We next show that $P_1(C_v,t)$ equals $L_v(J,t)$ of (27). For any prime ℓ coprime to q, the Kummer sequence and the perfect pairing (Poincaré duality)

$$H^1_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell) \times H^1_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell(1)) \xrightarrow{\cup} H^2_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell$$

provide isomorphisms of $\mathrm{Gal}(\bar{K}/K)$-representations

$$H^1_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} T_\ell J_K \otimes \mathbb{Q}_\ell, \quad H^1_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{Hom}(H^1_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell(1)), \mathbb{Q}_\ell).$$

So we obtain an isomorphism

$$H^1_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{Hom}(T_\ell J_K \otimes \mathbb{Q}_\ell, \mathbb{Q}_\ell) \tag{46}$$

of $\mathrm{Gal}(\bar{K}/K)$ representations. Since $H^1_{et}(C_v \times \bar{\mathbb{F}}_q, \mathbb{Q}_\ell)$ isomorphic to the subspace of $H^1_{et}(C_K \times \bar{K}, \mathbb{Q}_\ell)$ of invariants under the inertia subgroup [1, Lemma 1.2], it follows from (46) and (27) that $P_1(C_v,t)$ is $L_v(J,t)$.

Finally, one has the elementary identity [4] [18, Proposition 3.3]; see also [30, p.484]

$$P_2(C_v,t) = \prod_{i \in I} (1-(qt)^{r_i}).$$

This completes the proof. □

6.3. Relating Ш(J/F) and Br(X)

For any arithmetic surface $X \to S$, let δ be the index of X_0 over F and α be the order of the (finite) cokernel of the natural map $\mathrm{Pic}^0(X_0) \hookrightarrow J(F)$. For any finite place v of S, we put δ_v and δ'_v for the (local) index and period of $X \times F_v$ over the local field F_v. The following result is due to Geisser [17, Theorem 1.1]; there is also a recent proof by Flach-Siebel [14].

Theorem 37. *Assume that $Br(X)$ is finite. The following equality holds (up to powers of 2):*

$$[Br(X)]\alpha^2\delta^2 = [Ш(J/F)] \prod_{v \in S} \delta'_v \delta_v. \tag{47}$$

7. The proof of the main theorem in the general case

7.1. Preliminary steps

We are in a position to prove Theorem 2 in the general case. Let $\Sigma = \{v \in S \mid X_v \text{ is not smooth}\}$; let G_v denote the set of irreducible components of X_v.

[4] This proposition, first stated on p. 176 of [18], has a typo which is corrected in its restatement on p. 193.

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Lemma 38. *If*

$$Q_2(s) = \prod_{v\in\Sigma} \frac{(1-q_v^{1-s})}{\prod_{i\in G_v}(1-q_v^{r_i(1-s)})},$$

then

$$Q_2^*(1) = \frac{1}{\prod_{v\in\Sigma}((\log q_v)^{\#G_v-1} \cdot \prod_{i\in G_v} r_i)} = \prod_{v\in\Sigma} \frac{c_v}{\Delta_{ar}(R_v)\ .\delta_v \cdot \delta'_v} = P_{J,fin} \cdot \prod_{v\in\Sigma} \frac{1}{\Delta_{ar}(R_v)\ .\delta_v \cdot \delta'_v}. \tag{48}$$

Proof. The first equality is clear; the second uses Corollary 35; the third follows from (26) using $c_v = 1$ for $v \notin \Sigma$. □

Proposition 39. *One has*

$$\chi(B(1,1)) = \frac{P_{J,fin}}{P_J} = \frac{1}{P_{J,\infty}} = \frac{|d_F|^{g/2}}{P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}.$$

Proof. One has the isomorphism of integral structures (its Euler characteristic is 1)

$$H_B^1(J_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\sim} H_B^1(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}}.$$

As the notion of Néron model is local on the base [4, Proposition 4, page 13], Theorem 33 shows that the kernel and cokernel of $\mathrm{Lie}(\phi): H^1(X,\mathcal{O}) \to \mathrm{Lie}(\mathcal{J})$ are torsion $\mathcal{O}_F$-modules of the same length and the kernel is exactly the torsion of $H^1(X,\mathcal{O}_X)$. This implies that the Euler characteristic of

$$H^1(X,\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\mathrm{Lie}(\phi)} \mathrm{Lie}(\mathcal{J}) \otimes_{\mathbb{Z}} \mathbb{C}$$

is 1: $\chi(\mathrm{Lie}(\phi)) = 1$. Here the integral structures are $(H^1(X,\mathcal{O}),[H^1(X,\mathcal{O})_{tor}])$ and $(\mathrm{Lie}(\mathcal{J}),1)$. We obtain that the Euler characteristic of

$$H_B^1(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\gamma_X} H^1(X,\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C}$$

is equal to that of

$$\gamma_{\mathcal{J}}: H_B^1(J_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\sim} H_B^1(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\gamma_X} H^1(X,\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\mathrm{Lie}(\phi)} \mathrm{Lie}(\mathcal{J}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Proposition 25 and Theorem 23 show

$$\chi(\gamma_X) = \chi(\gamma_{\mathcal{J}}) = \frac{P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}{\sqrt{|d_F|}^g}. \tag{49}$$

Our convention for $B(1,1)$ is that $\mathrm{Ker}(\gamma_M)$ is in even degree, say, degree 0. Thus,

$$\underbrace{H_B^1(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}}}_{\text{degree 1}} \to \underbrace{H^1(X,\mathcal{O}) \otimes \mathbb{C}}_{\text{degree 2}}, \qquad det(B(1,1)) = \frac{\sqrt{|d_F|}^g}{P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)} = \chi(B(1,1)).$$

□

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7.2. Global index

Since X is regular, the natural map $\mathrm{Pic}(X)\to\mathrm{Pic}(X_0)$ is surjective: a natural section (as sets) is provided by sending a divisor on X_0 to its Zariski closure in X. So the index δ of X_0 over F is the order of the cokernel of the composite map $\mathrm{Pic}(X)\to\mathrm{Pic}(X_0)\xrightarrow{d}\mathbb{Z}$.

7.2.1. Calculation of $\chi(C(1))$. The torsion in $C(1)$ of (22) satisfies

$$\chi(C(1)_{tor})=\frac{w\cdot[Br(X)]}{[\mathrm{Pic}(X)_{tor}]\cdot[\mathrm{Pic}(X)_{tor}]}. \tag{50}$$

We can rewrite $C(1)_{\mathbb{C}}$ as

$$\begin{aligned}0\to \underset{degree\ 0}{\mathcal{O}_F^\times\otimes\mathbb{C}}\to\underset{degree\ 1}{\mathrm{Coker}(\gamma_{M_0})}\to\underset{degree\ 2}{\mathrm{Hom}(\mathrm{Pic}(X),\mathbb{C})}\to\underset{degree\ 3}{\mathrm{Pic}(X)\otimes\mathbb{C}}\to\underset{degree\ 4}{\mathrm{Ker}(\gamma_{M_2})}\\ \to\underset{degree\ 5}{\mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{C})}\to 0.\end{aligned} \tag{51}$$

Using the exact sequence

$$0\to\mathrm{Pic}^0(X)\to\mathrm{Pic}(X)\xrightarrow{d}\mathbb{Z}\to\frac{\mathbb{Z}}{\delta\mathbb{Z}}\to 0, \tag{52}$$

the sequence $C(1)$ breaks up into

- the sequence (14) with determinant R
$$0\to\underset{degree\ 0}{\mathcal{O}_F^\times\otimes\mathbb{C}}\to\underset{degree\ 1}{\mathrm{Coker}(\gamma_{M_0})}\to\underset{degree\ 2}{\mathrm{Hom}(\mathbb{Z},\mathbb{C})}\to 0;$$
- the sequence with determinant δ
$$0\to\mathrm{Hom}(\mathbb{Z},\mathbb{C})\to\mathrm{Hom}(\mathrm{Pic}(X),\mathbb{C})\to\mathrm{Hom}(\mathrm{Pic}^0(X),\mathbb{C})\to 0;$$
- and the sequence with determinant $det(\psi)$
$$h:\underset{degree\ 2}{\mathrm{Hom}(\mathrm{Pic}^0(X),\mathbb{C})}\to\underset{degree\ 3}{\mathrm{Pic}^0(X)\otimes\mathbb{C}}$$
where h is the Arakelov intersection pairing [27, Conjecture 2.3.7] as in (11);
- the sequence with determinant δ
$$0\to\mathrm{Pic}^0(X)\otimes\mathbb{C}\to\mathrm{Pic}(X)\otimes\mathbb{C}\xrightarrow{d}\mathbb{C}\to 0;$$
- the sequence (12) with determinant R
$$0\to\mathbb{C}\to\underset{degree\ 4}{\mathrm{Ker}(\gamma_{M_2})}\to\underset{degree\ 5}{\mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{Z})\otimes\mathbb{C}}\to 0.$$

It follows from (52) that, in (51), the image of the lattice in degree 3 has index δ in the lattice in degree 4; dually, the image of the lattice in degree 1 has index δ in the lattice in degree 3. Thus,

$$det(C(1))=\frac{R\cdot R}{\delta\cdot\delta\cdot det(h)}.$$

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So $\chi(C(1))$ is given as

$$= \frac{det(C(1))}{\chi(C(1)_{tor})} = \frac{R^2}{\delta^2 \cdot \Delta_{ar}(\mathrm{Pic}^0(X)) \cdot [\mathrm{Pic}^0(X)_{tor}]^2} \cdot \frac{[\mathrm{Pic}^0(X)_{tor}]^2}{w \cdot [\mathrm{Br}(X)]}$$
$$= \frac{R^2}{\delta^2 \cdot \Delta_{ar}(\mathrm{Pic}^0(X)) \cdot w \cdot [\mathrm{Br}(X)]}, \tag{53}$$

where, as in Subsection 2.3.5,

$$\Delta_{ar}(\mathrm{Pic}^0(X)) = \frac{det(h)}{[\mathrm{Pic}^0(X)_{tor}]^2}. \tag{54}$$

Our next task is to calculate $\Delta_{ar}(\mathrm{Pic}^0(X))$ and relate it to the Néron–Tate pairing (56) on $J(F)$.

7.2.2. Calculation of $\Delta_{ar}(\mathbf{Pic}^0(X))$. This is based on localisation sequences on X and S.

Let $U = S - \Sigma$. So the map $X_U = \pi^{-1}(U) \to U$ is smooth. For any finite $\Sigma' \subset S$ containing Σ, we put $U' = S - \Sigma'$ and $X_{U'} = X - \pi^{-1}(U')$.

Lemma 40.

(i) *The maps*

$$Pic(S) \to Pic(X), \qquad Pic(U') \to Pic(X_{U'})$$

are injective.

(ii) *There is an exact sequence*

$$0 \to \oplus_{v\in\Sigma} \frac{\mathbb{Z}^{G_v}}{\mathbb{Z}} \to \frac{Pic^0(X)}{Pic(S)} \to Pic^0(X_0) \to 0. \tag{55}$$

Proof.

(i) From the Leray spectral sequence for $\pi : X \to S$ and the étale sheaf $\mathbb{G}_m$ on X, we get the exact sequence

$$0 \to H^1(S,\pi_*\mathbb{G}_m) \to H^1(X,\mathbb{G}_m) \to H^0(S,R^1\pi_*\mathbb{G}_m) \to \mathrm{Br}(S).$$

Now use that $\pi_*\mathbb{G}_m$ is the sheaf $\mathbb{G}_m$ on S. This provides the injectivity of the first map. A similar argument provides the injectivity of the second.

(ii) We can compare the localisation sequences for $X_{U'} \subset X$ and $U' \subset S$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Gamma(X,\mathbb{G}_m) & \longrightarrow & \Gamma(U',\mathbb{G}_m) & \longrightarrow & \bigoplus_{v\in Z'} \mathbb{Z} & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & \mathrm{Pic}(U') & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X,\mathbb{G}_m) & \longrightarrow & \Gamma(X_{U'},\mathbb{G}_m) & \longrightarrow & \bigoplus_{v\in Z'} \mathbb{Z}^{G_v} & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X_{U'}) & \longrightarrow & 0
\end{array}$$

Note that S and X are regular and for any regular scheme Y, one has an isomorphism $\mathrm{Cl}(Y) = \mathrm{Pic}(Y)$ between the class group and the Picard group. The natural map between

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the localisation sequences is injective on all terms and, by assumption, is an isomorphism on the first and second terms. This provides the exact sequence

$$0 \to \oplus_{v\in\Sigma'} \frac{\mathbb{Z}^{G_v}}{\mathbb{Z}} \to \frac{\mathrm{Pic}(X)}{\mathrm{Pic}(S)} \to \frac{\mathrm{Pic}(X_{U'})}{\mathrm{Pic}(U')} \to 0.$$

In particular, we get this sequence for Σ and U. Using (52), we obtain the exact sequence

$$0 \to \oplus_{v\in\Sigma} \frac{\mathbb{Z}^{G_v}}{\mathbb{Z}} \to \frac{\mathrm{Pic}^0(X)}{\mathrm{Pic}(S)} \to \frac{\mathrm{Pic}^0(X_U)}{\mathrm{Pic}(U)} \to 0.$$

By assumption, X_v is geometrically irreducible for any $v \notin \Sigma$. So for any $U' = S - \Sigma'$ with $U' \subset U$, the induced maps

$$\frac{\mathrm{Pic}^0(X_U)}{\mathrm{Pic}(U)} \to \frac{\mathrm{Pic}^0(X_{U'})}{\mathrm{Pic}(U')}$$

are isomorphisms. Taking the limit over Σ' gives us an exact sequence

$$0 \to \oplus_{v\in\Sigma} \frac{\mathbb{Z}^{G_v}}{\mathbb{Z}} \to \frac{\mathrm{Pic}^0(X)}{\mathrm{Pic}(S)} \to \mathrm{Pic}^0(X_0) \to 0.$$

This proves the lemma. □

For any $v \in S$, recall $\Delta_{ar}(R_v)$ from Corollary 35, where

$$R_v = \frac{\mathbb{Z}^{G_v}}{\mathbb{Z}}$$

and, as before, G_v is the set of irreducible components of X_v. Let us define (see Subsection 2.3.5)

$$\Delta_{NT}(J(F)) = \frac{\Theta_{NT}(J)}{[J(F)_{tor}] \cdot [J(F)_{tor}]} \tag{56}$$

using the Néron–Tate pairing (35) on $J(F)$, analogous to $\Delta_{ar}(\mathrm{Pic}^0(X))$ from (54).

Proposition 41. *If α is the order of the cokernel of the natural map $Pic^0(X_0) \hookrightarrow J(F)$, then one has*

$$\Delta_{ar}(Pic^0(X)) = \pm\frac{\alpha^2}{h^2} \cdot \Delta_{NT}(J(F)) \cdot \prod_{v\in\Sigma} \Delta_{ar}(R_v). \tag{57}$$

Proof. By [21, Proposition 3.3], the Arakelov intersection pairing h is negative-definite on $\mathrm{Pic}^0(X) \otimes \mathbb{Q}$ and there exists a map $\kappa : \mathrm{Pic}^0(X_0) \to \mathrm{Pic}^0(X)$ such that

$$(y,y') \mapsto h(\kappa(y),\kappa(y'))$$

gives the intersection pairing on $\mathrm{Pic}^0(X_0)$ which, by Faltings–Hriljac [8], [21, Theorem 3.1], is the negative of the Néron–Tate pairing (35) on $J(F)$. Note that this means

$$\Delta_{NT}(J(F)) = \pm\Delta_{ar}(J(F)). \tag{58}$$

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So the sequence (55) splits over $\mathbb{Q}$ as an orthogonal direct sum with respect to the Arakelov intersection pairing

$$(\mathrm{Pic}^0(X_0)\otimes\mathbb{Q}) \oplus (\bigoplus_{v\in\Sigma}\frac{\mathbb{Q}^{G_v}}{\mathbb{Q}}) \cong \mathrm{Pic}^0(X)\otimes\mathbb{Q}. \tag{59}$$

The map κ is defined as follows: given any element y of $\mathrm{Pic}^0(X_0)$, consider its Zariski closure $\bar{y}$ in X. As the intersection pairing is negative-definite [29, §9.1, Theorem 1.23] on R_v, the linear mapping $R_v \to \mathbb{Z}$ defined by $z \mapsto z.\bar{y}$ is represented by a unique element $\kappa_v(y) \in R_v\otimes\mathbb{Q}$. Clearly, the element

$$\kappa(y) = \bar{y} - \sum_{v\in\Sigma}\kappa_v(y)$$

is orthogonal to $\oplus_{v\in\Sigma}R_v \subset \mathrm{Pic}^0(X)$; so the assignment $y\mapsto\kappa(y)$ provides (59).

Now (7) shows

$$\Delta_{ar}(\frac{\mathrm{Pic}^0(X)}{\mathrm{Pic}(S)}) = \Delta_{ar}(\mathrm{Pic}^0(X_0)) \cdot \prod_{v\in\Sigma}\Delta_{ar}(R_v).$$

Using $h = [\mathrm{Pic}(S)]$, this becomes

$$\Delta_{ar}(\mathrm{Pic}^0(X)) \cdot h^2 = \Delta_{ar}(\frac{\mathrm{Pic}^0(X)}{\mathrm{Pic}(S)}) = \Delta_{ar}(\mathrm{Pic}^0(X_0)) \cdot \prod_{v\in\Sigma}\Delta_{ar}(R_v). \tag{60}$$

As $\mathrm{Pic}^0(X_0)\hookrightarrow J(F)$ is a subgroup of index α, we see $\Delta_{ar}(\mathrm{Pic}^0(X_0)) = \alpha^2\Delta_{ar}(J(F)) = \pm\alpha^2\cdot\Delta_{NT}(J(F))$. □

7.3. The zeta function of X

Our first step is to rewrite the zeta function $\zeta(X,s)$

$$\zeta(X,s) = \prod_{v\in S}\zeta(X_v,q_v^{-s}), \qquad \zeta(X_v,t) = \frac{P_1(X_v,t)}{P_0(X_v,t)P_2(X_v,t)}.$$

Proposition 42. *One has*

$$\zeta(X,s) = \frac{\zeta(S,s)\cdot\zeta(S,s-1)}{L(J,s)}\cdot Q_2(s), \qquad \zeta^*(X,1) = \frac{\zeta^*(S,1)\cdot\zeta^*(S,0)}{L^*(J,1)}\cdot Q_2^*(1). \tag{61}$$

Proof. By Lemma 36, we see that

(i) the factors P_0 combine to give $\zeta(S,s)$;
(ii) the factors P_1 combine to give $L(J,s)$;
(iii) $P_2(X_v,t)$ is the expected factor $(1-q_vt)$ for $v\notin\Sigma$; for $v\in\Sigma$ and is given in Lemma 36. So

$$\prod_{v\in S}\frac{1}{P_2(X_v,q_v^{-s})} = Q_2(s)\cdot\prod_{v\in S}\frac{1}{1-q_v^{(1-s)}} = Q_2(s)\cdot\zeta(S,s-1).$$

□

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7.4. Completion of the proof of Theorem 2

We prove the main result of this article.

Proof (of Theorem 2). We can restate Conjecture 20 for J/F using (56) as

$$\frac{1}{L^*(J,1)} = \frac{|d_F|^{g/2}}{P_{J,fin} \cdot P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)} \cdot \frac{1}{\Delta_{NT}(J(F)) \cdot [\text{Ш}(J/F)]}. \tag{62}$$

From Proposition 42, we see that Conjecture 20 for J/F is equivalent to the equality

$$\begin{aligned}
\zeta^*(X,1) &= \zeta^*(S,1) \cdot \zeta^*(S,0) \cdot \frac{1}{L^*(J,1)} \cdot Q_2^*(1) \\
&\overset{(15)}{=} \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d_F|}} \cdot \zeta^*(S,0) \cdot \frac{1}{L^*(J,1)} \cdot Q_2^*(1) \\
&\overset{(13)}{=} \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d_F|}} \cdot \frac{hR}{w} \cdot \frac{1}{L^*(J,1)} \cdot Q_2^*(1) \\
&\overset{(62)}{=} \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d_F|}} \cdot \frac{hR}{w} \cdot \frac{|d_F|^{g/2}}{P_{J,fin} \cdot P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)} \\
&\quad \cdot \frac{1}{\Delta_{ar}(J(F)) \cdot [\text{Ш}(J/F)]} \cdot Q_2^*(1) \\
&\overset{(48)}{=} \frac{2^{r_1}(2\pi)^{r_2}h^2R^2 \cdot |d_F|^{g/2} \cdot P_{J,fin}}{w^2\sqrt{|d_F|} \cdot P_{J,fin} \cdot P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta) \cdot \Delta_{ar}(J(F)) \cdot [\text{Ш}(J/F)]} \\
&\quad \cdot \prod_{v\in\Sigma}\left(\frac{1}{\Delta_{ar}(R_v)\cdot\delta_v\cdot\delta_v'}\right) \\
&= \frac{2^{r_1}(2\pi)^{r_2}}{w\sqrt{|d_F|}} \cdot \frac{h^2\cdot R^2}{w} \cdot \frac{|d_F|^{g/2}}{P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)} \cdot \frac{1}{\Delta_{ar}(J(F)) \cdot [\text{Ш}(J/F)]} \\
&\quad \cdot \prod_{v\in\Sigma}\left(\frac{1}{\Delta_{ar}(R_v)\cdot\delta_v\cdot\delta_v'}\right).
\end{aligned} \tag{63}$$

Conjecture 1 states that

$$\zeta^*(X,1) = \chi(X,1) = \frac{\chi_{A,C}(X,1)}{\chi_B(X,1)} = \chi_A(X,1) \cdot \chi_{A'}(X,1) \cdot \chi(C(1)) \cdot \frac{\chi(B(1,1))}{\chi(B(0,1))\chi(B(2,1))}.$$

Using (24), (25), (19) and Proposition 39,

$$\chi(B(0,1)) = \frac{\sqrt{|d_F|}}{(2\pi)^{r_2}}, \quad \chi(B(1,1)) = \frac{|d_F|^{g/2}}{P_{J,\infty}(\eta) \cdot \mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)},$$

$$\chi(B(2,1)) = 1 = \chi_A(X,1), \quad \chi_{A'}(X,1) = \frac{1}{w},$$

we have

$$\begin{aligned}\chi(X,1) &= \frac{\chi_{A,C}(X,1)}{\chi_B(X,1)}\\ &= \chi_A(X,1)\cdot\chi_{A'}(X,1)\cdot\chi(C(1))\cdot\frac{\chi(B(1,1))}{\chi(B(0,1))\chi(B(2,1))}\\ &= 1.\ \frac{1}{w}\cdot\chi(C(1))\cdot\frac{(2\pi)^{r_2}}{\sqrt{|d_F|}}\cdot\frac{|d_F|^{g/2}}{P_{J,\infty}(\eta)\cdot\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}\\ &= \chi(C(1))\cdot\frac{(2\pi)^{r_2}}{w\sqrt{|d_F|}}\cdot\frac{|d_F|^{g/2}}{P_{J,\infty}(\eta)\cdot\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}\\ &\overset{(53)}{=} \frac{R^2}{\delta^2\cdot\Delta_{ar}(\mathrm{Pic}^0(X))\cdot w\cdot[\mathrm{Br}(X)]}\cdot\frac{(2\pi)^{r_2}}{w\sqrt{|d_F|}}\cdot\frac{|d_F|^{g/2}}{P_{J,\infty}(\eta)\cdot\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}.\end{aligned}\tag{64}$$

To prove Theorem 2, we need to show that $\chi(X,1)$ is equal to

$$\frac{2^{r_1}(2\pi)^{r_2}}{w\sqrt{|d_F|}}\frac{h^2\cdot R^2}{w}\cdot\frac{|d_F|^{g/2}}{P_{J,\infty}(\eta)\cdot\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{a}_\eta)}\cdot\frac{1}{\Delta_{ar}(J(F))\cdot[\text{Ш}(J/F)]}\cdot\prod_{v\in\Sigma}\frac{1}{\Delta_{ar}(R_v)\cdot\delta_v\cdot\delta'_v}.$$

Thus, we see that Theorem 2 follows if we prove that (neglecting powers of 2)

$$\begin{aligned}\chi(C(1)) &\overset{?}{=} \frac{h^2\cdot R^2}{w}\cdot\frac{1}{\Delta_{ar}(J(F))\cdot[\text{Ш}(J/F)]}\cdot\prod_{v\in\Sigma}\frac{1}{\Delta_{ar}(R_v)\cdot\delta_v\cdot\delta'_v}\\ &\overset{?}{=} \frac{R^2}{w}\cdot\frac{h^2}{\Delta_{ar}(J(F))}\cdot\prod_{v\in\Sigma}\frac{1}{\Delta_{ar}(R_v)}\cdot\frac{1}{[\text{Ш}(J/F)]}\cdot\prod_{v\in\Sigma}\frac{1}{\delta_v\cdot\delta'_v}.\end{aligned}\tag{65}$$

By (53), we have

$$\chi(C(1)) = \frac{R^2}{\delta^2\cdot\Delta_{ar}(\mathrm{Pic}^0(X))\cdot w\cdot[\mathrm{Br}(X)]}.$$

Thus, to prove Theorem 2, we only need to check if

$$\frac{1}{\delta^2\cdot\Delta_{ar}(\mathrm{Pic}^0(X))\cdot[\mathrm{Br}(X)]}\overset{?}{=}\frac{h^2}{\Delta_{ar}(J(F))}\cdot\prod_{v\in\Sigma}\frac{1}{\Delta_{ar}(R_v)}\cdot\frac{1}{[\text{Ш}(J/F)]}\cdot\prod_{v\in\Sigma}\frac{1}{\delta_v\cdot\delta'_v}.$$

We can verify this readily using Proposition 41, which says

$$\Delta_{ar}(\mathrm{Pic}^0(X)) = \frac{\alpha^2\cdot\Delta_{ar}(J(F))\cdot\prod_{v\in\Sigma}\Delta_{ar}(R_v)}{h^2},\tag{66}$$

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and Theorem 37, which says (note $\delta_v = 1 = \delta'_v$ if $v \notin \Sigma$)

$$[\mathrm{Br}(X)]\alpha^2\delta^2 = [\text{Ш}(J/F)]\prod_{v\in S}\delta'_v\delta_v. \tag{67}$$

This completes the proof of Theorem 2 in the general case. □

Remark 43. Consider the case $g=0$. As J is trivial, Theorem 2 says Conjecture 1 is true. Let us see directly that this is true (up to powers of 2): (63) shows that

$$\zeta^*(X,1) = \frac{2^{r_1}(2\pi)^{r_2}}{w\sqrt{|d_F|}} \cdot \frac{h^2 \cdot R^2}{w} \cdot \prod_{v\in\Sigma}\frac{1}{\Delta_{ar}(R_v) \cdot \delta_v \cdot \delta'_v},$$

but (64) and (65) show

$$\begin{aligned}\chi(X,1) = \frac{R^2}{\delta^2 \cdot \Delta_{ar}(\mathrm{Pic}^0(X)) \cdot w \cdot [\mathrm{Br}(X)]} \cdot \frac{(2\pi)^{r_2}}{w\sqrt{|d_F|}} = \frac{h^2}{\alpha^2 \cdot \prod_{v\in\Sigma}\Delta_{ar}(R_v)}\\ \cdot \frac{R^2}{\delta^2 \cdot w \cdot [\mathrm{Br}(X)]} \cdot \frac{(2\pi)^{r_2}}{w\sqrt{|d_F|}}.\end{aligned}$$

As α, δ_v, δ and δ'_v divide 2 and (66) shows that $[\mathrm{Br}(X)]$ is a power of 2, it follows that the equality $\zeta^*(X,1) = \chi(X,1)$ is valid up to a finite power of 2.

7.5. Comparison with the Artin–Tate conjecture

Theorem 2 is the analogue for arithmetic surfaces of Conjecture (d) [36, p. 427] in the spirit of [26]. More precisely, one has the following.

Theorem 44. *Conjecture 1 for X is equivalent to the following identity (up to a finite power of 2):*

$$\frac{\zeta^*(S,0)\cdot\zeta^*(S,1)}{\zeta^*(X,1)} = \chi(\gamma_X)\cdot[Br(X)]\cdot\Delta_{ar}\left(\frac{Pic^0(X)}{Pic(S)}\right)\cdot\delta^2. \tag{68}$$

Here δ is the global index of X as in Subsection 7.2. The term $\chi(\gamma_X)$ is an *Archimedean period*: it is the Euler characteristic (as in Subsection 2.3) of

$$\gamma_X : H^1_B(X_{\mathbb{C}},\mathbb{Z}(1))^+_{\mathbb{C}} \xrightarrow{\sim} H^1(X,\mathcal{O})\otimes_{\mathbb{Z}}\mathbb{C}$$

with respect to the integral structures provided by $H^1_B(X_{\mathbb{C}},\mathbb{Z}(1))^+$ and the abelian group underlying $H^1(X,\mathcal{O})$.

Proof. We can start with the left-hand side and rewrite it as follows:

$$\begin{aligned}
&\frac{\zeta^*(S,0)\cdot\zeta^*(S,1)}{\zeta^*(X,1)}\\
&\quad=\frac{\zeta^*(S,0)\cdot\zeta^*(S,1)}{\chi(X,1)}=\frac{\zeta^*(S,0)\cdot\zeta^*(S,1)\cdot\chi_B(X,1)}{\chi_A(X,1)\cdot\chi_{A'}(X,1)\cdot\chi(C(1))}\\
&\quad=\frac{\zeta^*(S,0)\cdot\zeta^*(S,1)\cdot\chi(\gamma_X)}{1\cdot\frac{1}{w}\cdot\chi(C(1))\cdot\frac{(2\pi)^{r_2}}{\sqrt{|d_F|}}}\quad\text{by (24),(25),(19)}\\
&\quad=\frac{\chi(\gamma_X)\cdot\zeta^*(S,0)\cdot\zeta^*(S,1)\cdot w^2\cdot\delta^2\cdot\Delta_{ar}(\mathrm{Pic}^0(X))\cdot[\mathrm{Br}(X)]\cdot\sqrt{|d_F|}}{R^2\cdot(2\pi)^{r_2}}\quad\text{by (53)}\\
&\quad=\chi(\gamma_X)\cdot\delta^2\cdot h^2\cdot\Delta_{ar}(\mathrm{Pic}^0(X))\cdot[\mathrm{Br}(X)]\cdot 2^{r_1}\quad\text{by (15),(13)}\\
&\quad=\chi(\gamma_X)\cdot\delta^2\cdot\Delta_{ar}\left(\frac{\mathrm{Pic}^0(X)}{\mathrm{Pic}(S)}\right)\cdot[\mathrm{Br}(X)]\quad\text{by (60).}
\end{aligned}$$

This proves the stated equivalence. □

Remark 45 (T. Suzuki). The identity (67) is an analogue for arithmetic surfaces of the Artin–Tate conjecture [36, Conjecture C]. Let us show this by rewriting the Artin–Tate conjecture for function fields [28, Conjecture 2]; we shall use the notation of [28] from now on.

So now let S be a smooth proper curve over a finite field F_q, and let X be a smooth proper surface with a flat proper morphism $\pi : X \to S$ whose generic fibre is a smooth geometrically connected curve X_0 over Spec $\mathbb{F}_q(S)$. By combining (2), (12) and (13) of [28], we obtain

$$\frac{\zeta(S,1-s)\cdot\zeta(S,s)}{\zeta(X,s)}=\frac{P_2(X,q^{-s})}{P_1(B,q^{-s})P_1(B,q^{1-s})(1-q^{1-s})^2}.$$

Hence, the Artin–Tate conjecture [28, Conjecture 2] is equivalent to the identity

$$\frac{\zeta^*(S,0)\cdot\zeta^*(S,1)}{\zeta^*(X,1}=\frac{P_2^*(X,\frac{1}{q})\cdot q^{\dim\ B}}{[B(\mathbb{F}_q)]^2\cdot(\log q)^2}=\frac{[\mathrm{Br}(X)]\cdot\Delta_{ar}(NS(X))\cdot q^{\chi(S,R^1\pi_*\mathcal{O}_X)}}{[B(\mathbb{F}_q)]^2\cdot(\log q)^2}$$

by (14), (3) and (4) of [28]. On the other hand, by [28, Proposition 13 and Corollary 5 (ii)], we have

$$\frac{\Delta_{ar}(NS(X))}{(\log q)^2}=\Delta_{ar}\left(\frac{NS(X)_0}{\pi^*NS(S)}\right)\cdot\delta^2=[B(\mathbb{F}_q)]^2\cdot\Delta_{ar}\left(\frac{\mathrm{Pic}(X)_0}{\pi^*\mathrm{Pic}(S)}\right)\cdot\delta^2.$$

Thus, the Artin–Tate conjecture for X is equivalent to the identity

$$\frac{\zeta^*(S,0)\cdot\zeta^*(S,1)}{\zeta^*(X,1)}=q^{\chi(S,R^1\pi_*\mathcal{O}_X)}\cdot[\mathrm{Br}(X)]\cdot\delta^2\cdot\Delta_{ar}\left(\frac{\mathrm{Pic}(X)_0}{\pi^*\mathrm{Pic}(S)}\right).$$

The factor $q^{\chi(S,R^1\pi_*\mathcal{O}_X)}$ plays the role of $\chi(\gamma_X)$ in (67).

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