Notes on Decimal Coinage and Approximation.

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Just over sixty years ago—in December 1841—a Commission on Weights and Measures made the following proposals towards the establishment of a decimal coinage in this country: (1) the sovereign to be the unit; (2) a coin worth two shillings to be introduced under a distinct name; (3) a coin equal to the hundredth part of a pound to be established; (4) the farthing to be considered as the thousandth part of a pound; (5) other coins bearing a simple relation to these (including the shilling and sixpence) to be circulated.

One of these recommendations was given effect to in 1849 by the issue of a coin bearing the inscription "one florin—one tenth of a pound."

In 1853 another Commission reported strongly in favour of the same coinage, the terms "cent" and "mil" being applied to the hundredth and thousandth part of a pound respectively. Although the adoption of this system has since been urged repeatedly by Chambers of Commerce and other public bodies, nothing further has been done towards carrying the other proposals into effect. In fact, the very reason for the introduction of the florin seems to have been forgotten, for on those issued since 1893 the inscription has been altered to "one florin—two shillings," all reference to its being the *decimal* part of a pound being carefully avoided.*

It need scarcely be pointed out that the adoption of a decimal system of coinage would not only simplify the working of all questions involving money, but it might also be expected to lead to the further simplification in Arithmetic which would arise from the decimalising of our complete system of weights and measures.

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* The florin may often be used with advantage in calculations, *e.g.*, 63 articles at 16/- each = 63 @ 8 florins each = 504 florins = £50 „ 4 florins = £50 „ 8/-.

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It seems to have been expected by many who advocated this change in the coinage that one of the most important results of it would be the greater facility in the use of "decimals" * which would arise. Thus De Morgan, writing in 1841, says:—"Much as a decimal coinage is to be wished for in almost every respect, we doubt if any advantage accruing from it would equal that of its indirect consequence, the forcing of the attention of people in general to the subject of decimal fractions." This apparently has happened in the United States. There a decimal coinage was introduced as early as 1786, and now decimal fractions are so widely used that it is stated that the term vulgar or common fractions is quite a misnomer.

To advocate the completion of the decimalisation of our coinage is not the object of the present paper—though that, perhaps, is a subject well worthy of the consideration of a Mathematical Society. It is rather to offer some suggestions which may lead to the more general use of "decimals" in the treatment of questions involving money, and hence (from the very fact that we have *not* a decimal coinage) to the use of approximation and contracted methods generally.

That the present state of knowledge of decimals and of approximations leaves *very* much to be desired must be a matter of common knowledge to all who have occasion to deal with beginners in Physical Laboratories, where the metric (decimal) system is generally used. If those who have not the benefit of this experience require proof of this statement, they will find it in the reports of examiners in all the leading examinations of which Arithmetic forms a part. Thus: In the Report on the Cambridge Local Examination of last year it is stated that the chief defect among the juniors was the unnecessary reduction of decimals to vulgar fractions, and the employment of the latter instead of the former in the question on decimal coinage! Among the seniors, the examiners report that there was a *grievous waste of labour* due to not using decimals. Again, in Sir Henry Craik's report on last year's Leaving Certificate Examinations, we read: "The fact that many candidates appear to consider

* The term "decimals" is used throughout to include all numbers expressed decimally whether they be integral, fractional, or "mixed": thus 34, ·34, and 3·4 are all in this sense decimals.

that the proper way to add and subtract decimal fractions consists in first expressing them as vulgar fractions shows a want of appreciation of the advantages of decimal notation." On the Higher Grade papers it is reported: "There was a tendency to use long methods, and few candidates had a grasp of contracted methods. Thus, when required to find an answer correct to the nearest penny, many found the answer to the millionth of a penny, or *wasted time* by long work with vulgar fractions." It is well to emphasise this *waste* of time and labour; for one of the reasons often given for not teaching approximation in classes in Arithmetic is *want* of time. I hope to show later that the early teaching of decimals will lead to an appreciable gain in the available time.

The question at once arises: how early should "decimals" be introduced? The complaint about reducing decimal fractions to vulgar fractions would be entirely obviated if the usual order of teaching were reversed, and the treatment of the four simple rules were extended to include numbers stretching on both sides of the units place before commencing the formal treatment of vulgar fractions.* The following sketch shows *one* way in which this may be done.

While the pupil is being drilled in becoming mechanically perfect in the four simple rules as applied to integers, he should also receive constant practice in so interpreting the notation he is using that the decimal nature of it is thoroughly understood, and the benefits of it are fully appreciated. Our money system may be most efficiently† used to illustrate the advantages belonging to that part of it which is decimal, and the disadvantages pertaining to that part which is

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* It is sometimes objected to the teaching of decimal fractions before vulgar fractions that this is not the historical order of development. The same argument would require us to go back a step further and operate only with fractions having unity for numerator. No one, however, nowadays proposes to substitute (say) $\frac{1}{2}+\frac{1}{26}$ for $\frac{7}{13}$ before operating with it. Of course it is not suggested that pupils should not use fractions such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, &c., till they have mastered decimal notation completely; what is proposed is that decimals should be taught without any reference to vulgar fractions, *e.g.*, ·1 is not to be defined as $\frac{1}{10}$.

† In many of the schools of Germany decimal fractions are introduced before vulgar fractions, their system of weights and measures being used in illustration, *e.g.*, 7 metres 2 decimetres = 7·2 metres.

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non-decimal. For example, £567 may be paid in any one of the following ways:—

	hundred-pound-notes,	ten-pound-notes,	sovereigns
by (1)	5 ,	6 ,	7
or (2)	5 ,	,	67
or (3)		56 ,	7
or (4)			567

It is here to be noted that, whether we use one, two, or three denominations, the figures 5, 6, 7 are not altered. Contrast this now with £5 „ 6 „ 7 in the non-decimal part of the money system. This, also, may be paid in one of four ways:—

	sovereigns,	shillings,	pence
by (1)	5 ,	6 ,	7
or (2)	5 .		79
or (3)		106 ,	7
or (4)			1279

but the figures are not the same in all the four ways, as was the case in the decimal part of the system.

The decimal part of the system having obviously advantages over the non-decimal part, the former may with profit be extended by the use of florins instead of shillings. As above, 567 florins may be considered as 56 sovereigns and 7 florins, or as 5 ten-pound-notes 6 sovereigns 7 florins, &c., &c.

In all such cases of decimal systems it is not necessary to use more than one denomination. A point is placed to the right of the figure denoting that denomination. This is called the decimal point. [It might with more propriety be called the *units* point, as it marks the position of the units figure. See footnote, p. 56.] Thus, we might write 567 florins as 56·7 sovereigns, or as 5·67 ten-pound-notes, &c.

The decimal or units point is generally omitted when the denomination mentioned is the lowest involved, *e.g.*, 567 florins is written for 567· florins. The figures left and right of the *units* figure receive (when necessary) corresponding names:

one place to left and right, tens and tenths,
two places „ „ hundreds and hundredths,
three places „ „ thousands and thousandths, &c., &c.

e.g., 300· is read three hundred,
0·03 is read three hundredths.

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The units place is here printed in heavy type as well as being indicated by the · following it.

Let us consider now the four simple rules with this extended notation.

In *addition* there is obviously no change needed : like denominations are to be added, and this may be insured by arranging the *units figures* in a vertical column. (Do not say the *decimal points.*)

Similarly with *subtraction.*

I assume that in *multiplication* the pupil has been shown that the figures of the multiplier may be used in any order, and that the most reasonable procedure is to begin with the most important figure, *i.e.*, the first or left hand figure. Thus, in 1341 × 432 we *may* arrange

```
thus     1341        or thus     1341        &c., &c.
          432                     432
         ----                    ----
        4023                     2682
         2682                  5364
       5364                     4023
       ------                  ------
       579312                  579312
```

but *ought* to use as the standard form

```
      1341
       432
      ----
    5364
     4023
      2682
    ------
    579312
```

We observe that the units figures of multiplier and multiplicand (and product) are again in a vertical column ; that the figures of the product obtained, from 4 (which is *two* places to the *left* of the units place) are placed *two* places to the *left* of the figures of the multiplicand from which they are derived ; that the figures of the product obtained from 3 (which is *one* place to the *left* of the units place) are placed *one* place to the *left* of the figures of the multiplicand from which they are derived ; that the figures of the product obtained from 2 (which is neither to the *left* nor to the *right* of the units place) are placed neither to the *left* nor to the *right* of the figures of the multiplicand from which they are derived.

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No difficulty will now be experienced with

```
   1341
    432·1
5364
  4023
    2682
     134·1
579446·1
```

All units are again in a vertical column; the figures of the product obtained from 1 (which is *one* place to the *right* of the units place) are placed *one* place to the *right* of the figures of the multiplicand from which they are derived.

Similarly we arrange 1·341 × 43·2

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  1·341
 43·2
 53·64
  4·023
   ·2682
 57·9312
```

Hence we see that in all cases of the multiplication of "decimals," whether integral, fractional, or "mixed," there is only one method, viz., that of the standard form used for whole numbers.

No "rule" in decimal fractions gives so much trouble in teaching and in practice as that of *division*. Here, also, it is advisable that the standard form adopted for integers should not require to be altered on the introduction of fractions. We may proceed as follows: Take as an example 1728 ÷ 48 and arrange thus

```
         36 = Quotient
  48)1728
     144
      288
      288
```

Explanation: The divisor 48 is contained in 172 three times. The product of this 3 with the *units* figure of the divisor being 24, the 4 is (as usual) placed under the 2 of 172. Now we saw above that, when multiplying by the *units* figure, the decimal positions of the

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figures of the product are the same as those of the multiplicand from which they are derived. *Hence the decimal position of the* 3 *must be the same as that of the* 2 *or* 4. Place it therefore in the same vertical column as the 2 and 4. Continue as usual, and all the figures of the quotient will be vertically over the corresponding figures of the dividend—units over units, tens over tens, &c., &c. Similarly in examples involving the decimal or units point, *e.g.*, 1·728 ÷ 4·8 and 1728 ÷ 0·48.

Arrange thus

```
          ·36  = Quotient              3600· = Quotient
    4·8)1·728                     0·48)1728·
        1·44                           144
        ·288                            288·
        ·288                            288·
```

The method in every case is the same: the 3 of the quotient is placed vertically over the product obtained from it and the *units* figure of the divisor; the units figures of quotient and dividend are then in the same vertical column. It is, of course, not necessary to carry the decimal point all through the working. In fact as, in the method suggested, the units figures are always in a vertical column, the position of the units needs to be indicated only once.

The methods of contraction for addition, subtraction, and multiplication call for no remark except that "much time should not be spent in ensuring accuracy in the last figure. Give and take should be permitted to save time." *

In most of the text books dealing with approximation, the calculations are carried out two places beyond the desired degree of accuracy. It will be found on trial that this involves much unnecessary labour. See examples on pages 57, 59, 60, and 61.

The case of approximation in division is much simplified if, instead of (as is usually done) fixing the number of figures to be used as divisor, we fix the number required for dividend.

e.g., Divide ·125179 by ·03216 to "one decimal place." †

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* Professor Everett—Discussion on the Teaching of Mathematics at the British Association, Glasgow, 1901.

† From the point of view of the present paper this phrase is unfortunate, as all the places in a number are equally entitled to be called decimal. The expression, however, is firmly established. It may be accepted and extended *to the left*. *E.g.*, in 516·37, the decimal places of 7, 3, 6, 1, 5, are respectively 2, 1, 0, −1, −2. [The characteristic of a logarithm of a number is minus the decimal place of the most important (or first significant) figure of the number.]

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Here the product of the 3 (in the *second* decimal place of the divisor) by the *first* decimal place of the quotient, will give the *third* decimal in ·125179; hence we need retain only ·125 for dividend, and as much of divisor as is necessary to go under these figures. In this case ·032.

Thus		
3·9 = Quotient		3·9 = Quotient
·03,210)·125179		0·032)0·125
96		0·096
29		0·029

First 32 is used as divisor, and the partial quotient is 3; the partial product 96 is placed as usual, and the position of the units figure of this (viz **0**), determines the position of the 3 of quotient. The 2 is now struck off from divisor, and the division is continued, &c., &c. On the right the work is shown in full, the units figures being indicated by heavier type.

Another example: ·125179 ÷ 1236 to 5 decimal places. Here the 1 of divisor is in decimal place − 3 (see footnote, p. 56); hence retain in dividend 5 + (− 3) = 2 decimal places and arrange

·00010 = Quotient
1,236)·125179
12
0

Expressions of the form $\frac{3·962 \times ·7189}{7439·2}$ are of frequent occurrence.

Suppose it is required to evaluate this to 5 decimal places. The decimal position of the most important figure of the divisor (viz., 7) is − 3, hence it is only necessary to retain 5 + (− 3) decimal places in the working:

3·96 | 2		
·71 | 89		·00038 Ans.
2·77		7,439·2)2·84
4		2·23
3		61
2·84		

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To apply these methods to questions involving money, we need only to write sums of money in pounds, florins, and mils, instead of pounds, shillings, pence, and farthings. Since there are 96 farthings in a florin, and (by definition), 100 mils in a florin, we may begin with a class of junior pupils by writing sums of money in pounds, florins, and farthings, and comparing the results with those obtained from the usual notation. The results of addition, of subtraction, and of division will in general closely agree, but greater differences will result from multiplication. For complete accuracy we have only to notice that since

	100 mils	= 1 florin	= 96 farthings,	
we have	25 mils	= 1 sixpence	= 24 farthings ;	
and since	25	= 1	+	24,

therefore the number of mils in any sum of money = the number of farthings + the number of sixpences in the sum. This method is general, but, as will be seen, needs to be applied only to sums less than 1/-.

e.g.,

10/6 = 5 florins 24 farthings = £·524 approximately
= £·525 exactly
8/4 = 4 florins 16 farthings = £·416 approximately
= £·416$\frac{2}{3}$ exactly (since 4d. = $\frac{2}{3}$ of 6d.)
9/ = 4$\frac{1}{2}$ florins = 4 florins 50 mils
= £·450.

In general, parts of a mil may be neglected.* To change from

* Parts of a mil must be considered when the sum of money is to be multiplied, and the product is not to be divided by a number as large as, or greater than, the multiplier; also, when the sum of money is a divisor, and more than 3 decimal places are required by the method of page 57.

In these cases the parts of a mil may be obtained as a decimal, by treating the last 2 decimal places (or their excess over 25, 50, or 75) as pence, and reducing *these* to mils for the next two places, and so on.

E.g., 6/2$\frac{1}{2}$ = £·310 approx., = £·31041 more nearly, = £·3104166 still more nearly, = £·310416666, &c., &c.

The proof is left to the reader.

the decimal form to £. s. d. read the money in florins and mils, and remember that 25 mils = 24 farthings.

e.g.,

£·718 = 7 florins 18 mils = 14 shillings and 18 farthings approx.
= 14/4½.

£·336 = 3 florins 36 mils = 6 shillings and 36 farthings approx.
= 6 shillings and 35 farthings more nearly
= 6/8¾.

With very little practice the change from one system to the other may be done mentally. The teaching should be entirely oral and no steps of the working should *ever* be written down, although it is necessary to do so here in explaining the method.

In order to give some idea of the waste of time and labour involved in the use of the ordinary non-decimal system of money, I append three examples treated in both ways.

1. Divide £36104 „ 9 „ 6 by 9416.

Quotient = £3 „ 16 „ 8¼	Quotient = £3·834 = £3 „ 16 „ 8¼
9416)£36104 „ 9 „ 6	9,4,1,6)£36104·~~4~~~~7~~~~5~~
28248	28248
7856	7856
20	7533
157129	323
9416	282
62969	41
56496	
6473	
12	
77682	
75328	
2354	
4	
9416	
9416	

2. How many articles costing £2 ‥ 17 ‥ 4½ each can be purchased for £2194 ‥ 11 ‥ 10½ ?

		765 = Ans.
£2 ‥ 17 ‥ 4½	£2194 ‥ 11 ‥ 10½	£2,·8,6~~5~~)£2194·~~5~~~~9~~~~3~~
20	20	2008
57	43891	186
12	12	172
688	526702	14
2	2	14
1377	)1053405(765 = Ans.	
	9639	
	8950	
	8262	
	6885	
	6885	

3. A bankrupt's debts amount to £9089 ‥ 1 ‥ 4 and his estate realises £5254 ‥ 12 ‥ 4. How much can he pay in the pound?

		£·578 = 11/6¾
£9089 ‥ 1 ‥ 4	£5254 ‥ 12 ‥ 4	£9,08~~9~~·~~0~~~~6~~~~6~~)£5254·~~6~~~~1~~~~6~~
20	20	4545
181781	105092	709
3	3	636
545344	)315277(11/6¾ Ans.	73
	20	
	6305540	
	545344	
	852100	
	545344	
	306756	
	12	
	3681072	
	3272064	
	409008	
	4	
	1636032	
	1636032	

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4. Find to a penny the cost of 23 tons 13 cwt. 2 qr. 14 lbs. @ £3 ,, 14 ,, 8½ per ton.

At £1 per ton this weight would cost £23 + 13/ + 6d. + 1½d. Hence by multiplication and using footnote on page 58, we have

£23·681	25
3·735	4166
71·044	
16·577	
710	
118	
. 9	
£88·458	= £88 ,, 9 ,, 2.

The testing of the accuracy of this result is left to the reader.