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BOUNDS ON PRICES FOR ASIAN OPTIONS VIA FOURIER METHODS

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Abstract

The problem of pricing arithmetic Asian options is nontrivial, and has attracted much interest over the last two decades. This paper provides a method for calculating bounds on option prices and approximations to option deltas in a market where the underlying asset follows a geometric Lévy process. The core idea is to find a highly correlated, yet more tractable proxy to the event that the option finishes in-the-money. The paper provides a means for calculating the joint characteristic function of the underlying asset and proxy processes, and relies on Fourier methods to compute prices and deltas. Numerical studies show that the lower bound provides accurate approximations to prices and deltas, while the upper bound provides good though less accurate results.

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1. Introduction

In this paper, we consider the problem of finding approximations to the price of arithmetic Asian-style options via lower and upper bounds. Asian options have a payoff that depends on the average price of the underlying asset over the term of the contract. This average may be calculated in a variety of ways under either discrete or continuous monitoring. The averaging feature provides these options with attractive features including resistance to market manipulation in thinly traded markets. These characteristics make them attractive to hedgers and speculators and result in them being one of the most actively traded exotic options in the world.

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Asian-style options have been studied extensively in the literature from many different angles, including:

- moment matching by Milevsky and Posner [24], Stace [29], Forde and Jacquier [12], Novikov et al. [26];
- recurrent integration by Fusai and Meucci [16], March [22];
- Laplace transform and series expansion by Geman and Yor [17], Dufresne [11], Dassios and Nagaradjasarma [10], Cai and Kou [6];
- partial differential equation (PDE) and partial integro-differential equation (PIDE) by Rogers and Shi [28], Alziary et al. [2], Večeř [31], Fouque and Han [14], Foufas and Larson [13];
- Monte Carlo method by Boyle [4], Kemna and Vorst [18], Boyle and Potapchik [5];
- chaos expansion by Funahashi and Kijima [15];
- upper and lower bounds by Curran [9], Rogers and Shi [28], Thompson [30], Lord [20], Lemmens et al. [19].

For a thorough survey, we refer the reader to the paper by Cai and Kou [6].

We follow the work of Thompson [30], who, building on the work of Curran [9] and Rogers and Shi [28], investigated the problem where the underlying asset followed a geometric Brownian motion (GBM). However, it is well known that some characteristics of financial assets cannot be replicated under GBM. For example, the implied volatility surfaces of options under GBM are flat, unlike the “smiles” and “skews” exhibited in the market (see Cont and Tankov [8, Sections 1.1–1.2]). This has prompted the use of more sophisticated models including stochastic volatility and exponential Lévy processes that better reproduce these market characteristics. Lemmens et al. [19] proposed a method to calculate bounds on prices of discretely monitored fixed-strike Asian options, where the underlying asset was modelled by exponential Lévy processes.

Here, also in the framework of exponential Lévy processes, we follow a different approach to calculate lower bounds. The suggested method is presented for discretely and continuously monitored options with both fixed and floating-strike payoff features. Our numerical studies show that the price approximations provided by the lower bound are close to the true prices of these options, obtained by using the considerably more time consuming Monte Carlo method.

2. The Asian option pricing problem

Consider a stock price process $S = (S_t, 0 \le t \le T)$, a maturity time T and a filtered-probability space $(\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t\ge 0})$ on which it is assumed that all random variables in this paper are defined. In its general form, the price C_T of an arithmetic Asian call option can be written as

$$C_T = E[e^{-R_T} F_T(S)], \tag{2.1}$$

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where the payoff function F is given by

$$F_T(S) = \left(\int_0^T S_u\,\mu(du) - g(S_T)\right)^+ \tag{2.2}$$

$$= \left(\int_0^T (S_u - g(S_T))\,\mu(du)\right)^+ \tag{2.3}$$

if we demand that $\int_0^T \mu(du) = 1$. Here $R_t = \int_0^t r_u\,du$, where r is a nonnegative, possibly random process, $x^+ = \max(x, 0)$ and g is a positive function which is typically given by $g(S_T) = K$ (*fixed-strike*) or $g(S_T) = S_T$ (*floating-strike*). The price of a put option can be expressed in a similar fashion or can be obtained through put–call parity formulae (see, for instance, Večeř [31]). Below, we adopt the notation

$$\overline{h} = \int_0^T h_u\,\mu(du), \qquad h \in H,$$

where H is the class of adapted and integrable processes $h = (h_t, 0 \le t \le T)$. This allows the option price given by (2.1)–(2.3) to be expressed as

$$C_T = E[e^{-R_T}(\overline{S} - g(S_T))^+] \tag{2.4}$$

$$= E[e^{-R_T}\overline{(S - g(S_T))}^+]. \tag{2.5}$$

By changing the definition of μ, various flavours of this type of option can be defined. For instance, a continuously monitored arithmetic Asian option can be priced by taking

$$\mu(du) = \frac{1}{T}\,du \tag{2.6}$$

and a discretely monitored option by setting

$$\mu(du) = \frac{1}{N}\sum_{j=1}^{N} \delta_{t_j}(du), \tag{2.7}$$

where δ is the Dirac measure, and the set of monitoring times $\{t_1, \ldots, t_N\}$ satisfies $0 = t_0 < t_1 < \cdots < t_N = T$.

3. Lower and upper bounds

In this section, we derive lower and upper bounds for arithmetic Asian options. The core idea is to replace the event that the option finishes in-the-money with a highly correlated yet more tractable proxy, and then to optimise over the parameter triggering this proxy event.

Theorem 3.1. *The option price given by* (2.4) *or* (2.5) *can be written as*

$$C_T = \sup_{z\in\mathbb{R}, h\in H} E[e^{-R_T}(\overline{S} - g(S_T))\mathbb{I}_{\{\overline{h}>z\}}] \tag{3.1}$$

$$= \inf_{h\in H} E[e^{-R_T}\overline{(S - g(S_T)(1 + h - \overline{h}))^+}], \tag{3.2}$$

where both the supremum and the infimum are attained by choosing $h_u = S_u/g(S_T)$ *and* $z = 1$.

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Proof. We slightly modify the arguments of Novikov and Kordzakhia [25], who looked only at the fixed-strike case, to cater for a wider variety of Asian options. To prove (3.1), we begin with (2.4) and note that for any $h \in H$,

$$\begin{aligned}
C_T &= E[e^{-R_T}(\overline{S} - g(S_T))^+] \\
&\geq E[e^{-R_T}(\overline{S} - g(S_T))^+ \mathbb{I}_{\{\overline{h}>z\}}] \\
&= E[e^{-R_T}(\overline{S} - g(S_T))\mathbb{I}_{\{\overline{h}>z\}} + (\overline{S} - g(S_T))^- \mathbb{I}_{\{\overline{h}>z\}}] \\
&\geq E[e^{-R_T}(\overline{S} - g(S_T))\mathbb{I}_{\{\overline{h}>z\}}],
\end{aligned} \tag{3.3}$$

where the first inequality is due to the indicator function $\mathbb{I}_{\{y>x\}}$ taking the values zero or one, and the second inequality is due to the nonnegativity of $x^- = \max(-x, 0)$. These are similar arguments to those made by others who have studied lower bounds. Taking $h_u = S_u/g(S_T)$ and $z = 1$ in the right-hand side of (3.3) gives

$$\begin{aligned}
&E[e^{-R_T}(\overline{S} - g(S_T))\mathbb{I}_{\{\overline{S/g(S_T)}>1\}}] \\
&\quad = E[e^{-R_T}(\overline{S} - g(S_T))\mathbb{I}_{\{\overline{S}>g(S_T)\}}] \\
&\quad = E[e^{-R_T}(\overline{S} - g(S_T))^+] \\
&\quad = C_T.
\end{aligned}$$

So, we can claim that $C_T = \sup_{z\in\mathbb{R},h\in H} E[e^{-R_T}(\overline{S} - g(S_T))\mathbb{I}_{\{\overline{h}>z\}}]$, which is (3.1), where the supremum is attained with the aforementioned choices for h and z.

For (3.2), we again start with (2.4) and write

$$\begin{aligned}
C_T &= E[e^{-R_T}(\overline{S} - g(S_T))^+] \\
&= E[e^{-R_T}(\overline{S} - g(S_T)\overline{(1 + h - \overline{h})})^+] \\
&= E[e^{-R_T}(\overline{S} - \overline{g(S_T)(1 + h - \overline{h})})^+] \\
&= E[e^{-R_T}(\overline{S - g(S_T)(1 + h - \overline{h})})^+] \\
&\leq E[e^{-R_T}\overline{(S - g(S_T)(1 + h - \overline{h}))^+}],
\end{aligned} \tag{3.4}$$

where the second equality is due to $\overline{1 + h - \overline{h}} = 1$ and the inequality is due to the convexity of x^+ via Jensen's inequality. Substituting $h_u = S_u/g(S_T)$ into the right-hand side of (3.4), we get

$$\begin{aligned}
&E[e^{-R_T}\overline{(S - g(S_T)(1 + S/g(S_T) - \overline{S/g(S_T)}))}^+] \\
&\quad = E[e^{-R_T}\overline{(S - g(S_T)(1 + S/g(S_T) - \overline{S}/g(S_T)))^+}] \\
&\quad = E[e^{-R_T}\overline{(\overline{S} - g(S_T))^+}] \\
&\quad = E[e^{-R_T}(\overline{S} - g(S_T))^+] \\
&\quad = C_T;
\end{aligned}$$

this yields $C_T = \inf_{h\in H} E[e^{-R_T}\overline{(S - g(S_T)(1 + h - \overline{h}))^+}]$, which is equation (3.2) with the infimum attained for the same choice of h as for the supremum. □

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The supremum forms the basis for the lower bound and the infimum for the upper bound. Numerical studies in the case of asset prices driven by GBM demonstrate that the upper bound provides option price approximations that are close to that of the true price (see Novikov and Kordzakhia [25]). However, the lower bound (LB) produces more accurate results; therefore, we concentrate on the lower bound for the remainder of the paper.

4. Exponential Lévy processes

Obviously, if the distribution of $\overline{S}$ is known, there is no need to resort to Theorem 3.1. Unfortunately, except for the case where r is constant and S follows a GBM (see Geman and Yor [17]) or a square-root process (see Dassios and Nagaradjasarma [10]), this distribution is not known. However, if we consider the processes $X_t = \log(S_t/S_0)$ and $Y_t = X_t - X_T$, it turns out that in many cases we can find the joint characteristic functions of $(X_t, \overline{X})$ and $(X_t, \overline{Y})$, respectively. The reason for our interest in these processes will be made clear shortly.

In this section, we consider the cases of fixed and floating-strike arithmetic Asian options corresponding to the choices of $g(S_T) = K$ and $g(S_T) = S_T$, respectively, in Theorem 3.1.

Definition 4.1. Let the stock price process be modelled via $S_t = S_0 e^{X_t}$, with the market having the following properties:

(1) X_t is a Lévy process on $(\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t\geq 0})$;
(2) $E[|X_t|], E[|\overline{X}|] < \infty$ for $t \in [0, T]$;
(3) the interest rate r is constant and $R_t = rt$;
(4) a constant dividend rate q is paid continuously and $Q_t = qt$;
(5) S_t satisfies the martingale condition $E[S_t] = S_0 e^{R_t - Q_t}$ [27, page 481];
(6) the pricing of continuously monitored Asian options uses the choice of μ in (2.6);
(7) the pricing of discretely monitored options with the set of monitoring times $\{t_1, \ldots, t_N\}$ satisfying $0 = t_0 < t_1 < \cdots < t_N = T$ uses the measure μ in (2.7);
(8) the joint characteristic functions of $(X_t, \overline{X})$ and $(X_t, \overline{Y})$, where $\overline{Y} = \overline{X} - X_T$, can be inverted.

4.1. Fixed-strike options

The following theorem suggests a choice of the process h suitable in the case of fixed-strike options.

Theorem 4.1. *Assume that Definition 4.1 applies. By choosing $h_u = aX_u$, $a \in \mathbb{R}$, the bounds can be written as*

$$C_T \geq LB = S_0 \sup_{z\in\mathbb{R}} E\Big[e^{-R_T}\Big(\overline{e^X} - \frac{K}{S_0}\Big)\mathbb{I}_{\{\overline{X}>z\}}\Big], \tag{4.1}$$

$$C_T \leq UB = S_0 \inf_{a\in\mathbb{R}} E\Big[e^{-R_T}\overline{\Big(e^X - \frac{K}{S_0}(1 + aX - a\overline{X})\Big)^+}\Big].$$

Proof. Use $g(S_T) = K$ and $h_u = aX_u$ in Theorem 3.1. □

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4.1.1. *Lower bounds* We adapt the two-step procedure of Thompson [30] to perform the optimisation in (4.1):

- Step 1 – calculate the optimal value of z, $\underline{z}$;
- Step 2 – use $\underline{z}$ to evaluate the expectation.

The next theorem provides the means for calculating the optimum value of z, $\underline{z}$ (Step 1).

THEOREM 4.2. *Under the conditions of Definition 4.1, the value of z, $\underline{z}$ that maximizes the LB (4.1) for a fixed-strike arithmetic Asian option satisfies*

$$\frac{1}{\pi f_{\overline{X}}(\underline{z})}\int_0^T\int_0^\infty \Re(\varphi(-i,\zeta;u)e^{-i\zeta\underline{z}})\,d\zeta\,\mu(du)=\frac{K}{S_0}, \tag{4.2}$$

where $\varphi(\xi,\zeta;t)=E[e^{i\xi X_t+i\zeta\overline{X}}]$ is the characteristic function of $(X_t,\overline{X})$ and $f_{\overline{X}}(y)=(1/\pi)\int_0^\infty \Re(\varphi(0,\zeta;T)e^{-i\zeta y})\,d\zeta$ is the density of $\overline{X}$.

PROOF. We reproduce the proof of Alexander et al. [1]. Let $f_{X,\overline{X}}$ and $f_{\overline{X}}$ be the densities of $(X_t,\overline{X})$ and $\overline{X}$, respectively. Following the lead of Thompson [30], we differentiate the expression to be maximised in (4.1) with respect to the parameter z.

$$\begin{aligned}
&\frac{\partial}{\partial z}S_0E\Big[e^{-R_T}\Big(\overline{e^X}-\frac{K}{S_0}\Big)\mathbb{I}_{\{\overline{X}>z\}}\Big]\\
&=\frac{\partial}{\partial z}S_0E\Big[e^{-R_T}\int_0^T\Big(e^{X_u}-\frac{K}{S_0}\Big)\mu(du)\mathbb{I}_{\{\overline{X}>z\}}\Big]\\
&=S_0e^{-R_T}\frac{\partial}{\partial z}\int_0^T E\Big[\Big(e^{X_u}-\frac{K}{S_0}\Big)\mathbb{I}_{\{\overline{X}>z\}}\Big]\mu(du)\\
&=S_0e^{-R_T}\frac{\partial}{\partial z}\int_0^T\int_{-\infty}^\infty\int_{-\infty}^\infty\Big(e^x-\frac{K}{S_0}\Big)\mathbb{I}_{\{y>z\}}f_{X,\overline{X}}(x,y;u)\,dx\,dy\,\mu(du)\\
&=S_0e^{-R_T}\int_0^T\frac{\partial}{\partial z}\int_z^\infty\int_{-\infty}^\infty\Big(e^x-\frac{K}{S_0}\Big)f_{X,\overline{X}}(x,y;u)\,dx\,dy\,\mu(du)\\
&=-S_0e^{-R_T}\int_0^T\int_{-\infty}^\infty\Big(e^x-\frac{K}{S_0}\Big)f_{X,\overline{X}}(x,z;u)\,dx\,\mu(du)\\
&=-S_0e^{-R_T}f_{\overline{X}}(z)\int_0^T\int_{-\infty}^\infty\Big(e^x-\frac{K}{S_0}\Big)\frac{f_{X,\overline{X}}(x,z;u)}{f_{\overline{X}}(z)}\,dx\,\mu(du)\\
&=-S_0e^{-R_T}f_{\overline{X}}(z)\int_0^T E\Big[\Big(e^{X_u}-\frac{K}{S_0}\Big)\Big|\overline{X}=z\Big]\mu(du),
\end{aligned}$$

the use of Fubini's theorem being justified by Definition 4.1(2). Setting this to zero gives

$$\int_0^T E[e^{X_u}|\overline{X}=\underline{z}]\,\mu(du)=\frac{K}{S_0} \tag{4.3}$$

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by the integrability condition on μ. As shown by Lemmens et al. [19], the conditional expectation can be recovered from the characteristic function φ as

$$E[e^{X_u}|\overline{X}=\underline{z}] = \frac{1}{2\pi f_{\overline{X}}(\underline{z})}\int_{-\infty}^{\infty}\varphi(-i,\zeta;u)e^{-i\zeta\underline{z}}\,d\zeta. \tag{4.4}$$

Letting x^* be the complex conjugate of x and using the relationship $e^{ix}=(e^{-ix})^*$, the integrand in (4.4) can be re-written as

$$\begin{aligned}\varphi(-i,\zeta;u)e^{-i\zeta\underline{z}} &= e^{-i\zeta\underline{z}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,z;u)e^{ix(-i)}e^{iy\zeta}\,dx\,dy\\ &= (e^{i\zeta\underline{z}})^*\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,z;u)e^{ix(-i)}(e^{-iy\zeta})^*\,dx\,dy\\ &= (e^{i\zeta\underline{z}})^*\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,z;u)e^{ix(-i)}e^{iy(-\zeta)}\,dx\,dy\Big)^*\\ &= (e^{i\zeta\underline{z}})^*(\varphi(-i,-\zeta;u))^*\\ &= (\varphi(-i,-\zeta;u)e^{i\zeta\underline{z}})^*.\end{aligned}$$

Using this symmetry, we can re-write (4.4) as

$$E[e^{X_u}|\overline{X}=\underline{z}] = \frac{1}{\pi f_{\overline{X}}(\underline{z})}\int_{0}^{\infty}\Re(\varphi(-i,\zeta;u)e^{-i\zeta\underline{z}})\,d\zeta,$$

providing advantages for numerical evaluation. So, (4.3) becomes

$$\frac{1}{\pi f_{\overline{X}}(\underline{z})}\int_0^T\int_0^{\infty}\Re(\varphi(-i,\zeta;u)e^{-i\zeta\underline{z}})\,d\zeta\,\mu(du) = \frac{K}{S_0}$$

and $f_{\overline{X}}$ can be recovered from its characteristic function $\varphi_{\overline{X}}$ via

$$\begin{aligned}f_{\overline{X}}(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_{\overline{X}}(\zeta)e^{-iy\zeta}\,d\zeta\\ &= \frac{1}{\pi}\int_0^{\infty}\Re(\varphi_{\overline{X}}(\zeta)e^{-iy\zeta})\,d\zeta\\ &= \frac{1}{\pi}\int_0^{\infty}\Re(\varphi(0,\zeta;T)e^{-iy\zeta})\,d\zeta,\end{aligned}$$

where the second equality follows from similar reasoning. This completes the proof of the theorem. □

With the optimal z, $\underline{z}$, being the solution of the inverse problem in Theorem 4.2 (Step 1), we now devise a procedure to calculate the expectation in (4.1) (Step 2), and thereby arrive at the lower bound. To do so, we apply a two-dimensional version of the exponential damping method of Borovkov and Novikov [3], a general idea introduced by Carr and Madan [7].

THEOREM 4.3. *Let $f_{X,\overline{X}}$ be the joint density of $(X_t,\overline{X})$ and assume that we can find $\alpha_1>0$, $\alpha_2<-1$ and $\beta<0$ such that $E[e^{-\alpha_1X_t-\beta\overline{X}}]$ and $E[e^{-\alpha_2X_t-\beta\overline{X}}]$ are finite for all*

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$t \in [0, T]$. *Under the conditions of Definition 4.1, the lower bound (4.1) of a fixed-strike arithmetic Asian option can be expressed as*

$$LB = \frac{e^{-R_T} S_0}{2\pi^2} \int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \Re(\hat{h}_1(\xi, \zeta; \underline{z}) \varphi(-\xi + i\alpha_1, -\zeta + i\beta; u) \\ + \hat{h}_2(\xi, \zeta; \underline{z}) \varphi(-\xi + i\alpha_2, -\zeta + i\beta; u))\, d\xi\, d\zeta\, \mu(du), \tag{4.5}$$

where

$$\hat{h}_1(\xi, \zeta; \underline{z}) = \left(\frac{K}{(\alpha_1 + i\xi) S_0} - \frac{1}{\alpha_1 + 1 + i\xi} \right) \frac{e^{\underline{z}(\beta + i\zeta)}}{\beta + i\zeta},$$
$$\hat{h}_2(\xi, \zeta; \underline{z}) = \left(\frac{1}{\alpha_2 + 1 + i\xi} - \frac{K}{(\alpha_2 + i\xi) S_0} \right) \frac{e^{\underline{z}(\beta + i\zeta)}}{\beta + i\zeta},$$
$$\varphi(\xi, \zeta; t) = E[e^{i\xi X_t + i\zeta \overline{X}}]$$

and $\underline{z}$ *has been calculated from (4.2) of Theorem 4.2.*

Proof. We repeat the proof of Alexander et al. [1]. Using $\underline{z}$ from (4.2) in (4.1) yields

$$\begin{aligned}
\mathrm{LB} &= e^{-R_T} S_0 \sup_{z \in \mathbb{R}} E\left[\left(\overline{e^X} - \frac{K}{S_0} \right) \mathbb{I}_{\{\overline{X} > z\}} \right] \\
&= e^{-R_T} S_0 E\left[\int_0^T \left(e^{X_u} - \frac{K}{S_0} \right) \mu(du) \mathbb{I}_{\{\overline{X} > \underline{z}\}} \right] \\
&= e^{-R_T} S_0 \int_0^T E\left[\left(e^{X_u} - \frac{K}{S_0} \right) \mathbb{I}_{\{\overline{X} > \underline{z}\}} \right] \mu(du) \\
&= e^{-R_T} S_0 \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^x - \frac{K}{S_0} \right) \mathbb{I}_{\{y > \underline{z}\}} f_{X, \overline{X}}(x, y; u)\, dx\, dy\, \mu(du) \\
&= e^{-R_T} S_0 \int_0^T \Psi(u; \underline{z})\, \mu(du),
\end{aligned} \tag{4.6}$$

where

$$\Psi(u; \underline{z}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^x - \frac{K}{S_0} \right) \mathbb{I}_{\{y > \underline{z}\}} f_{X, \overline{X}}(x, y; u)\, dx\, dy,$$

the use of Fubini's theorem being justified by Definition 4.1(2).

Since we are integrating over all of x, we must split the integral into two and damp each piece separately. So, write

$$\Psi(u; \underline{z}) = \Psi_1(u; \underline{z}) + \Psi_2(u; \underline{z}),$$

where

$$\Psi_1(u; \underline{z}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^x - \frac{K}{S_0} \right) \mathbb{I}_{\{x < 0\}} \mathbb{I}_{\{y > \underline{z}\}} f_{X, \overline{X}}(x, y; u)\, dx\, dy$$

and

$$\Psi_2(u; \underline{z}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(e^x - \frac{K}{S_0} \right) \mathbb{I}_{\{x > 0\}} \mathbb{I}_{\{y > \underline{z}\}} f_{X, \overline{X}}(x, y; u)\, dx\, dy.$$

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Next, define the exponentially damped functions

$$h_1(x,y;\underline{z}) = e^{\alpha_1 x+\beta y}\Big(e^x - \frac{K}{S_0}\Big)\mathbb{I}_{\{x<0\}}\mathbb{I}_{\{y>\underline{z}\}}$$

and

$$h_2(x,y;\underline{z}) = e^{\alpha_2 x+\beta y}\Big(e^x - \frac{K}{S_0}\Big)\mathbb{I}_{\{x>0\}}\mathbb{I}_{\{y>\underline{z}\}},$$

where $\alpha_1 > 0$, $\alpha_2 < -1$ and $\beta < 0$ guarantee that these functions are absolutely integrable and, therefore, their Fourier transforms exist. For instance, the Fourier transform $\hat{h}_1$ of h_1 is

$$\begin{aligned}
\hat{h}_1(\xi,\zeta;\underline{z}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(x,y;\underline{z})e^{ix\xi+iy\zeta}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\alpha_1 x+\beta y}\Big(e^x - \frac{K}{S_0}\Big)\mathbb{I}_{\{x<0\}}\mathbb{I}_{\{y>\underline{z}\}}e^{ix\xi+iy\zeta}\,dx\,dy \\
&= \Big(\frac{K}{(\alpha_1+i\xi)S_0} - \frac{1}{\alpha_1+1+i\xi}\Big)\frac{e^{\underline{z}(\beta+i\zeta)}}{\beta+i\zeta},
\end{aligned}$$

which is readily shown through direct integration. The Fourier transform of h_2, $\hat{h}_2$, can be found in a likewise fashion.

It will also be convenient to define the subsidiary functions

$$H_1(x,y;u) = e^{-\alpha_1 x-\beta y}f_{X,\overline{X}}(x,y;u)$$

and

$$H_2(x,y;u) = e^{-\alpha_2 x-\beta y}f_{X,\overline{X}}(x,y;u),$$

which are also absolutely integrable by the assumptions made at the beginning of the theorem. The Fourier transforms of these functions can be written in terms of the joint characteristic function φ as

$$\hat{H}_1(\xi,\zeta;u) = \varphi(\xi+i\alpha_1,\zeta+i\beta;u)$$

and

$$\hat{H}_2(\xi,\zeta;u) = \varphi(\xi+i\alpha_2,\zeta+i\beta;u).$$

Now

$$\begin{aligned}
\Psi_1(u;\underline{z}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(e^x - \frac{K}{S_0}\Big)\mathbb{I}_{\{x<0\}}\mathbb{I}_{\{y>\underline{z}\}}f_{X,\overline{X}}(x,y;u)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(x,y;\underline{z})H_1(x,y;u)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{h}_1(\xi,\zeta;\underline{z})e^{-ix\xi-iy\zeta}\,d\xi\,d\zeta H_1(x,y;u)\,dx\,dy \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{h}_1(\xi,\zeta;\underline{z})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_1(x,y;u)e^{-ix\xi-iy\zeta}\,dx\,dy\,d\xi\,d\zeta
\end{aligned}$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{h}_1(\xi,\zeta;\underline{z})\hat{H}_1(-\xi,-\zeta;u)\,d\xi\,d\zeta$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{h}_1(\xi,\zeta;\underline{z})\varphi(-\xi+i\alpha_1,-\zeta+i\beta;u)\,d\xi\,d\zeta, \tag{4.7}$$

where the application of the Fourier inversion theorem to $\hat{h}_1$ is justified as $\hat{h}_1$ is absolutely integrable (see Alexander et al. [1] for details). As in Theorem 4.2, we can exploit some symmetries of the integrand in (4.7) to provide advantages when evaluated numerically. Because $\hat{h}_1$ is the Fourier transform of a real-valued function, we immediately have

$$\hat{h}_1(\xi,\zeta;\underline{z}) = (\hat{h}_1(-\xi,-\zeta;\underline{z}))^*,$$

while for φ, note that

$$\begin{aligned}
\varphi(-\xi+i\alpha_1,-\zeta+i\beta;u) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,y;u)e^{ix(-\xi+i\alpha_1)+iy(-\zeta+i\beta)}\,dx\,dy\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,y;u)e^{-\alpha_1 x-\beta y}e^{-ix\xi}e^{-iy\zeta}\,dx\,dy\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,y;u)e^{-\alpha_1 x-\beta y}(e^{ix\xi})^*(e^{iy\zeta})^*\,dx\,dy\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,y;u)e^{-\alpha_1 x-\beta y}(e^{ix\xi}e^{iy\zeta})^*\,dx\,dy\\
&= \bigg(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,\overline{X}}(x,y;u)e^{ix(\xi+i\alpha_1)+iy(\zeta+i\beta)}\,dx\,dy\bigg)^*\\
&= (\varphi(\xi+i\alpha_1,\zeta+i\beta;u))^*,
\end{aligned}$$

using the relationship $e^{-ix} = (e^{ix})^*$. Thus, the integrand in (4.7) satisfies

$$\begin{aligned}
\hat{h}_1(\xi,\zeta;\underline{z})\varphi(-\xi+i\alpha_1,-\zeta+i\beta;u) &= (\hat{h}_1(-\xi,-\zeta;\underline{z}))^*(\varphi(\xi+i\alpha_1,\zeta+i\beta;u))^*\\
&= (\hat{h}_1(-\xi,-\zeta;\underline{z})\varphi(\xi+i\alpha_1,\zeta+i\beta;u))^*,
\end{aligned}$$

allowing us to re-write Ψ_1 as

$$\Psi_1(u;\underline{z}) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re(\hat{h}_1(\xi,\zeta;\underline{z})\varphi(-\xi+i\alpha_1,-\zeta+i\beta;u))\,d\xi\,d\zeta$$

and, using similar arguments, to write Ψ_2 as

$$\Psi_2(u;\underline{z}) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re(\hat{h}_2(\xi,\zeta;\underline{z})\varphi(-\xi+i\alpha_2,-\zeta+i\beta;u))\,d\xi\,d\zeta.$$

Making the substitution of the re-worked inner double integral $\Psi = \Psi_1 + \Psi_2$ in (4.6) completes the proof. □

Remark 4.1. By differentiating (4.5) with respect to S_0, an approximation to the delta of the option price can be calculated.

4.1.2. *Joint characteristic function* Theorems 4.2 and 4.3 give us the means to calculate the lower bound (4.1) in Theorem 4.1 whenever we can find the joint characteristic function of $(X_t, \overline{X})$. The following general result provides a means for doing so.

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THEOREM 4.4. *Let X be per Definition 4.1. Define the characteristic exponent*

$$\psi(\xi) = \frac{1}{t}\log(E[e^{i\xi X_t}]), \quad \xi \in \mathbb{R},$$

and set

$$G(t) = \int_0^t \mu(du)$$

as a left-continuous nonrandom function of finite variation. Then, for $\zeta \in \mathbb{R}$,

$$E[e^{i\xi X_t + i\zeta\overline{X}}] = \exp\left\{\int_0^T \psi(\xi\mathbb{I}_{\{t\geq u\}} + \zeta(G(T) - G(u)))\,du\right\}.$$

PROOF.

$$\begin{aligned}
E[e^{i\xi X_t + i\zeta\overline{X}}] &= E\left[\exp\left\{i\xi X_t + i\zeta\int_0^T X_u\,\mu(du)\right\}\right] \\
&= E\left[\exp\left\{i\xi\int_0^T \mathbb{I}_{\{t\geq u\}}\,dX_u + i\zeta\int_0^T X_u\,dG(u)\right\}\right] \\
&= E\left[\exp\left\{i\xi\int_0^T \mathbb{I}_{\{t\geq u\}}\,dX_u + i\zeta\left(X_T G(T) - \int_0^T G(u)\,dX_u\right)\right\}\right] \\
&= E\left[\exp\left\{i\xi\int_0^T \mathbb{I}_{\{t\geq u\}}\,dX_u + i\zeta\int_0^T (G(T) - G(u))\,dX_u\right\}\right] \\
&= E\left[\exp\left\{i\int_0^T (\xi\mathbb{I}_{\{t\geq u\}} + \zeta(G(T) - G(u)))\,dX_u\right\}\right] \\
&= E\left[\exp\left\{i\int_0^T f(u)\,dX_u\right\}\right],
\end{aligned}$$

where $f(u) = \xi\mathbb{I}_{\{t\geq u\}} + \zeta(G(T) - G(u))$. But, if f is nonrandom and continuous [21] or left-continuous [8, Lemma 15.1], then

$$E\left[\exp\left\{i\int_0^T f(u)\,dX_u\right\}\right] = \exp\left\{\int_0^T \psi(f(u))\,du\right\}. \tag{4.8}$$

Substituting f into (4.8) completes the proof. □

COROLLARY 4.1. *The joint characteristic functions of $(X_t, \overline{X})$ for fixed-strike options with continuous monitoring and discrete monitoring are given by*

$$\varphi(\xi, \zeta; t) = \exp\left\{\int_0^t \psi\left(\xi + \zeta\frac{T-u}{T}\right)du + \int_t^T \psi\left(\zeta\frac{T-u}{T}\right)du\right\} \quad \textit{and} \tag{4.9}$$

$$\varphi(\xi, \zeta; t_j) = \exp\left\{\sum_{k=1}^{j}\psi\left(\xi + \zeta\frac{N+1-k}{N}\right)(t_k - t_{k-1}) + \sum_{k=j+1}^{N}\psi\left(\zeta\frac{N+1-k}{N}\right)(t_k - t_{k-1})\right\}, \tag{4.10}$$

respectively, where the set of monitoring times $\{t_1, \ldots, t_N\}$ satisfies $0 = t_0 < t_1 < \cdots < t_N = T$.

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Proof. For the continuous case, taking $\mu(du) = (1/T)\,du$ in Theorem 4.4 gives (4.9). Unfortunately, if we take $\mu(du) = (1/N)\sum_{j=1}^{N}\delta_{t_j}(du)$ in Theorem 4.4, then G, and thereby f, is right-continuous. This is because we demand that $\int_{t_{j-1}}^{t_j} X_u\delta_{t_j}(du) = X_{t_j}$; that is, we treat all integrals involving Dirac measures as right-continuous. So, the theorem cannot be used. Instead, we prove this case separately. First note that

$$\begin{aligned}
i\xi X_{t_j} + i\zeta\overline{X} &= i\xi X_{t_j} + i\zeta\frac{1}{N}\sum_{k=1}^{N} X_{t_k}\\
&= i\xi\sum_{k=1}^{j}(X_{t_k} - X_{t_{k-1}}) + i\zeta\frac{1}{N}\sum_{k=1}^{N}(X_{t_k} - X_{t_{k-1}})(N+1-k)\\
&= i\sum_{k=1}^{j}(X_{t_k} - X_{t_{k-1}})\Big(\xi + \frac{N+1-k}{N}\zeta\Big) + i\sum_{k=j+1}^{N}(X_{t_k} - X_{t_{k-1}})\frac{N+1-k}{N}\zeta.
\end{aligned}$$

Then, using the properties of independence and stationarity of increments,

$$\begin{aligned}
\varphi(\xi,\zeta;t_j) &= E[e^{i\xi X_{t_j}+i\zeta\overline{X}}]\\
&= E\Big[\exp\Big\{i\xi X_{t_j} + i\zeta\frac{1}{N}\sum_{k=1}^{N} X_{t_k}\Big\}\Big]\\
&= E\Big[\exp\Big\{i\sum_{k=1}^{j}(X_{t_k} - X_{t_{k-1}})\Big(\xi + \frac{N+1-k}{N}\zeta\Big) + i\sum_{k=j+1}^{N}(X_{t_k} - X_{t_{k-1}})\frac{N+1-k}{N}\zeta\Big\}\Big]\\
&= \prod_{k=1}^{j} E[e^{i(X_{t_k}-X_{t_{k-1}})(\xi+((N+1-k)/N)\zeta)}]\prod_{k=j+1}^{N} E[e^{i(X_{t_k}-X_{t_{k-1}})((N+1-k)/N)\zeta}]\\
&= \prod_{k=1}^{j} e^{\psi(\xi+((N+1-k)/N)\zeta)(t_k-t_{k-1})}\prod_{k=j+1}^{N} e^{\psi(((N+1-k)/N)\zeta)(t_k-t_{k-1})}\\
&= \exp\Big\{\sum_{k=1}^{j}\psi\Big(\xi+\zeta\frac{N+1-k}{N}\Big)(t_k-t_{k-1}) + \sum_{k=j+1}^{N}\psi\Big(\zeta\frac{N+1-k}{N}\Big)(t_k-t_{k-1})\Big\},
\end{aligned}$$

which is (4.10). We point out that a similar formula was also derived by Lemmens et al. [19]. □

4.2. Floating-strike options Now we echo the last section for the floating-strike case, starting with the bounds.

Theorem 4.5. *Assume that Definition 4.1 applies. Letting $Y_t = X_t - X_T$ and choosing $h_u = aY_u$, $a \in \mathbb{R}$, the bounds are*

$$C_T \geq LB = S_0 \sup_{z \in \mathbb{R}} E\Big[e^{-R_T}\Big(\overline{e^X} - \frac{S_T}{S_0}\Big)\mathbb{I}_{\{\overline{Y} > z\}}\Big], \tag{4.11}$$

$$C_T \leq UB = S_0 \inf_{a \in \mathbb{R}} E\Big[e^{-R_T}\overline{\Big(e^X - \frac{S_T}{S_0}(1 + aY - a\overline{Y})\Big)^+}\Big].$$

Proof. Use $g(S_T) = S_T$ and $h_u = aY_u$ in Theorem 3.1. □

4.2.1. *Lower bounds* The following theorem facilitates the calculation of the optimal z for the floating-strike case.

Theorem 4.6. *Under the conditions of Definition 4.1, the value of z, $\underline{z}$, that maximises the LB (4.11) for a floating-strike arithmetic Asian option satisfies*

$$\int_0^T \int_0^\infty \Re(\varphi(-i, \zeta; u)e^{-i\zeta\underline{z}})\, d\zeta\, \mu(du) = \int_0^\infty \Re(\varphi(-i, \zeta; T)e^{-i\zeta\underline{z}})\, d\zeta, \tag{4.12}$$

where the characteristic function of $(X_t, \overline{Y})$ is

$$\varphi(\xi, \zeta; t) = E[e^{i\xi X_t + i\zeta\overline{Y}}].$$

Proof. The proof is similar to that of Theorem 4.2 and can be found in Alexander et al. [1]. □

Having calculated the optimal value of z, $\underline{z}$, we can now evaluate the expectation in (4.11) and thereby the lower bound. The next theorem allows us to do so.

Theorem 4.7. *Let $f_{X,\overline{Y}}$ be the joint density of $(X_t, \overline{Y})$ and assume that we can find $\alpha_1 > -1$, $\alpha_2 < -1$ and $\beta < 0$ such that $E[e^{-\alpha_1 X_t - \beta\overline{Y}}]$ and $E[e^{-\alpha_2 X_t - \beta\overline{Y}}]$ are finite for $t \in [0, T]$. Under the conditions of Definition 4.1, the lower bound (4.11) of a floating-strike arithmetic Asian option can be expressed as*

$$\begin{aligned} LB = \frac{e^{-R_T}S_0}{2\pi^2}\Big(&\int_0^T \int_{-\infty}^\infty \int_0^\infty \Re(\hat{h}_1(\xi, \zeta; \underline{z})\varphi(-\xi + i\alpha_1, -\zeta + i\beta; u) \\ &+ \hat{h}_2(\xi, \zeta; \underline{z})\varphi(-\xi + i\alpha_2, -\zeta + i\beta; u))\, d\xi\, d\zeta\, \mu(du) \\ &- \int_{-\infty}^\infty \int_0^\infty \Re(\hat{h}_1(\xi, \zeta; \underline{z})\varphi(-\xi + i\alpha_1, -\zeta + i\beta; T) \\ &+ \hat{h}_2(\xi, \zeta; \underline{z})\varphi(-\xi + i\alpha_2, -\zeta + i\beta; T))\, d\xi\, d\zeta\Big), \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \hat{h}_1(\xi, \zeta; \underline{z}) &= \frac{-1}{\alpha_1 + 1 + i\xi}\frac{e^{\underline{z}(\beta + i\zeta)}}{\beta + i\zeta}, \\ \hat{h}_2(\xi, \zeta; \underline{z}) &= \frac{1}{\alpha_2 + 1 + i\xi}\frac{e^{\underline{z}(\beta + i\zeta)}}{\beta + i\zeta}, \\ \varphi(\xi, \zeta; t) &= E[e^{i\xi X_t + i\zeta\overline{Y}}], \end{aligned}$$

$\alpha_1 > -1$, $\alpha_2 < -1$, $\beta < 0$ and $\underline{z}$ has been calculated from (4.12) of Theorem 4.6.

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PROOF. The proof is similar to that of Theorem 4.3 and can be found in Alexander et al. [1]. □

REMARK 4.2. By differentiating (4.13) with respect to S_0, an approximation to the delta of the option price can be calculated.

4.2.2. *Joint characteristic function* Next, a general method that allows the calculation of the joint characteristic function of $(X_t, \overline{Y})$ for the floating-strike case is presented.

THEOREM 4.8. *Let X be per Definition 4.1. Define the characteristic exponent*

$$\psi(\xi) = \frac{1}{t}\log(E[e^{i\xi X_t}]), \quad \xi \in \mathbb{R}$$

and set $G(t) = \int_0^t \mu(du)$ *as a left-continuous nonrandom function of finite variation. Then, for* $\zeta \in \mathbb{R}$,

$$E[e^{i\xi X_t + i\zeta\overline{Y}}] = \exp\biggl\{\int_0^T \psi(\xi\mathbb{I}_{\{t\geq u\}} + \zeta(G(T) - G(u) - 1))\,du\biggr\}.$$

PROOF. The proof is similar to that of Theorem 4.4. □

Finally, we provide formulae for the calculation of the joint characteristic function of $(X_t, \overline{Y})$.

COROLLARY 4.2. *The joint characteristic functions of* $(X_t, \overline{Y})$ *for fixed-strike options with continuous monitoring and discrete monitoring are given by*

$$\varphi(\xi, \zeta; t) = \exp\left\{\int_0^t \psi\Bigl(\xi - \zeta\frac{u}{T}\Bigr)\,du + \int_t^T \psi\Bigl(-\zeta\frac{u}{T}\Bigr)\,du\right\} \quad \textit{and} \tag{4.14}$$

$$\varphi(\xi, \zeta; t_j) = \exp\left\{\sum_{k=1}^{j} \psi\Bigl(\xi + \zeta\frac{1-k}{N}\Bigr)(t_k - t_{k-1}) + \sum_{k=j+1}^{N} \psi\Bigl(\zeta\frac{1-k}{N}\Bigr)(t_k - t_{k-1})\right\},$$

respectively, where the set of monitoring times $\{t_1, \ldots, t_N\}$ *satisfies* $0 = t_0 < t_1 < \cdots < t_N = T$.

PROOF. The proof is similar to the proof of Corollary 4.1. □

5. Numerical examples

In this section, we use the lower bound equations given in Theorems 4.3 and 4.7, and the expressions for the joint characteristic functions provided by Corollaries 4.1 and 4.2, to price fixed and floating-strike arithmetic Asian options for a variety of underlying processes. We do this for the continuously monitored case (by choosing μ as in (2.6)) and for the discretely monitored case (taking μ as in (2.7)). The values of the parameters common for each process are $S_0 = 100$, $r = 0.05$, $q = 0$, $K = 100$ and $T = 1$.

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All prices are validated using Monte Carlo methods with 1 000 000 paths each of 500 discretisation points. In the tables displaying the results the term "fixed$_N$" refers to a discretely monitored fixed-strike Asian option with $N \in \mathbb{N}$ monitoring points; the case $N = \infty$ indicates a continuously monitored option. Floating-strike options are notated in a similar fashion. For the case $N = 10$ we have used the monitoring points $t_j \in \{0.1, 0.15, 0.2, 0.45, 0.5, 0.6, 0.8, 0.85, 0.95, 1.0\}$ while for $N = 20$ and $N = 50$ equally spaced monitoring points have been used.

Of our three examples, closed form expressions for the joint characteristic functions of $(X_t, \overline{X})$ and $(X_t, \overline{Y})$ under continuous monitoring could be found for GBM and Merton's jump-diffusion models (see [23]), but not for the exponential normal inverse Gaussian (NIG) process. This was due to the complexity of the characteristic exponent and the challenges that this entailed for symbolic integration. In this case, we performed the integration in (4.9) and (4.14) numerically. This of course was not an issue in the discretely monitored case, as here the joint characteristic functions involve sums of the known characteristic exponents.

All expressions for ψ and γ are either directly from, or are small modifications of, formulae from Chapters 13 and 15 of Pascucci [27].

Example 5.1 (GBM). Let $X_t = \gamma t + \sigma W_t$, where W is a Wiener process. The characteristic exponent of X_t is

$$\psi(\xi) = i\gamma\xi - \tfrac{1}{2}\sigma^2\xi^2$$

and the martingale condition (see Definition 4.1(5)) requires that

$$\gamma = r - q - \tfrac{1}{2}\sigma^2.$$

For the fixed-strike continuous monitoring case, the joint characteristic function of $(X_t, \overline{X})$ is

$$\varphi(\xi, \zeta; t) = \exp\left\{i\gamma\left(t\xi + \frac{1}{2}T\zeta\right) - \frac{1}{2}\sigma^2\left(t\xi^2 + t\frac{2T - t}{T}\xi\zeta + \frac{1}{3}T\zeta^2\right)\right\},$$

while, for the discrete monitoring case,

$$\begin{aligned}\varphi(\xi, \zeta; t_j) = \exp\bigg\{&\sum_{k=1}^{j}\left(i\gamma\left(\xi + \frac{N+1-k}{N}\zeta\right) - \frac{1}{2}\sigma^2\left(\xi + \frac{N+1-k}{N}\zeta\right)^2\right)(t_k - t_{k-1})\\ &+ \sum_{k=j+1}^{N}\left(i\gamma\frac{N+1-k}{N}\zeta - \frac{1}{2}\sigma^2\left(\frac{N+1-k}{N}\zeta\right)^2\right)(t_k - t_{k-1})\bigg\},\end{aligned}$$

which can be derived via Corollary 4.1. Note that in the case of equidistant monitoring points, this can be simplified to

$$\begin{aligned}\varphi(\xi, \zeta; t_j) = \exp\bigg\{&i\gamma\left(j\xi + \frac{N+1}{2}\zeta\right)\frac{T}{N} - \frac{1}{2}\sigma^2\bigg(j\xi^2 + j\frac{2N-j+1}{N}\xi\zeta\\ &+ \frac{2N^2+3N+1}{6N}\zeta^2\bigg)\frac{T}{N}\bigg\}.\end{aligned}$$

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Likewise, the characteristic function of $(X_t, \overline{Y})$ in the floating-strike case under continuous monitoring can be derived using Corollary 4.2 as

$$\varphi(\xi,\zeta;t) = \exp\Big\{i\gamma\Big(t\xi - \frac{1}{2}T\zeta\Big) - \frac{1}{2}\sigma^2\Big(t\xi^2 - \frac{t^2}{T}\xi\zeta + \frac{1}{3}T\zeta^2\Big)\Big\},$$

while for discrete monitoring,

$$\begin{aligned}\varphi(\xi,\zeta;t_j) = \exp\Big\{&\sum_{k=1}^{j}\Big(i\gamma\Big(\xi + \frac{1-k}{N}\zeta\Big) - \frac{1}{2}\sigma^2\Big(\xi + \frac{1-k}{N}\zeta\Big)^2\Big)(t_k - t_{k-1})\\ &+ \sum_{k=j+1}^{N}\Big(i\gamma\frac{1-k}{N}\zeta - \frac{1}{2}\sigma^2\Big(\frac{1-k}{N}\zeta\Big)^2\Big)(t_k - t_{k-1})\Big\}\end{aligned}$$

simplifies to

$$\begin{aligned}\varphi(\xi,\zeta;t_j) = \exp\Big\{&i\gamma\Big(j\xi - \frac{N-1}{2}\zeta\Big)\frac{T}{N} - \frac{1}{2}\sigma^2\Big(j\xi^2 - j\frac{j-1}{N}\xi\zeta\\ &+ \frac{2N^2-3N+1}{6N}\zeta^2\Big)\frac{T}{N}\Big\}\end{aligned}$$

for equidistant monitoring points.

To demonstrate the method, we now collate the steps for calculating the lower bound price for discretely monitored fixed-strike options. First, we use (2.7) in (4.2) of Theorem 4.2 and solve the quadratic optimisation problem

$$\underline{z} = \operatorname*{argmin}_{z\in\mathbb{R}}\Big(\frac{1}{\pi N f_{\overline{X}}(z)}\int_0^\infty \sum_{j=1}^{N}\Re(\varphi(-i,\zeta;t_j)e^{-i\zeta z})\,d\zeta - \frac{K}{S_0}\Big)^2,$$

where

$$f_{\overline{X}}(y) = \frac{1}{\pi}\int_0^\infty \Re(\varphi(0,\zeta;T)e^{-i\zeta y})\,d\zeta.$$

We then use (2.7) in Theorem 4.3 with $\underline{z}$ from above and calculate the lower bound as

$$\begin{aligned}\mathrm{LB} = \frac{e^{-R_T}S_0}{2\pi^2 N}\int_{-\infty}^{\infty}\int_0^\infty \sum_{j=1}^{N}&\Re(\hat{h}_1(\xi,\zeta;\underline{z})\varphi(-\xi+i\alpha_1,-\zeta+i\beta;t_j)\\ &+ \hat{h}_2(\xi,\zeta;\underline{z})\varphi(-\xi+i\alpha_2,-\zeta+i\beta;t_j))\,d\xi\,d\zeta,\end{aligned}$$

where

$$\hat{h}_1(\xi,\zeta;\underline{z}) = \Big(\frac{K}{(\alpha_1+i\xi)S_0} - \frac{1}{\alpha_1+1+i\xi}\Big)\frac{e^{\underline{z}(\beta+i\zeta)}}{\beta+i\zeta},$$

$$\hat{h}_2(\xi,\zeta;\underline{z}) = \Big(\frac{1}{\alpha_2+1+i\xi} - \frac{K}{(\alpha_2+i\xi)S_0}\Big)\frac{e^{\underline{z}(\beta+i\zeta)}}{\beta+i\zeta}$$

and $\alpha_1 > 0$, $\alpha_2 < -1$ and $\beta < 0$. For the results shown in all the tables below, we chose the parameter values $\alpha_1 = 1$, $\alpha_2 = -2$ and $\beta = -1$. The results for the GBM case are presented in Table 1 using the parameter $\sigma = 0.2$. ◇

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TABLE 1. GBM lower bounds.

	Lower bound		Monte Carlo	
Type	Price	$\underline{z}$	Price	Standard error
fixed_{10}	6.2324	−2.1329E−03	6.2283	4.2187E−03
fixed_{20}	5.9986	−2.0947E−03	5.9925	4.0621E−03
fixed_{50}	5.8571	−2.0429E−03	5.8560	3.9774E−03
fixed_{∞}	5.7627	−2.0044E−03	5.7765	3.9245E−03
floating_{10}	3.0017	−1.7396E−03	3.0049	2.4269E−03
floating_{20}	3.2906	−1.9086E−03	3.2908	2.6601E−03
floating_{50}	3.3594	−1.9779E−03	3.3604	2.7143E−03
floating_{∞}	3.4044	−2.0201E−03	3.3995	2.7438E−03

EXAMPLE 5.2 (Merton jump diffusion). Let

$$X_t = \gamma t + \sigma W_t + \sum_{j=1}^{N_t} Z_j,$$

where W is a Wiener process, N_t is a Poisson process with intensity λ and $Z_j \sim \mathcal{N}(m, \theta)$ and independent. In this instance, the characteristic exponent of X_t is

$$\psi(\xi) = i\gamma\xi - \tfrac{1}{2}\sigma^2\xi^2 + \lambda(e^{im\xi - \theta^2\xi^2/2} - 1),$$

where

$$\gamma = r - q - \tfrac{1}{2}\sigma^2 - \lambda(e^{m+\theta^2/2} - 1).$$

The joint characteristic function of $(X_t, \overline{X})$ is

$$\begin{aligned}
\varphi(\xi, \zeta; t) &= E[e^{i\xi X_t + i\zeta\overline{X}}] \\
&= \exp\Big\{i\gamma\Big(t\xi + \frac{1}{2}T\zeta\Big) - \frac{1}{2}\sigma^2\Big(t\xi^2 + t\Big(2 - \frac{t}{T}\Big)\xi\zeta + \frac{1}{3}T\zeta^2\Big)\Big\} \\
&\quad \times \exp\Big\{-T\lambda + T\lambda\frac{\sqrt{\pi}}{\sqrt{2}\theta\zeta}e^{-m^2/2\theta^2}\Big(\operatorname{erf}\Big\{\frac{\theta^2(\xi+\zeta) - im}{\sqrt{2}\theta}\Big\} \\
&\quad - \operatorname{erf}\Big\{\frac{\theta^2(T(\xi+\zeta) - t\zeta) - imT}{\sqrt{2}T\theta}\Big\} + i\operatorname{erfi}\Big\{\frac{m}{\sqrt{2}\theta}\Big\} \\
&\quad - i\operatorname{erfi}\Big\{\frac{i(T-t)\theta^2\zeta + mT}{\sqrt{2}T\theta}\Big\}\Big)\Big\},
\end{aligned}$$

which can be shown via Corollary 4.1 and with the assistance of a computer algebra system such as Mathematica. The joint characteristic function of $(X_t, \overline{Y})$ has a similar form and can be derived using Corollary 4.2 (see Alexander et al. [1]). The results are recorded in Table 2, where we have used the parameter values $\sigma = 0.15$, $\lambda = 1.75$, $m = -0.1$ and $\theta = 0.02$.

TABLE 2. Merton lower bounds.

	Lower bound		Monte Carlo	
Type	Price	$\underline{z}$	Price	Standard error
fixed_{10}	6.2351	−2.1542E−03	6.2411	4.8959E−03
fixed_{20}	6.0000	−2.1175E−03	6.0022	4.6724E−03
fixed_{50}	5.8581	−2.0638E−03	5.8444	4.5513E−03
fixed_{∞}	5.7634	−2.0241E−03	5.7730	4.4837E−03
floating_{10}	3.0162	−1.7237E−03	3.0191	3.5586E−03
floating_{20}	3.3056	−1.8945E−03	3.3054	3.9099E−03
floating_{50}	3.3748	−1.9643E−03	3.3702	3.9809E−03
floating_{∞}	3.4207	−2.0069E−03	3.4210	4.0428E−03

TABLE 3. NIG lower bounds.

	Lower bound		Monte Carlo	
Type	Price	$\underline{z}$	Price	Standard error
fixed_{10}	6.2121	−2.0568E−03	6.2140	4.3091E−03
fixed_{20}	5.9770	−2.0252E−03	5.9855	4.1544E−03
fixed_{50}	5.8356	−1.9741E−03	5.8324	4.0542E−03
fixed_{∞}	5.7413	−1.9362E−03	5.7411	3.9994E−03
floating_{10}	2.9820	−1.6723E−03	2.9791	2.4355E−03
floating_{20}	3.2685	−1.8430E−03	3.2645	2.6695E−03
floating_{50}	3.3368	−1.9108E−03	3.3316	2.7219E−03
floating_{∞}	3.3821	−1.9522E−03	3.3784	2.7560E−03

EXAMPLE 5.3 (Normal inverse Gaussian). Let $X_t = \gamma\tau_t + \sigma W_{\tau_t}$, where W is a Wiener process subordinated by the process $\tau_t \sim IG(t, t^2/\nu)$. The characteristic exponent of X_t is

$$\psi(\xi) = \frac{1 - \sqrt{1 + \nu\xi(-2i\gamma + \xi\sigma^2)}}{\nu} \quad \text{and} \quad \gamma = r - q - \frac{1}{2}\sigma^2 - \frac{1}{2}r^2\nu.$$

We used $\sigma = 0.2$ and $\nu = 0.025$ in our example and display the results in Table 3.

6. Conclusions

We provide a framework for calculating prices and deltas of Asian-type options through lower and upper bound approximations. The core idea in this paper is to replace the event that the option finishes in-the-money with a highly correlated proxy that is easier to deal with.

The main result is an extension of the method of Thompson [30] in pricing Asian options via lower bound approximations. This method can be used for a wide class of processes and in cases where the relevant joint densities of the underlying asset and the proxy processes are not known. We provide a means for calculating the characteristic

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functions of these distributions, in both continuously and discretely monitored cases, and a method involving Fourier techniques to compute the lower bound prices and delta approximations. These Fourier techniques use a two-dimensional version of the exponential damping method of Borovkov and Novikov [3]. Our numerical studies demonstrate higher accuracy compared to Monte Carlo outputs.

We also propose a novel upper bound in the case of GBM, which provides close approximations to option prices, though these approximations are not as accurate as the lower bounds. The upper bound is necessarily more complicated than the lower bound as it involves an "optionality" term analogous to, though easier to deal with than, the original pricing problem. Developing methods to efficiently compute these upper bounds is the subject of some future work.

The damping and Fourier methods themselves introduce numerical difficulties through the oscillatory nature of the integrands they involve. The degree of these difficulties is likely to be responsive to changes in the values of the damping parameters. These issues are currently being investigated.

The methodology and notation used to set up the problem are quite general and can be extended to other average-type options including volume weighted average price (VWAP) and, through vectorisation, basket options involving European or even Asian-type payoffs. This work is ongoing.

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