20
Diagrammatics

The diagrammatic method is one of the most powerful tools of theoretical physics. It allows us to efficiently organize perturbative computations in statistical physics, quantum many-body theory and quantum field theory. The main feature of this method is a representation of individual terms of a perturbative expansion as *diagrams* (*graphs*). Diagrams consist of *vertices* representing terms in the perturbation, *lines* representing pairings between vertices and, possibly, *external legs*.

There exist several kinds of diagrams. We will try to present them in a systematic way.

In Sect. 20.1 we present a diagrammatic formalism whose goal is to organize integration of polynomials with respect to a Gaussian measure. This formalism is used extensively in classical statistical physics. It also plays an important role in quantum physics, especially in the Euclidean approach, since many quantum quantities can be expressed in terms of Gaussian integrals over classical variables.

We use the term "Gaussian integration" in a rather broad sense. Beside commuting "bosonic" variables, we also consider anti-commuting "fermionic" variables, where we use the Berezin integral with respect to a Gaussian weight. Even in the case of commuting variables, the "Gaussian integral" is not necessarily meant in the sense of measure theory. It denotes an algebraic operation performed on polynomials (or formal power series), which in the case of a positive definite covariance coincides with the usual integral with a Gaussian weight. However, we allow the covariance to be complex, or even negative definite, and do not insist that the operation have a measure theoretic meaning.

We distinguish two kinds of spaces on which we perform the integrals: real and complex. As in many other places in our work, we treat these two cases in parallel. Of course, the difference between the real (i.e. neutral), and the complex (i.e. charged) formalism is mainly that of a different notation. In particular, charged lines need to be equipped with an arrow, whereas neutral lines need not.

The terminology that we use is inspired by quantum field theory. Therefore, the variables that enter the integral are associated with "particles"; they are divided into "bosons" and "fermions", each subdivided into "neutral" and "charged" particles.

In the main part of the chapter we describe the diagram formalism used in quantum many-body physics and quantum field theory. As a preparation, we include a brief Sect. 20.2 devoted to the basic terminology of perturbation theory

for quantum dynamics. We focus on the concept of the *scattering operator* and of the *energy shift of the ground state*.

We discuss first the situation of a time-dependent perturbation of a fixed free Hamiltonian. In this case, the usual scattering operator is guaranteed to exist, e.g. if the perturbation decays in time sufficiently fast.

If the perturbation is time-independent, one can still try to use the usual definition of the scattering operator. It is well known that this definition works well in quantum mechanics, where the free Hamiltonian is the Laplacian and the perturbation is a short-range potential. However, in quantum field theory the standard definition of the scattering operator is usually inapplicable, even on the level of formal expressions. This is related to the fact that the interacting Hamiltonian has a different ground state than the free Hamiltonian.

There exists a different formalism for scattering theory, which has more applicability and in some situations can be used in quantum field theory. The main idea of this formalism is the so-called *adiabatic switching of the interaction*. More precisely, we multiply the interaction with a time-dependent coupling constant $\mathrm{e}^{-\epsilon|t|}$ and introduce the scattering operator depending on the parameter ϵ. Then we take the limit of the scattering operator as $\epsilon \searrow 0$, dividing it by its expectation value with respect to a distinguished vector (typically, the non-interacting vacuum). This procedure is associated with the names of Gell-Mann and Low, and is usually (more or less implicitly) taken as the basic definition of the scattering operator in quantum field theory.

This procedure works, at least on the perturbative level, for sufficiently regular perturbations localized in space. If we assume that the perturbation is translation invariant, which is the usual assumption in quantum field theory, the situation becomes more complicated. In particular, one needs to perform the so-called wave function renormalization. We will not discuss this topic.

Starting with Sect. 20.3, we describe diagrams used in many-body quantum theory and quantum field theory. Our main aim is the computation of the scattering operator and the energy shift of the ground state.

It seems natural to divide diagrams into two categories. The first are the so-called *Friedrichs diagrams* and the second *Feynman diagrams*.

Friedrichs diagrams appear naturally when we want to compute the Wick symbol of a product of Wick-ordered operators. An algorithm for its computation is usually called the *Wick theorem*. It can be given a graphical interpretation, which we describe in Sect. 20.3.

In this formalism, a vertex represents a Wick monomial. It has two kinds of legs, those representing annihilation operators and those representing creation operators. We draw the former on the right of a vertex and the latter on the left.

A typical Hamiltonian in many-body quantum physics and in quantum field theory can be written as the sum of a quadratic term of the form $\mathrm{d}\Gamma(h)$ for some one-particle Hamiltonian h and an interaction given by a Wick polynomial. One can use Friedrichs diagrams to compute the scattering operator for such

Hamiltonians, as we describe in Sect. 20.4. A characteristic feature of this formalism is the presence of time labels on all vertices and the fact that diagrams with different time orderings are considered distinct.

Naively, this formalism seems very natural and physically intuitive. In fact, Friedrichs diagrams would provide natural illustrations for typical computations of the early years of quantum field theory (even though diagrams were apparently not used in that time). Weinberg calls a formalism essentially equivalent to that of Friedrichs diagrams the *old-fashioned perturbation theory*.

Since the late 1940s, a different diagram formalism has been developed. Since then it has dominated the calculations of quantum field theory. It originated especially in the work of Feynman, and therefore is called the formalism of *Feynman diagrams*. Again, the main goal is to compute the scattering operator for a Hamiltonian of the form $\mathrm{d}\Gamma(h)$ perturbed by a quantization of a Wick polynomial. It is convenient to express this perturbation using the neutral or charged formalism.

In Sect. 20.5. we describe Feynman diagrams used to compute the vacuum expectation value of the scattering operator. They can be essentially interpreted as a special case of the diagrams described in Sect. 20.1 used to compute Gaussian integrals. In this formalism, the order of times associated with individual vertices does not play any role. This allows us to cut down on the number of diagrams, as compared with Friedrichs diagrams. In the case of relativistic theories, each Feynman diagram is manifestly covariant, which is not the case for Friedrichs diagrams. Therefore, Feynman diagrams are usually preferred for practical computations over Friedrichs diagrams.

Feynman diagrams used to compute the Wick symbol of the scattering operator have in addition external legs. These legs are either incoming or outgoing. The former are then paired with creation operators and the latter with annihilation operators. Again, the temporal order of vertices is not relevant.

The main goal of this chapter is a formal description of the diagrammatic method. We will disregard the problems of convergence. We will often treat vector spaces as if they were finite-dimensional, even if in applications they are usually infinite-dimensional.

We try to describe the graphical method using a rigorous formalized language. This is perhaps not always the most natural thing to do. One can argue that an informal account involving a more colloquial language is more convenient in this context. Nevertheless, some readers may appreciate a formalized description.

20.1 Diagrams and Gaussian integration

In this section we present a diagrammatic formalism used to describe the integration and the Wick transformation w.r.t. a Gaussian measure.

We start with a description of purely graphical and combinatorial elements of the formalism. We will introduce the analytic part later.

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We will freely use basic terminology of set theory. In particular, we will always include the possibility that a set can be empty. $\#X$ will denote the number of elements of the set X. Recall also the following definition:

Definition 20.1 *Let $\{A_j\}_{j\in J}$ be a family of sets indexed by a set J. The* disjoint union of the sets A_j, $j \in J$, *is defined as*

$$\bigsqcup_{j\in J} A_j := \big\{(j,a) \ : \ j \in J,\ a \in A_j\big\}.$$

Let us stress that the notion of a disjoint union of sets does not coincide with the notion of the union of disjoint sets.

At some places in this section it will be convenient to totally order sets that we consider. In the case of bosonic particles, the end result does not depend on this ordering. For fermions, however, some quantities may depend on the order, but only modulo even permutations. In order to express this dependence, we introduce the following definition:

Definition 20.2 *Let A be a set of n elements. Let p, q be bijections $\{1,\dots,n\} \to A$. We say that they are equivalent if $q^{-1} \circ p$ is an even permutation. There are precisely two equivalence classes of this relation.*

We say that the set A is oriented *if one of these equivalence classes is chosen. This equivalence class is then called the* orientation of A. *We say that a total order of A is* admissible *if it is given by an element of the orientation of A.*

Let $\{A_i\}_{i\in I}$ be a finite family of oriented disjoint sets, each with an even number of elements. Then $\bigcup_{i\in I} A_i$ has a natural orientation.

20.1.1 Vertices

Definition 20.3 *Let* Pr *be a set. Its elements are called* (species of) particles. Pr *is subdivided into disjoint sets* $\mathrm{Pr_s}$ *and* $\mathrm{Pr_a}$, *whose elements are called* (species of) bosons, *resp.* fermions. *We assume that the set* $\mathrm{Pr_a}$ *is oriented. For $p \in \mathrm{Pr_s}$ we set $\epsilon_p = 1$, and for $p \in \mathrm{Pr_a}$ we set $\epsilon_p = -1$.*

Definition 20.4 $\mathrm{Pr_s}$, *resp.* $\mathrm{Pr_a}$ *are subdivided into disjoint sets*

$$\mathrm{Pr_s} = \mathrm{Pr_s^n} \cup \mathrm{Pr_s^c}, \quad \textit{resp.} \quad \mathrm{Pr_a} = \mathrm{Pr_a^n} \cup \mathrm{Pr_a^c}.$$

We set

$$\mathrm{Pr^n} := \mathrm{Pr_s^n} \cup \mathrm{Pr_a^n}, \quad \mathrm{Pr^c} := \mathrm{Pr_s^c} \cup \mathrm{Pr_a^c}.$$

Elements of $\mathrm{Pr^n}$, *resp.* $\mathrm{Pr^c}$ *are called* (species of) neutral, *resp.* charged particles.

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Definition 20.5 *A* multi-degree *is a triple of functions*

$$\begin{aligned}\mathrm{Pr}^{\mathrm{n}} \ni p &\mapsto m_p \in \{0,1,2,\dots\},\\ \mathrm{Pr}^{\mathrm{c}} \ni q &\mapsto m_q^{(+)} \in \{0,1,2,\dots\},\\ \mathrm{Pr}^{\mathrm{c}} \ni q &\mapsto m_q^{(-)} \in \{0,1,2,\dots\}.\end{aligned}$$

For brevity a multi-degree is typically denoted by a single letter, e.g. m.

We say that m *is* fermion-even *if*

$$\sum_{p\in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{a}}} m_p + \sum_{q\in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{a}}} \big(m_q^{(+)} + m_q^{(-)}\big) \quad \textit{is even.} \tag{20.1}$$

Definition 20.6 *A* vertex, *denoted* F, *is a finite set* $\mathrm{Lg}(F)$ *equipped with a map*

$$\mathrm{Lg}(F) \ni l \mapsto \mathrm{pr}(l) \in \mathrm{Pr} \tag{20.2}$$

and a partition into three disjoint subsets

$$\mathrm{Lg}^{\mathrm{n}}(F),\ \mathrm{Lg}^{(+)}(F),\ \mathrm{Lg}^{(-)}(F)$$

such that the image of $\mathrm{Lg}^{\mathrm{n}}(F)$ *under (20.2) is contained in* Pr^{n} *and the images of* $\mathrm{Lg}^{(+)}(F)$ *and* $\mathrm{Lg}^{(-)}(F)$ *are contained in* Pr^{c}.

Elements of $\mathrm{Lg}^{\mathrm{n}}(F)$ *are called* neutral legs of the vertex F. *Elements of* $\mathrm{Lg}^{(+)}(F)$, *resp.* $\mathrm{Lg}^{(-)}(F)$ *are called* charge creating, *resp.* charge annihilating legs of the vertex F.

We set

$$\begin{aligned}\mathrm{Lg}_p(F) &:= \big\{l \in \mathrm{Lg}^{\mathrm{n}}(F) \ :\ \mathrm{pr}(l) = p\big\},\quad p \in \mathrm{Pr}^{\mathrm{n}},\\ \mathrm{Lg}_q^{(\pm)}(F) &:= \big\{l \in \mathrm{Lg}^{(\pm)}(F) \ :\ \mathrm{pr}(l) = q\big\},\quad q \in \mathrm{Pr}^{\mathrm{c}}.\end{aligned}$$

We assume that the sets $\mathrm{Lg}_p(F)$, $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{a}}$, *and* $\mathrm{Lg}_q^{(\pm)}(F)$, $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{a}}$, *are oriented.*

The multi-degree of F *is defined by*

$$\begin{aligned}m_p(F) &:= \#\mathrm{Lg}_p(F),\quad p \in \mathrm{Pr}^{\mathrm{n}},\\ m_q^{(\pm)}(F) &:= \#\mathrm{Lg}_q^{(\pm)}(F),\quad q \in \mathrm{Pr}^{\mathrm{c}}.\end{aligned}$$

Graphically a leg is depicted by a line segment attached to the vertex at one end. The shape or the decoration of a leg corresponds to the particle type. For example, traditionally, photon legs are represented by wavy lines, while electron legs are represented by straight lines. Moreover lines corresponding to charge creating, resp. charge annihilating legs are decorated with an arrow pointing away, resp. towards the origin of the line. Neutral legs have no arrows at all.

A vertex F is depicted by a dot with the legs of $\mathrm{Lg}(F)$ originating at the dot. Each kind of a vertex is represented by a different dot.

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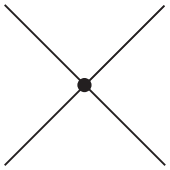

Figure 20.1 A vertex with four neutral legs.

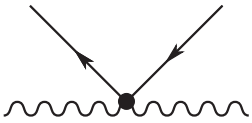

Figure 20.2 A vertex with two electron legs and two photon legs.

20.1.2 Diagrams

Vertices are linked with one another to form *diagrams*. A rigorous definition of a diagram (used in the Gaussian integration) is given below.

Definition 20.7 *Let* $\{F_j\}_{j\in J}$ *be a finite family of vertices. Set*

$$\mathrm{Lg}\Big(\prod_{j\in J} F_j\Big) := \bigsqcup_{j\in J} \mathrm{Lg}(F_j).$$

Elements of $\mathrm{Lg}\Big(\prod_{j\in J} F_j\Big)$ *are called* legs of $\prod_{j\in J} F_j$. *For* $\mathrm{l} = (j, l) \in \mathrm{Lg}\Big(\prod_{j\in J} F_j\Big)$ *we define* $\mathrm{nr}(\mathrm{l}) := j$ *and* $\mathrm{pr}(\mathrm{l}) := \mathrm{pr}(l)$.

$\mathrm{Lg}\Big(\prod_{j\in J} F_j\Big)$ *is the union of disjoint sets*

$$\mathrm{Lg}^{\pi}\Big(\prod_{j\in J} F_j\Big) := \bigsqcup_{j\in J} \mathrm{Lg}^{\pi}(F_j), \quad \pi = \mathrm{n}, (+), (-).$$

Elements of $\mathrm{Lg}^{\mathrm{n}}\Big(\prod_{j\in J} F_j\Big)$ *are called* neutral legs of $\prod_{j\in J} F_j$. *Elements of* $\mathrm{Lg}^{(+)}\Big(\prod_{j\in J} F_j\Big)$, *resp.* $\mathrm{Lg}^{(-)}\Big(\prod_{j\in J} F_j\Big)$ *are called* charge annihilating, *resp.* charge creating legs of $\prod_{j\in J} F_j$.

A labeled diagram over $\prod_{j\in J} F_j$ *is a pair* $D = (\mathrm{Lg}(D), \mathrm{Ln}(D))$, *where*

(1) $\mathrm{Lg}(D)$ *is a subset of* $\mathrm{Lg}\Big(\prod_{j\in J} F_j\Big)$, *whose elements are called* legs *or* external lines of D;

(2) $\mathrm{Ln}(D)$ *is a partition of* $\mathrm{Lg}\Big(\prod_{j\in J} F_j\Big)\backslash\mathrm{Lg}(D)$ *into pairs such that, if* $\{\mathrm{l}, \mathrm{l}'\} \in \mathrm{Ln}(D)$, *then*
   (i) $\mathrm{pr}(\mathrm{l}) = \mathrm{pr}(\mathrm{l}')$;
   (ii) *if* $\mathrm{l} \in \mathrm{Lg}^{(-)}\Big(\prod_{j\in J} F_j\Big)$, *then* $\mathrm{l}' \in \mathrm{Lg}^{(+)}\Big(\prod_{j\in J} F_j\Big)$.
   Pairs in $\mathrm{Ln}(D)$ *are called* links *or* internal lines of D.

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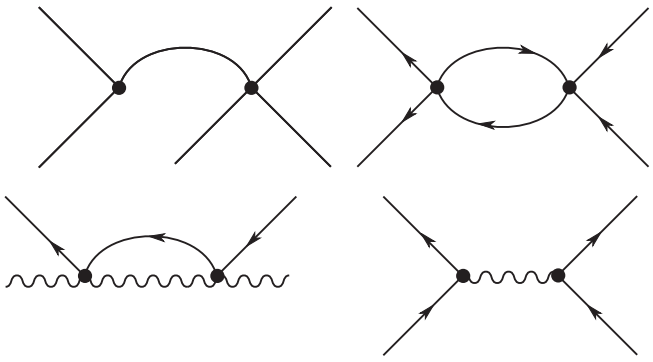

Figure 20.3 Various diagrams.

We set

$$\mathrm{Lg}^{\pi}(D) := \mathrm{Lg}(D) \cap \mathrm{Lg}^{\pi}\Big(\coprod_{j\in J} F_j\Big), \qquad \pi = \mathrm{n}, (+), (-).$$

We define $\mathrm{Ln}^{\mathrm{n}}(D)$ *to be the set of links consisting of neutral particles and* $\mathrm{Ln}^{\mathrm{c}}(D)$ *to be the set of links consisting of charged particles. Sometimes we will treat charged links as ordered pairs, writing* $(\mathrm{l}^{(+)}, \mathrm{l}^{(-)}) \in \mathrm{Ln}^{\mathrm{c}}(D)$ *with* $\mathrm{l}^{(\pm)} \in \mathrm{Lg}^{(\pm)}\Big(\coprod_{j\in J} F_j\Big)$.

We set

$$\mathrm{Lg}_p(D) := \big\{l \in \mathrm{Lg}^{\mathrm{n}}(D) \ : \ \mathrm{pr}(l) = p\big\}, \quad p \in \mathrm{Pr}^{\mathrm{n}},$$
$$\mathrm{Lg}_q^{(\pm)}(D) := \big\{l \in \mathrm{Lg}^{(\pm)}(D) \ : \ \mathrm{pr}(l) = q\big\}, \quad q \in \mathrm{Pr}^{\mathrm{c}}.$$

The multi-degree of D *is defined by*

$$m_p(D) := \#\mathrm{Lg}_p(D), \quad p \in \mathrm{Pr}^{\mathrm{n}},$$
$$m_q^{(\pm)}(D) := \#\mathrm{Lg}_q^{(\pm)}(D), \quad q \in \mathrm{Pr}^{\mathrm{c}}.$$

The number of vertices of D *is denoted by*

$$\mathrm{vert}(D) := \#J.$$

The set of all labeled diagrams over $\Big(\coprod_{j\in J} F_j\Big)$ *will be denoted* $\widetilde{\mathrm{Dg}}\Big(\coprod_{j\in J} F_j\Big)$.

Thus to draw a diagram over vertices $\{F_j\}_{j\in J}$, we first draw the vertices themselves, then join some of the legs. We are allowed to join only pairs of legs that belong to the same particle species. In the case of charged particles, we are only allowed to join a charge creating with a charge annihilating leg. Neutral lines have no arrow, whereas charged lines are decorated with an arrow.

Definition 20.8 *Let* D *be a diagram over* $\{F_j\}_{j\in J}$. *We say that* D has no self-lines *if* $\{\mathrm{l}, \mathrm{l}'\} \in \mathrm{Ln}(D)$ *implies* $\mathrm{nr}(\mathrm{l}) \neq \mathrm{nr}(\mathrm{l}')$. *The set of all labeled diagrams over* $\{F_j\}_{j\in J}$ *without self-lines will be denoted* $\mathrm{Dg}\Big(\coprod_{j\in J} F_j\Big)$.

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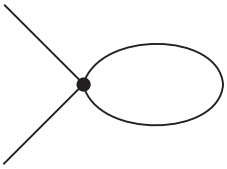

Figure 20.4 A diagram with a self-line.

Thus, in a diagram without self-lines, there are no lines that start and end at the same vertex.

Diagrams can be multiplied:

Definition 20.9 *Consider two diagrams* $D \in \widetilde{\mathrm{Dg}}\Big(\coprod\limits_{i\in I} F_i\Big)$ *and* $D' \in \widetilde{\mathrm{Dg}}\Big(\coprod\limits_{j\in J} F_j\Big)$. *Clearly,* $\mathrm{Lg}(D)$ *and* $\mathrm{Lg}(D')$ *can be considered as subsets of* $\mathrm{Lg}\Big(\coprod\limits_{i\in I} F_i \times \coprod\limits_{j\in J} F_j\Big)$. *Likewise,* $\mathrm{Ln}(D)$ *and* $\mathrm{Ln}(D')$ *can be considered as sets of pairs in* $\mathrm{Lg}\Big(\coprod\limits_{i\in I} F_i \times \coprod\limits_{j\in J} F_j\Big)$.

The product of D and D', *denoted* $DD' = D'D$, *is defined as the diagram over* $\coprod\limits_{i\in I} F_i \times \coprod\limits_{j\in J} F_j$ *such that*

$$\mathrm{Lg}(DD') := \mathrm{Lg}(D) \cup \mathrm{Lg}(D'), \qquad \mathrm{Ln}(DD') := \mathrm{Ln}(D) \cup \mathrm{Ln}(D').$$

Thus, graphically, multiplication of diagrams consists simply in their juxtaposition. Clearly, a product of diagrams with no self-lines is a diagram with no self-lines.

A vertex is an example of a diagram with no self-lines. The diagram whose set of legs equals $\mathrm{Lg}\Big(\coprod\limits_{j\in J} F_j\Big)$, and whose set of lines is empty equals $\coprod\limits_{j\in J} F_j$. This explains the notation used in Def. 20.7.

20.1.3 Connected diagrams

The following concepts have self-explanatory names.

Definition 20.10 *A diagram* $D \in \widetilde{\mathrm{Dg}}\Big(\coprod\limits_{j\in J} F_j\Big)$ *is called* connected *if for all* $j, j' \in J$ *there exist*

$$\{\mathrm{l}_n, \mathrm{l}'_n\}, \dots, \{\mathrm{l}_1, \mathrm{l}'_1\} \in \mathrm{Ln}(D)$$

such that $\mathrm{nr}(\mathrm{l}_n) = j$, $\mathrm{nr}(\mathrm{l}'_k) = \mathrm{nr}(\mathrm{l}_{k-1})$, $k = n, \dots, 1$, $\mathrm{nr}(\mathrm{l}'_1) = j'$.

For $A \subset \widetilde{\mathrm{Dg}}\Big(\coprod\limits_{j\in J} F_j\Big)$, *we set*

$$A_{\mathrm{con}} := \{D \in A \ : \ D \ \ \textit{is connected}\ \}.$$

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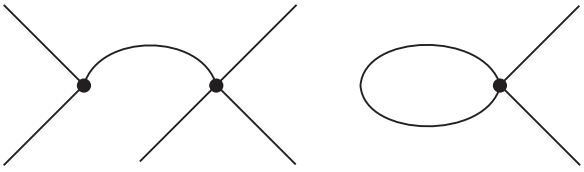

Figure 20.5 A disconnected diagram.

Clearly, each diagram can be decomposed into a product of connected diagrams. This decomposition is unique up to a permutation of factors.

Definition 20.11 *We say that a diagram* D has no external legs *if* $\mathrm{Lg}(D) = \emptyset$. *If* $A \subset \widetilde{\mathrm{Dg}}\Big(\coprod_{j\in J} F_j\Big)$, *then we set*

$$\begin{aligned} A_{\mathrm{nl}} &:= \{D \in A \ : \ D \ \ \textit{has no external legs}\},\\ A_{\mathrm{cnl}} &:= (A_{\mathrm{nl}})_{\mathrm{con}},\\ A_{\mathrm{link}} &:= (A\backslash A_{\mathrm{nl}})_{\mathrm{con}}. \end{aligned}$$

20.1.4 Particle spaces and Gaussian integration

Now we introduce the analytical part of the diagram formalism.

Definition 20.12 (1) *For any* $p \in \mathrm{Pr}^{\mathrm{n}}$, *let* $\mathcal{V}_p$ *be a real vector space equipped with a form* $\sigma_p \in L(\mathcal{V}_p^{\#}, \mathcal{V}_p)$, *where* σ_p *is non-degenerate symmetric if* $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{s}}$ *and non-degenerate anti-symmetric if* $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{a}}$. *We set*

$$\mathcal{V}^{\mathrm{n}} := \bigoplus_{p\in \mathrm{Pr}^{\mathrm{n}}} \mathcal{V}_p, \qquad \epsilon^{\mathrm{n}} := \bigoplus_{p\in \mathrm{Pr}^{\mathrm{n}}} \epsilon_p \mathbb{1}_{\mathcal{V}_p}, \qquad \sigma^{\mathrm{n}} := \bigoplus_{p\in \mathrm{Pr}^{\mathrm{n}}} \sigma_p.$$

(2) *For any* $q \in \mathrm{Pr}^{\mathrm{c}}$, *let* $\mathcal{V}_q$ *be a complex vector space equipped with a form* $\sigma_q \in L(\mathcal{V}_q^{\#}, \overline{\mathcal{V}}_q)$, *where* σ_q *is non-degenerate Hermitian if* $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{s}}$ *and non-degenerate anti-Hermitian if* $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{a}}$. *We set*

$$\mathcal{V}^{\mathrm{c}} := \bigoplus_{q\in \mathrm{Pr}^{\mathrm{c}}} \mathcal{V}_q, \qquad \epsilon^{\mathrm{c}} := \bigoplus_{q\in \mathrm{Pr}^{\mathrm{c}}} \epsilon_q \mathbb{1}_{\mathcal{V}_q}, \qquad \sigma^{\mathrm{c}} := \bigoplus_{q\in \mathrm{Pr}^{\mathrm{c}}} \sigma_q.$$

(3) *Set*

$$\mathcal{V} := \mathbb{C}\mathcal{V}^{\mathrm{n}} \oplus \mathcal{V}^{\mathrm{c}} \oplus \overline{\mathcal{V}}^{\mathrm{c}}, \qquad \epsilon = \epsilon^{\mathrm{n}}_{\mathbb{C}} \oplus \epsilon^{\mathrm{c}} \oplus \bar{\epsilon}^{\mathrm{c}}.$$

We will treat $\mathcal{V}^{\mathrm{n}}$, $\mathcal{V}^{\mathrm{c}}$ and $\mathcal{V}$ as super-spaces (see Subsect. 1.1.15). In particular, we can define the set of holomorphic polynomials over $\mathcal{V}$, denoted $\mathrm{Pol}(\mathcal{V})$. As usual, if $G \in \mathrm{Pol}(\mathcal{V})$, then $G(0)$ denotes the zero-th order component of G.

Definition 20.13 *For* $G \in \mathrm{Pol}(\mathcal{V})$, *we define*

$$\int G := \left(\exp\Big(\sum_{p\in \mathrm{Pr}^{\mathrm{n}}} \frac{1}{2} \nabla_{v_p} \sigma_p \nabla_{v_p} + \sum_{q\in \mathrm{Pr}^{\mathrm{c}}} \nabla_{\overline{v}_q} \sigma_q \nabla_{v_q} \Big) G \right)(0). \tag{20.3}$$

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Applying respectively the identities (4.15), (4.16) and (2) of Prop. 7.19, we obtain the following interpretation of (20.3):

(1) If $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{s}}$, $G \in \mathrm{Pol}_{\mathrm{s}}(\mathbb{C}\mathcal{V}_p)$ and σ_p is positive definite, then (20.3) coincides with the usual integral over $\mathcal{V}_p$ w.r.t. the probability Gaussian measure with covariance σ_p, that is,

$$\int G = C \int G(v_p) \mathrm{e}^{-\frac{1}{2} v_p \sigma_p^{-1} v_p} \mathrm{d}v_p.$$

(2) If $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{s}}$, $G \in \mathrm{Pol}_{\mathrm{s}}(\mathcal{V}_q \oplus \overline{\mathcal{V}}_q)$ and σ_q is positive definite, (20.3) coincides with the usual integral over $\mathcal{V}_{q,\mathbb{R}} \simeq \mathrm{Re}(\mathcal{V}_q \oplus \overline{\mathcal{V}}_q)$ w.r.t. the probability Gaussian measure with covariance σ_q, that is,

$$\int G = C \int G(v_q, \overline{v}_q) \mathrm{e}^{-\overline{v}_q \sigma_q^{-1} v_q} \mathrm{d}\overline{v}_q \mathrm{d}v_q.$$

(3) If $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{a}}$, $G \in \mathrm{Pol}_{\mathrm{a}}(\mathbb{C}\mathcal{V}_p)$, then (20.3) coincides with the Berezin integral with the weight $\mathrm{e}^{-\frac{1}{2} v_p \sigma_p^{-1} v_p}$, that is,

$$\int G = C \int G(v_p) \mathrm{e}^{-\frac{1}{2} v_p \sigma_p^{-1} v_p} \mathrm{d}v_p.$$

(4) If $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{a}}$, $G \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{V}_q \oplus \overline{\mathcal{V}}_q)$, then (20.3) coincides with the Berezin integral with the weight $\mathrm{e}^{-\overline{v}_q \sigma_q^{-1} v_q}$, that is,

$$\int G = C \int G(v_q, \overline{v}_q) \mathrm{e}^{-\overline{v}_q \sigma_q^{-1} v_q} \mathrm{d}\overline{v}_q \mathrm{d}v_q.$$

In all these cases, C is the normalizing constant.

Definition 20.14 *Define the* Wick transform of $G \in \mathrm{Pol}(\mathcal{V})$ *by*

$$:G: := \exp\Big(- \sum_{p \in \mathrm{Pr}^{\mathrm{n}}} \frac{1}{2} \nabla_{v_p} \sigma_p \nabla_{v_p} - \sum_{q \in \mathrm{Pr}^{\mathrm{c}}} \nabla_{\overline{v}_q} \sigma_q \nabla_{v_q} \Big) G. \tag{20.4}$$

If $G = :G_1:$*, then* G_1 *will be sometimes called the* Wick symbol of G.

Note that Def. 20.14 generalizes the Wick transform from Def. 9.18, where it was a construction closely related to the Gram–Schmidt orthogonalization.

Clearly,

$$G = \exp\Big(\sum_{p \in \mathrm{Pr}^{\mathrm{n}}} \frac{1}{2} \nabla_{v_p} \sigma_p \nabla_{v_p} + \sum_{q \in \mathrm{Pr}^{\mathrm{c}}} \nabla_{\overline{v}_q} \sigma_q \nabla_{v_q} \Big) :G:, \tag{20.5}$$

$$\int :G: = G(0).$$

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20.1.5 Monomials

Let m be a multi-degree.

Definition 20.15 *Set*

$$\otimes^m(\mathcal{V}) := \bigotimes_{p\in \mathrm{Pr^n}} \otimes^{m_p}(\mathbb{C}\mathcal{V}_p) \otimes \bigotimes_{q\in \mathrm{Pr^c}} \otimes^{m_q^{(+)}}(\overline{\mathcal{V}}_q) \otimes \bigotimes_{q\in \mathrm{Pr^c}} \otimes^{m_q^{(-)}}(\mathcal{V}_q), \tag{20.6}$$

$$\Gamma^m(\mathcal{V}) := \bigotimes_{p\in \mathrm{Pr^n}} \Gamma^{m_p}_{\epsilon_p}(\mathbb{C}\mathcal{V}_p) \otimes \bigotimes_{q\in \mathrm{Pr^c}} \Gamma^{m_q^{(+)}}_{\epsilon_q}(\overline{\mathcal{V}}_q) \otimes \bigotimes_{q\in \mathrm{Pr^c}} \Gamma^{m_q^{(-)}}_{\epsilon_q}(\mathcal{V}_q), \tag{20.7}$$

$$\mathrm{Pol}^m(\mathcal{V}) := \bigotimes_{p\in \mathrm{Pr^n}} \mathrm{Pol}^{m_p}_{\epsilon_p}(\mathbb{C}\mathcal{V}_p) \otimes \bigotimes_{q\in \mathrm{Pr^c}} \mathrm{Pol}^{m_q^{(+)}}_{\epsilon_q}(\overline{\mathcal{V}}_q) \otimes \bigotimes_{q\in \mathrm{Pr^c}} \mathrm{Pol}^{m_q^{(-)}}_{\epsilon_q}(\mathcal{V}_q). \tag{20.8}$$

Elements of $\mathrm{Pol}^m(\mathcal{V})$ *are called* complex polynomials of multi-degree m.

Clearly, $\Gamma^m(\mathcal{V}) \subset \otimes^m(\mathcal{V})$ and $\mathrm{Pol}^m(\mathcal{V}) = \Gamma^m(\mathcal{V})^{\#}$.

Definition 20.16

$$\Theta^m := \bigotimes_{p\in \mathrm{Pr^n}} \Theta^{m_p}_{\epsilon_p} \otimes \bigotimes_{q\in \mathrm{Pr^c}} \Theta^{m_q^{(+)}}_{\epsilon_q} \otimes \bigotimes_{q\in \mathrm{Pr^c}} \Theta^{m_q^{(-)}}_{\epsilon_q}$$

denotes the usual projection of $\otimes^m(\mathcal{V})$ *onto* $\Gamma^m(\mathcal{V})$. *Therefore,* $\Theta^{m\#}$ *is the usual projection of* $\otimes^m(\mathcal{V})^{\#}$ *onto* $\mathrm{Pol}^m(\mathcal{V})$.

Let F be a vertex with multi-degree m. With every leg of the vertex we associate a space

$$\begin{aligned} \mathcal{V}_{\mathrm{l}} &\simeq \mathcal{V}_p,\ \mathrm{l}\in \mathrm{Lg}_p(F),\quad p\in \mathrm{Pr^n},\\ \overline{\mathcal{V}}_{\mathrm{k}} &\simeq \overline{\mathcal{V}}_q, \mathrm{k}\in \mathrm{Lg}^{(+)}_q(F), q\in \mathrm{Pr^c},\\ \mathcal{V}_{\mathrm{k}} &\simeq \mathcal{V}_q, \mathrm{k}\in \mathrm{Lg}^{(-)}_q(F), q\in \mathrm{Pr^c}. \end{aligned} \tag{20.9}$$

Within each family

$$\begin{aligned} &\mathrm{Lg}_p(F), p\in \mathrm{Pr^n},\\ &\mathrm{Lg}^{(+)}_q(F), q\in \mathrm{Pr^c},\\ &\mathrm{Lg}^{(-)}_q(F), q\in \mathrm{Pr^c}, \end{aligned}$$

we label the legs by consecutive integers. For fermionic particles we assume that the numbering is admissible. Note that apart from this condition, the numbering is arbitrary and plays only an auxiliary role. Thanks to this numbering, we have a natural bijection between the set of legs of the vertex F and the factors of (20.6). Thus $\otimes^m(\mathcal{V})$ can be identified with

$$\bigotimes_{\mathrm{l}\in \mathrm{Lg^n}(F)} \mathcal{V}_{\mathrm{l}} \otimes \bigotimes_{\mathrm{k}\in \mathrm{Lg}^{(+)}(F)} \overline{\mathcal{V}}_{\mathrm{k}} \otimes \bigotimes_{\mathrm{k}\in \mathrm{Lg}^{(-)}(F)} \mathcal{V}_{\mathrm{k}}, \tag{20.10}$$

and an element of $\otimes^m(\mathcal{V})^{\#}$ can be viewed as a multi-linear function on

$$\prod_{\mathrm{l}\in \mathrm{Lg^n}(F)} \mathcal{V}_{\mathrm{l}} \times \prod_{\mathrm{k}\in \mathrm{Lg}^{(+)}(F)} \overline{\mathcal{V}}_{\mathrm{k}} \times \prod_{\mathrm{k}\in \mathrm{Lg}^{(-)}(F)} \mathcal{V}_{\mathrm{k}}. \tag{20.11}$$

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In this way we can associate with a monomial in $\mathrm{Pol}^m(\mathcal{V})$ a vertex F of the same multi-degree. Therefore, we will use the same letter F to denote a monomial and its associated vertex, and we will usually not distinguish between them.

It is convenient to adopt natural names of the corresponding generic variables:

$$\begin{aligned} v_{\mathrm{l}} &\quad \text{for the generic variable in } \mathcal{V}_{\mathrm{l}},\ \mathrm{l} \in \mathrm{Lg}^{\mathrm{n}}(F),\\ \overline{v}_{\mathrm{k}} &\quad \text{for the generic variable in } \overline{\mathcal{V}}_{\mathrm{k}},\ \mathrm{k} \in \mathrm{Lg}^{(+)}(F),\\ v_{\mathrm{k}} &\quad \text{for the generic variable in } \mathcal{V}_{\mathrm{k}},\ \mathrm{k} \in \mathrm{Lg}^{(-)}(F). \end{aligned}$$

20.1.6 Evaluation of diagrams

Let $\{F_j\}_{j\in J}$ be a family of fermion-even monomials. Let $D \in \widetilde{\mathrm{Dg}}\Big(\coprod_{j\in J} F_j\Big)$, that is, let D be a diagram over $\coprod_{j\in J} F_j$.

Definition 20.17 *The* evaluation of D *is an element of* $\mathrm{Pol}^{m(D)}(\mathcal{V})$ *denoted by the same symbol* D *and given by*

$$\begin{aligned} D := \Theta^{m(D)\#} & \prod_{\ell=\{\mathrm{l},\mathrm{l}'\}\in \mathrm{Ln}^{\mathrm{n}}(D)} \nabla_{v_{\mathrm{l}}} \sigma_\ell \nabla_{v_{\mathrm{l}'}} \\ & \times \prod_{\kappa=(\mathrm{k}^{(+)},\mathrm{k}^{(-)})\in \mathrm{Ln}^{\mathrm{c}}(D)} \nabla_{\overline{v}_{\mathrm{k}^{(-)}}} \sigma_\kappa \nabla_{v_{\mathrm{k}^{(+)}}} \prod_{j\in J} F_j. \end{aligned} \tag{20.12}$$

Here, if $\ell = \{\mathrm{l},\mathrm{l}'\} \in \mathrm{Lg}^{\mathrm{n}}(D)$ *and* $p = \mathrm{pr}(\mathrm{l}) = \mathrm{pr}(\mathrm{l}')$, *then* σ_ℓ *denotes* σ_p.

Likewise, if $\kappa = (\mathrm{k}^{(+)},\mathrm{k}^{(-)}) \in \mathrm{Lg}^{\mathrm{c}}(D)$ *and* $q = \mathrm{pr}(\mathrm{k}^{(+)}) = \mathrm{pr}(\mathrm{k}^{(-)})$, *then* σ_κ *denotes* σ_q.

(20.12) should be interpreted as follows:

(1) We treat $\coprod_{j\in J} F_j$ as a multi-linear function depending on the variables

$$\begin{aligned} v_{\mathrm{l}} &\in \mathcal{V}_{\mathrm{l}},\quad \mathrm{l} \in \mathrm{Lg}^{\mathrm{n}}\Big(\coprod_{j\in J} F_j\Big),\\ \overline{v}_{\mathrm{k}} &\in \overline{\mathcal{V}}_{\mathrm{k}},\quad \mathrm{k} \in \mathrm{Lg}^{(+)}\Big(\coprod_{j\in J} F_j\Big),\\ v_{\mathrm{k}} &\in \mathcal{V}_{\mathrm{k}},\quad \mathrm{k} \in \mathrm{Lg}^{(-)}\Big(\coprod_{j\in J} F_j\Big). \end{aligned}$$

(2) We perform the differentiation indicated in (20.12), which produces a multi-linear function depending on

$$\begin{aligned} v_{\mathrm{l}} &\in \mathcal{V}_{\mathrm{l}},\ \mathrm{l} \in \mathrm{Lg}^{\mathrm{n}}(D),\\ \overline{v}_{\mathrm{k}} &\in \overline{\mathcal{V}}_{\mathrm{k}},\ \mathrm{k} \in \mathrm{Lg}^{(+)}(D),\\ v_{\mathrm{k}} &\in \mathcal{V}_{\mathrm{k}},\ \mathrm{k} \in \mathrm{Lg}^{(-)}(D). \end{aligned}$$

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(3) We label the set J by consecutive integers in an arbitrary way. This gives an obvious ordering of the legs in each family

$$\begin{aligned} &\mathrm{Lg}_p\Big(\coprod_{j\in J} F_j\Big), p\in \mathrm{Pr}^{\mathrm{n}},\\ &\mathrm{Lg}_q^{(+)}\Big(\coprod_{j\in J} F_j\Big), q\in \mathrm{Pr}^{\mathrm{c}},\\ &\mathrm{Lg}_q^{(-)}\Big(\coprod_{j\in J} F_j\Big), q\in \mathrm{Pr}^{\mathrm{c}}. \end{aligned} \tag{20.13}$$

(4) We order the set of particles. The ordering of fermionic particles should be admissible.
(5) The ordering determined in the previous two points allows us to identify the result of differentiation with an element of $\otimes^{m(D)}(\mathcal{V})^{\#}$.
(6) We symmetrize/anti-symmetrize, obtaining an element of $\mathrm{Pol}^{m(D)}(\mathcal{V})$.

Note that if $D_1,\dots,D_n$ are diagrams and $D = D_1\cdots D_n$, then the evaluation of D equals the product of the evaluations of D_i, $i=1,\dots,n$.

Remark that if D has no external legs, then as a monomial it is a number, hence $:D:=D$.

Note that the group of permutations of J leaves invariant the monomial associated with diagrams in $\mathrm{Dg}\Big(\coprod_{j\in J} F_j\Big)$ because all monomials F_j are fermion-even.

20.1.7 Gaussian integration of products of monomials

The following theorem shows that diagrams can be efficiently used to compute the Wick symbol and the Gaussian integral of products of monomials. In the bosonic case, for a positive definite σ, (20.15) and (20.17) are graphical interpretations of Thm. 9.25.

Theorem 20.18 *Let $\{F_1,\dots,F_r\}$ be fermion-even monomials. Then*

$$F_n\cdots F_1 = \sum_{D\in\widetilde{\mathrm{Dg}}(F_n,\dots,F_1)} :D:, \tag{20.14}$$

$$:F_n:\cdots:F_1: = \sum_{D\in\mathrm{Dg}(F_n,\dots,F_1)} :D:, \tag{20.15}$$

$$\int F_n\cdots F_1 = \sum_{D\in\widetilde{\mathrm{Dg}}(F_n,\dots,F_1)_{\mathrm{n1}}} D, \tag{20.16}$$

$$\int :F_n:\cdots:F_1: = \sum_{D\in\mathrm{Dg}(F_n,\dots,F_1)_{\mathrm{n1}}} D. \tag{20.17}$$

Proof (20.14) is a restatement of (20.5) applied to $F_n\cdots F_1$, where we repeatedly use the formula (3.36).

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(20.15) follows from (20.5) applied to $:F_n:\cdots:F_1:$ and (20.4) applied to F_i.
(20.16), resp. (20.17) follow from (20.14), resp. (20.15) by (20.3). □

In Fig. 20.6, we illustrate the above theorem by the diagrams needed to evaluate the identities

$$:\phi^4:\,:\phi^4: = :\phi^8: + 4:\phi^6: + 72:\phi^4: + 96:\phi^2: + 24,$$

$$(2\pi)^{-1/2}\int :\phi^4:\,:\phi^4:\mathrm{e}^{-\frac{1}{2}\phi^2}\,\mathrm{d}\phi = 24.$$

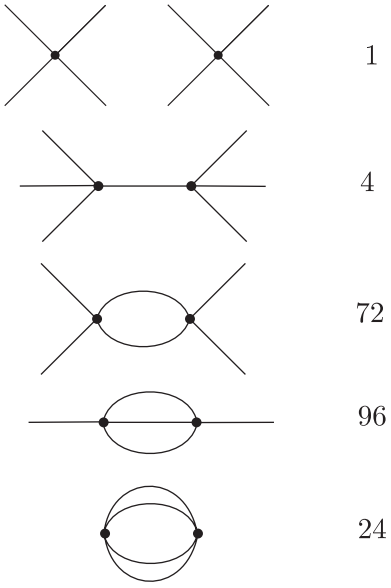

Figure 20.6 Diagrams for $:\phi^4:\,:\phi^4:$.

20.1.8 Identical diagrams

Let $\{F_r,\dots,F_1\}$ be a certain finite set of vertices. For brevity, this set will be denoted by $\mathfrak{V}$.

Recall that S_n denotes the group of permutations of n elements.

Suppose that $n_1,\dots,n_r \in \{0,1,\dots\}$. The group $\prod_{i=1}^{r} S_{n_i}$ acts in the obvious way on $\mathrm{Dg}\Big(\prod_{i=1}^{r} F_i^{n_i}\Big)$.

Definition 20.19 *We set*

$$\mathrm{Dg}\{\mathfrak{V}\} := \bigsqcup_{n_1,\dots,n_r=1}^{\infty} \mathrm{Dg}\Big(\prod_{i=1}^{r} F^{n_i}\Big)\Big/\prod_{i=1}^{r} S_{n_i}. \tag{20.18}$$

Elements of (20.18) will be called unlabeled diagrams with vertices in $\mathfrak{V}$.

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In other words, $\mathrm{Dg}\{\mathfrak{V}\}$ consists of classes of diagrams made with vertices from $\mathfrak{V}$, which differ just by a permutation. Typically, we will write $[D]$ for an unlabeled diagram, D being a labeled diagram and the square bracket denoting the equivalence class.

20.1.9 Gaussian integration of exponentials

Let $\mathfrak{V} = \{F_1, \dots, F_r\}$ be a set of fermion-even monomials and λ a coupling constant. Set $G := F_1 + \cdots + F_r$. Our main aim is to compute the Gaussian integral and the Wick symbol of $\exp(\lambda{:}G{:})$.

As indicated before, with each monomial F_i, $i = 1, \dots, r$, we associate a vertex of the same multi-degree, denoted by the same symbol F_i.

Let $D \in \mathrm{Dg}\Big(\prod_{i=1}^{r} F^{n_i}\Big)$. The evaluation of the diagram D (see Def. 20.17) does not depend on the action of the group $\prod_{i=1}^{r} S_{n_i}$. Hence, it is well defined for unlabeled diagrams.

Theorem 20.20

$$\exp(\lambda{:}G{:}) = \sum_{[D]\in \mathrm{Dg}\{\mathfrak{V}\}} \lambda^{\mathrm{vert}(D)} {:}D{:}. \tag{20.19}$$

Proof Clearly,

$$\begin{aligned}
\exp(\lambda{:}G{:}) &= \sum_{n_1,\dots,n_r=0}^{\infty} \frac{1}{n_1!\cdots n_r!} \lambda^{n_1+\cdots+n_r} {:}F_1{:}^{n_1} \cdots {:}F_r{:}^{n_r} \\
&= \sum_{n_1,\dots,n_r=0}^{\infty} \sum_{D\in \mathrm{Dg}(F_1^{n_1}\cdots F_r^{n_r})} \frac{\lambda^{\mathrm{vert}(D)}}{n_1!\cdots n_r!} {:}D{:} \\
&= \sum_{n_1,\dots,n_r=0}^{\infty} \sum_{[D]\in \mathrm{Dg}(F_1^{n_1}\cdots F_r^{n_r})/\prod_{i=1}^{r} S_{n_i}} \lambda^{\mathrm{vert}(D)} {:}D{:}.
\end{aligned}$$

□

Theorem 20.21 (Linked cluster theorem)

$$\exp(\lambda{:}G{:}) = {:}\exp\left(\sum_{[C]\in \mathrm{Dg}\{\mathfrak{V}\}_{\mathrm{con}}} \lambda^{\mathrm{vert}(C)} C\right){:}, \tag{20.20}$$

$$\log\Big(\int \exp(\lambda{:}G{:})\Big) = \sum_{[C]\in \mathrm{Dg}\{\mathfrak{V}\}_{\mathrm{cnl}}} \lambda^{\mathrm{vert}(C)} C, \tag{20.21}$$

$$\frac{\exp(\lambda{:}G{:})}{\int \exp(\lambda{:}G{:})} = {:}\exp\left(\sum_{[C]\in \mathrm{Dg}\{\mathfrak{V}\}_{\mathrm{link}}} \lambda^{\mathrm{vert}(C)} C\right){:}. \tag{20.22}$$

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Proof Let us prove (20.20). Let $C_j \in \mathrm{Dg}\Big(\prod_{i=1}^{r} F_i^{m_{ji}}\Big)$, $j = 1, \dots, p$, be distinct connected labeled diagrams. Let $k_j \in \{0, 1, \dots\}$, $j = 1, \dots, p$. Let $D := C_1^{k_1} \cdots C_p^{k_p} \in \mathrm{Dg}\Big(\prod_{i=1}^{r} F_i^{n_i}\Big)$. Clearly, $n_i = \sum_{j=1}^{p} m_{ji}$ and

$$\#[D] = \prod_{i=1}^{r} n_i!, \quad \#[C_j] = \prod_{i=1}^{r} m_{ji}!.$$

An elementary combinatorial argument shows that each diagram in $[D]$ represents $\prod_{i=1}^{r} n_i! \prod_{j=1}^{p} (m_{ji}!)^{-k_j} (k_j!)^{-1}$ times the same diagram in the Cartesian product $\prod_{j=1}^{p} [C_j]^{k_j}$. Therefore,

$$\begin{aligned} D &= \prod_{i=1}^{r} (n_i!)^{-1} \sum_{D' \in [D]} D' \\ &= \prod_{i=1}^{r} \prod_{j=1}^{p} (m_{ji}!)^{-k_j} (k_j!)^{-1} \prod_{l=1}^{k_j} \sum_{C'_{jl} \in [C_j]} C'_{jl} \\ &= \prod_{j=1}^{p} (k_j!)^{-1} C_j^{k_j}. \end{aligned}$$

Now (20.21) and (20.22) follow from (20.20). □

20.2 Perturbations of quantum dynamics

In this section we recall the terminology used in quantum physics in the context of a dynamics and its perturbations. We will consider first the case of time-dependent, and then time-independent perturbations.

We recall in particular the basic concepts of scattering theory. Its central notion is the scattering operator. There are several varieties of scattering operators. We recall the standard definition, which is successfully used in the context of Schrödinger operators with short-range potentials. Note, however, that the standard definition usually does not apply to quantum field theory, even on a formal level. We introduce also the adiabatic scattering operator, which is often used to develop the formalism of quantum field theory in standard textbooks.

Our presentation throughout this section will be rather formal. In order to make rigorous some of the formulas we give, one needs to make relatively complicated technical assumptions – we refrain from describing them.

Throughout the section $\mathcal{H}$ is a Hilbert space.

20.2.1 Time-ordered exponentials

Let $\mathbb{R} \ni t \mapsto B_i(t) \in B(\mathcal{H})$, $i = 1, \dots, n$, be time-dependent families of operators.

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Definition 20.22 *Let $t_n, \dots, t_1 \in \mathbb{R}$ be pairwise distinct. We define* Dyson's time-ordered product of $B_n(t_n)$,..., $B_1(t_1)$ *by*

$$\mathrm{T}\left(B_n(t_n)\cdots B_1(t_1)\right) := B_{i_n}(t_{i_n})\cdots B_{i_1}(t_{i_1}),$$

where $(i_n, \dots, i_1)$ is the permutation of $(n, \dots, 1)$ such that $t_{i_n} \geq \cdots \geq t_{i_1}$.

Consider now a single family of operators $\mathbb{R} \ni t \mapsto B(t)$.

Definition 20.23 *For $t_+ \geq t_-$, the* time-ordered exponential *is defined as*

$$\begin{aligned}\mathrm{Texp}\left(\int_{t_-}^{t_+} B(t)\mathrm{d}t\right) &:= \sum_{n=0}^{\infty} \int\limits_{t_+ \geq t_n \geq \cdots \geq t_1 \geq t_-} \cdots \int B(t_n)\cdots B(t_1)\mathrm{d}t_n \cdots \mathrm{d}t_1 \\ &= \sum_{n=0}^{\infty} \int_{t_-}^{t_+} \cdots \int_{t_-}^{t_+} \frac{1}{n!}\mathrm{T}\left(B(t_n)\cdots B(t_1)\right)\mathrm{d}t_n \cdots \mathrm{d}t_1.\end{aligned}$$

If $t_+ \leq t_-$, then we set

$$\begin{aligned}\mathrm{Texp}\left(\int_{t_-}^{t_+} B(t)\mathrm{d}t\right) &:= \left(\mathrm{Texp}\int_{t_+}^{t_-} B(t)\mathrm{d}t\right)^{-1} \\ &= \sum_{n=0}^{\infty} \int\limits_{t_- \leq t_1 \leq \cdots \leq t_n \leq t_+} \cdots \int (-1)^n B(t_1)\cdots B(t_n)\mathrm{d}t_1 \cdots \mathrm{d}t_n.\end{aligned}$$

For brevity, let us write

$$U(t_+, t_-) := \mathrm{Texp}\left(\int_{t_-}^{t_+} B(t)\mathrm{d}t\right).$$

Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_+}U(t_+, t_-) &= B(t_+)U(t_+, t_-), \\ \frac{\mathrm{d}}{\mathrm{d}t_-}U(t_+, t_-) &= -U(t_+, t_-)B(t_-), \\ U(t, t) &= \mathbb{1}, \\ U(t_2, t_1)U(t_1, t_0) &= U(t_2, t_0).\end{aligned}$$

If $B(t) = B$, then $U(t_+, t_-) = \mathrm{e}^{(t_+ - t_-)B}$.

20.2.2 Perturbation theory

Let H_0 be a self-adjoint operator. Let $\mathbb{R} \ni t \mapsto V(t)$ be a family of self-adjoint operators. Set $H(t) := H_0 + \lambda V(t)$. Consider the unitary evolution

$$U(t_+, t_-) = \mathrm{Texp}\left(-\mathrm{i}\int_{t_-}^{t_+} H(t)\mathrm{d}t\right).$$

Let $\mathbb{R} \ni t \mapsto A(t)$ be an operator-valued function.

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Definition 20.24 *The* operator $A(t)$ in the interaction picture *is defined as*

$$A_{\mathrm{I}}(t) := \mathrm{e}^{\mathrm{i}tH_0} A(t) \mathrm{e}^{-\mathrm{i}tH_0}.$$

The evolution in the interaction picture *is defined as*

$$U_{\mathrm{I}}(t_+, t_-) := \mathrm{e}^{\mathrm{i}t_+ H_0} U(t_+, t_-) \mathrm{e}^{-\mathrm{i}t_- H_0}.$$

Note that

$$U_{\mathrm{I}}(t_+, t_-) = \mathrm{Texp}\left(-\mathrm{i}\lambda \int_{t_-}^{t_+} V_{\mathrm{I}}(t)\mathrm{d}t\right).$$

In some cases we can take the limit $t_- \to -\infty$ or $t_+ \to \infty$. In particular this is the case if $V(t)$ decays in time sufficiently fast.

Definition 20.25 *The* Møller *or* wave operators*, resp. the* scattering operator *(if they exist) are defined as*

$$S^{\pm} := \mathrm{s} - \lim_{t \to \pm\infty} U_{\mathrm{I}}(0, t),$$
$$S = S^{+*} S^{-}.$$

Theorem 20.26 (1) *If $S^{\pm}$ exist, then they are isometric.*
(2) $S := \mathrm{w} - \lim_{(t_+, t_-) \to (+\infty, -\infty)} U_{\mathrm{I}}(t_+, t_-)$.
(3) *If* $\operatorname{Ran} S^{+} = \operatorname{Ran} S^{-}$*, then S is unitary.*

Note that the operators $U_{\mathrm{I}}(t_+, t_-)$, S^{-}, resp. S^{+*} can be viewed as special cases of the scattering operator, if we multiply $V(t)$ by $\mathbb{1}_{[t_-, t_+]}(t)$, $\mathbb{1}_{[-\infty, 0[}(t)$, resp. $\mathbb{1}_{]\infty, 0[}(t)$.

20.2.3 Standard Møller and scattering operators

Until the end of the section, H_0 and V are fixed self-adjoint operators and $H := H_0 + \lambda V$. We have $U(t_+, t_-) = \mathrm{e}^{-\mathrm{i}(t_+ - t_-)H}$ and $V_{\mathrm{I}}(t) = \mathrm{e}^{\mathrm{i}tH_0} V \mathrm{e}^{-\mathrm{i}tH_0}$.

Clearly, the Møller and scattering operators (if they exist) are

$$S^{\pm} = \mathrm{s} - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0},$$
$$S := \mathrm{w} - \lim_{(t_+, t_-) \to (+\infty, -\infty)} \mathrm{e}^{\mathrm{i}t_+ H_0} \mathrm{e}^{-\mathrm{i}(t_+ - t_-)H} \mathrm{e}^{-\mathrm{i}t_- H_0}.$$

Definition 20.27 *In what follows, we will call $S^{\pm}$ and S defined as above the* standard Møller *and* scattering operators.

Theorem 20.28 *Suppose that the standard Møller operators exist.*

(1) *The standard Møller operators satisfy $S^{\pm} H_0 = H S^{\pm}$.*
(2) *The standard scattering operator satisfies $H_0 S = S H_0$.*

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We have, at least formally,

$$S^+ = \mathrm{Texp}\left(-\int_{+\infty}^{0} \mathrm{i}\lambda V_{\mathrm{I}}(t)\mathrm{d}t\right),$$
$$S^- = \mathrm{Texp}\left(-\int_{-\infty}^{0} \mathrm{i}\lambda V_{\mathrm{I}}(t)\mathrm{d}t\right),$$
$$S = \mathrm{Texp}\left(-\int_{-\infty}^{+\infty} \mathrm{i}\lambda V_{\mathrm{I}}(t)\mathrm{d}t\right).$$

20.2.4 Stone formula

For $\epsilon > 0$, set

$$\delta_\epsilon(\xi) := \frac{\epsilon}{\pi}(\xi^2 + \epsilon^2)^{-1}.$$

which is a family of approximations of the delta function. For any $a < b$, we have the Stone formula,

$$\mathrm{s} - \lim_{\epsilon \searrow 0} \int_a^b \delta_\epsilon(\xi \mathbb{1} - H)\mathrm{d}\xi = \frac{1}{2}\left(\mathbb{1}_{[a,b]}(H) + \mathbb{1}_{]a,b[}(H)\right).$$

We will formally write

$$\delta(\xi\mathbb{1} - H_0) \quad \text{for} \quad \lim_{\epsilon \searrow 0} \delta_\epsilon(\xi\mathbb{1} - H),$$
$$\big((\xi \pm \mathrm{i}0)\mathbb{1} - H_0\big)^{-1} \quad \text{for} \quad \lim_{\epsilon \searrow 0}\big((\xi \pm \mathrm{i}\epsilon)\mathbb{1} - H_0\big)^{-1}.$$

These limits do not exist as bounded operators, but can sometimes be given a rigorous meaning as operators between appropriate weighted spaces.

20.2.5 Stationary formulas for Møller and scattering operators

We have the identities (see e.g. Yafaev (1992)):

$$S^\pm = \int \Big(\mathbb{1} + \big((\xi \mp \mathrm{i}0)\mathbb{1} - H\big)^{-1}\lambda V\Big)\delta(\xi\mathbb{1} - H_0)\mathrm{d}\xi$$
$$= \int \sum_{n=0}^{\infty} \lambda^n \Big(\big((\xi \mp \mathrm{i}0)\mathbb{1} - H_0\big)^{-1} V\Big)^n \delta(\xi\mathbb{1} - H_0)\mathrm{d}\xi.$$
$$S - \mathbb{1} = -2\pi \int \delta(\xi\mathbb{1} - H_0)\Big(\lambda V + \lambda^2 V\big((\xi + \mathrm{i}0)\mathbb{1} - H\big)^{-1} V\Big)\delta(\xi\mathbb{1} - H_0)\mathrm{d}\xi$$
$$= -2\pi \int \delta(\xi\mathbb{1} - H_0) \sum_{n=0}^{\infty} \lambda^n V\Big(\big((\xi + \mathrm{i}0)\mathbb{1} - H_0\big)^{-1} V\Big)^n \delta(\xi\mathbb{1} - H_0)\mathrm{d}\xi.$$

20.2.6 Problem with eigenvalues

It is easy to show the following fact:

Theorem 20.29 *If the standard Møller operators exist and $H_0\Psi = E\Psi$, then $H\Psi = E\Psi$.*

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In practice, the standard formalism of scattering theory is usually applied to Hamiltonians H_0 which only have absolutely continuous spectra. In such a case, Thm. 20.29 is irrelevant.

Thm. 20.29 becomes relevant in models inspired by quantum field theory. Suppose that the starting point of a model is a pair of Hamiltonians H_0 and H (possibly defined only on a formal level). In typical situations both Hamiltonians have a ground state, and these ground states are different. Thm. 20.29 then shows that the standard scattering theory is not applicable. Instead one can sometimes try other approaches, such as the adiabatic approach developed by Gell-Mann and Low, which we describe below.

20.2.7 Adiabatic dynamics

Definition 20.30 *The* adiabatically switched on interaction*, or simply the* adiabatic interaction*, is defined as* $V_\epsilon(t) := \mathrm{e}^{-\epsilon|t|}V$, $\epsilon > 0$. *The* adiabatic Hamiltonian *is* $H_\epsilon(t) := H_0 + \lambda V_\epsilon(t)$. *The corresponding dynamics is denoted by* $U_\epsilon(t_+, t_-)$ *and the corresponding dynamics in the interaction picture by* $U_{\epsilon\mathrm{I}}(t_+, t_-)$. *We also define the* adiabatic Møller *and* scattering operators

$$S_\epsilon^\pm := \mathrm{s} - \lim_{t\to\pm\infty} U_{\epsilon\mathrm{I}}(0,t),$$
$$S_\epsilon := S_\epsilon^+ S_\epsilon^{-*}.$$

Note that if the standard Møller operators exist and, if some mild additional assumptions hold, we have

$$S^\pm = \mathrm{s} - \lim_{\epsilon\searrow 0} S_\epsilon^\pm,$$
$$S = \mathrm{w} - \lim_{\epsilon\searrow 0} S_\epsilon.$$

As we argued in Subsect. 20.2.6, in quantum field theory the standard scattering theory usually fails. One needs to use non-standard definition of scattering operators. (Analogs of Møller operators are rarely used in quantum field theory anyway.) One possible modification of the definition of the scattering operator is given below. In this definition, Φ_0 is a distinguished unit vector, typically the ground state of H_0 (e.g. the free vacuum in quantum field theory).

Definition 20.31 *The* Gell-Mann–Low scattering operator *(if it exists) is*

$$S_{\mathrm{GL}} := \mathrm{w} - \lim_{\epsilon\searrow 0} \frac{S_\epsilon}{(\Phi_0|S_\epsilon\Phi_0)}.$$

20.2.8 Bound state energy

Suppose that Φ_0 and E_0, resp. Φ and E are eigenvectors and eigenvalues of H_0, resp H, so that

$$H_0\Phi_0 = E_0\Phi_0, \qquad H\Phi = E\Phi.$$

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We assume that Φ, E are small perturbations of Φ_0, E_0 when the coupling constant λ is small enough.

The following heuristic formulas can be sometimes rigorously proven:

$$E - E_0 = \lim_{t \to \pm\infty} \mathrm{i}^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \log(\Phi_0 | \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \Phi_0), \tag{20.23}$$

$$E - E_0 = \lim_{t \to \pm\infty} (2\mathrm{i})^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \log(\Phi_0 | \mathrm{e}^{-\mathrm{i}tH_0} \mathrm{e}^{\mathrm{i}2tH} \mathrm{e}^{-\mathrm{i}tH_0} \Phi_0). \tag{20.24}$$

To see why we can expect (20.23) to be true, we write

$$(\Phi_0 | \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \Phi_0) = |(\Phi_0 | \Phi)|^2 \mathrm{e}^{\mathrm{i}t(E-E_0)} + C(t).$$

Then, if we can argue that for large t the term $C(t)$ does not play a role, we obtain (20.23). (20.24) follows by essentially the same argument.

Note that (20.23) involves $\mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0}$, which can be called an approximate Møller operator, and (20.24) involves $\mathrm{e}^{-\mathrm{i}tH_0} \mathrm{e}^{\mathrm{i}2tH} \mathrm{e}^{-\mathrm{i}tH_0}$, an approximate scattering operator.

Still heuristic, but a little more satisfactory, are the formulas that give the energy shift in terms of the adiabatic Møller and scattering operators:

$$E - E_0 = \lim_{\epsilon \searrow 0} \mathrm{i}\epsilon\lambda\partial_\lambda \log(\Phi_0 | S_\epsilon^+ \Phi_0), \tag{20.25}$$

$$E - E_0 = \lim_{\epsilon \searrow 0} \frac{\mathrm{i}\epsilon\lambda}{2} \partial_\lambda \log(\Phi_0 | S_\epsilon \Phi_0). \tag{20.26}$$

(20.26) is called the *Sucher formula*.

20.3 Friedrichs diagrams and products of Wick monomials

The main aim of the remaining part of this chapter is to describe the perturbation theory for the dynamics of the form $\mathrm{d}\Gamma(h)$ plus the quantization of a (possibly time-dependent) Wick polynomial. We will describe two distinct formalisms for this purpose. In this and the next section we discuss the formalism of *Friedrichs diagrams*. The characteristic feature of these diagrams is the fact that the vertices are ordered in time.

One can argue that the formalism of Friedrichs diagrams was implicitly used in quantum field theory since its birth. Strangely, however, before the late 1940s it was not common to draw pictures to keep track of terms in the perturbation expansion. Apparently, the first to use pictorial representations of the perturbation theory was Stueckelberg and, on a larger scale, Feynman. Their diagrams, however, are different, and will be discussed in Sects. 20.5 and 20.6 under the name *Feynman diagrams*. In Feynman diagrams the order of the time label does not play a role, which usually is a serious advantage. Thus Feynman's invention is not limited to the use of pictorial diagrams. The idea of making pictures when one tries to do computations in perturbation theory is actually quite easy

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to come by. What was more important and less obvious was to group together various time orderings.

Even though Feynman diagrams dominate, especially in relativistic computations, Friedrichs diagrams are also useful in some situations. In particular, they can be used to compute leading singularities of certain terms of the perturbation expansion.

We divided the discussion of Friedrichs diagrams into two sections. In this section, the goal is to explain how to represent pictorially the symbol of the product of Wick monomials. In the next section we discuss how to compute the scattering operator using Friedrichs diagrams.

20.3.1 Friedrichs vertices

Just as in Sect. 20.1, we start with a description of purely graphical and combinatorial aspects of the formalism. It is quite similar to that of Sect. 20.1, and we will often use the same terms, sometimes with a slightly different meaning.

We assume that we have a set $\mathrm{Pr} = \mathrm{Pr_s} \sqcup \mathrm{Pr_a}$ describing particles, which are bosonic or fermionic, as in Def. 20.3. We assume that the set $\mathrm{Pr_a}$ is oriented.

We adapt the definition of the multi-degree to the context of this section.

Definition 20.32 *A* multi-degree *is a function*

$$\mathrm{Pr} \ni p \mapsto m_p \in \{0, 1, 2, \dots\}.$$

Definition 20.33 *A* Friedrichs vertex*, denoted W, is a pair of disjoint sets $\big(\mathrm{Lg}^+(W), \mathrm{Lg}^-(W)\big)$, each equipped with a function*

$$\mathrm{Lg}^{\pm}(W) \ni l \mapsto \mathrm{pr}(l) \in \mathrm{Pr}.$$

Elements of $\mathrm{Lg}^{\pm}(W)$ are called outgoing*, resp.* incoming legs of the vertex W*. The sets $\mathrm{Lg}^{\pm}_p(W)$, $p \in \mathrm{Pr_a}$, are oriented.*

The outgoing*, resp.* incoming multi-degree of W *is given by*

$$m^{\pm}_p(W) := \#\{l \in \mathrm{Lg}^{\pm}(W) \ : \ \mathrm{pr}(l) = p\}, \quad p \in \mathrm{Pr}.$$

We say that W is fermion-even *if*

$$\sum_{p \in \mathrm{Pr_a}} \big(m^+_p(W) + m^-_p(W)\big) \quad \textit{is even.} \tag{20.27}$$

A Friedrichs vertex W is graphically depicted by a dot with legs of $\mathrm{Lg}(W)$ originating at the dot. Incoming legs are on the right and the outgoing legs are on the left. Again, legs for different particle types are depicted by different graphical styles.

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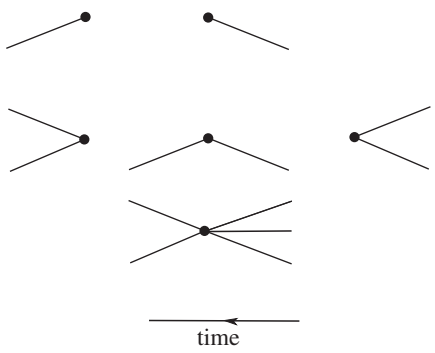

Figure 20.7 Various Friedrichs vertices.

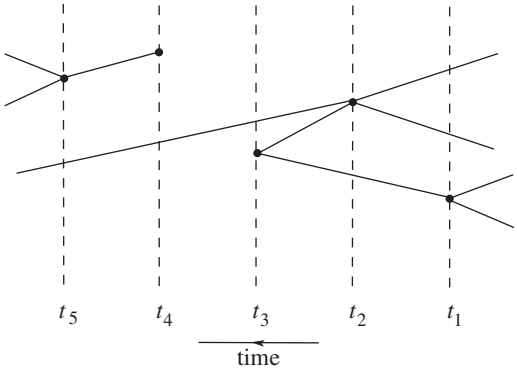

Figure 20.8 A disconnected Friedrichs diagram.

20.3.2 Friedrichs diagrams

Definition 20.34 *Let* $(W_n, \dots, W_1)$ *be a sequence of Friedrichs vertices. Set*

$$\mathrm{Lg}(W_n, \dots, W_1) := \bigsqcup_{n \geq j \geq 1} \mathrm{Lg}(W_j),$$
$$\mathrm{Lg}^{\pm}(W_n, \dots, W_1) := \bigsqcup_{n \geq j \geq 1} \mathrm{Lg}^{\pm}(W_j).$$

Clearly, $\mathrm{Lg}(W_n, \dots, W_1)$ *is the union of disjoint sets*

$$\mathrm{Lg}^{+}(W_n, \dots, W_1), \quad \mathrm{Lg}^{-}(W_n, \dots, W_1).$$

Elements of $\mathrm{Lg}^{\pm}(W_n, \dots, W_1)$ *are called* incoming, *resp.* outgoing legs of $(W_n, \cdots, W_1)$. *For* $\mathrm{l} = (j, k) \in \mathrm{Lg}^{\pm}(W_n, \dots, W_1)$ *we define* $\mathrm{nr}(\mathrm{l}) = j$ *and* $\mathrm{pr}(\mathrm{l}) = \mathrm{pr}(k)$. *(Note that* $j \in \{n, \dots, 1\}$ *and* $k \in \mathrm{Lg}(W_j)$*; see Def. 20.1.)*

A Friedrichs diagram *B* over $(W_n, \cdots, W_1)$ *is a triple* $\big(\mathrm{Lg}^{-}(B), \mathrm{Lg}^{+}(B), \mathrm{Ln}(B)\big)$, *where*

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(1) $\mathrm{Lg}^-(B)$ *is a subset of* $\mathrm{Lg}^-(W_n,\dots,W_1)$*;*
(2) $\mathrm{Lg}^+(B)$ *is a subset of* $\mathrm{Lg}^+(W_n,\dots,W_1)$*;*
(3) $\mathrm{Ln}(B)$ *is a partition of*

$$\big(\mathrm{Lg}^+(W_n,\dots,W_1)\backslash\mathrm{Lg}^+(B)\big)\cup\big(\mathrm{Lg}^-(W_n,\dots,W_1)\backslash\mathrm{Lg}^-(B)\big)$$

into two-element sets such that
(i) $\{\mathrm{l}_+,\mathrm{l}_-\}\in\mathrm{Ln}(B)$ *implies* $\mathrm{nr}(\mathrm{l}_+)\neq\mathrm{nr}(\mathrm{l}_-)$*,*
(ii) *if* $\{\mathrm{l}_+,\mathrm{l}_-\}\in\mathrm{Ln}(B)$ *and* $\mathrm{nr}(\mathrm{l}_+)>\mathrm{nr}(\mathrm{l}_-)$*, then* $\mathrm{l}_+\in\mathrm{Lg}^-(B)$*,* $\mathrm{l}_-\in\mathrm{Lg}^+(B)$*,*
(iii) $\{\mathrm{l}_+,\mathrm{l}_-\}\in\mathrm{Ln}(B)$ *implies* $\mathrm{pr}(\mathrm{l}_+)=\mathrm{pr}(\mathrm{l}_-)$*.*

The incoming *and* outgoing multi-degree of B *is defined by*

$$m_p^{\pm}(B):=\#\big\{\mathrm{l}\in\mathrm{Lg}^{\pm}(B)\ :\ \mathrm{pr}(\mathrm{l})=p\big\},\ p\in\mathrm{Pr}.$$

The number of vertices *of* B *is denoted by* $\mathrm{vert}(B)=n$*. The set of all Friedrichs diagrams over* $(W_n,\dots,W_1)$ *will be denoted* $\mathrm{FDg}(W_n,\dots,W_1)$*.*

Thus, to draw a Friedrichs diagram we first put the Friedrichs vertices in the correct order, and then we join some of the outgoing legs with later incoming legs of the same particle species.

Remark 20.35 *Note that the vertices in a Friedrichs diagram are* ordered*, contrarily to those appearing in Subsect. 20.1.2. Typically each vertex is associated with a time and vertices are ordered according to increasing times. To our knowledge, in the literature one can find three conventions concerning the time arrow in a diagram: time flows to the left, right or upwards. We adopt the convention that time flows to the left, because it agrees with the order of multiplication of operators.*

20.3.3 Connected Friedrichs diagrams

The following definitions are very similar to the analogous definitions of Sect. 20.1.

Definition 20.36 *A Friedrichs diagram* B *is called* connected *if for all* $j,j'\in\{n,\dots,1\}$ *there exist*

$$\{\mathrm{l}_m,\mathrm{l}'_m\},\dots,\{\mathrm{l}_1,\mathrm{l}'_1\}\in\mathrm{Ln}(B) \tag{20.28}$$

such that $\mathrm{nr}(\mathrm{l}_m)=j$*,* $\mathrm{nr}(\mathrm{l}'_k)=\mathrm{nr}(\mathrm{l}_{k-1})$*,* $k=m,\dots,2$*,* $\mathrm{nr}(\mathrm{l}'_1)=j'$*.*

If $A\subset\mathrm{FDg}(W_n,\dots,W_1)$*, then we set*

$$A_{\mathrm{con}}:=\{B\in A\ :\ B\ \ \textit{is connected}\ \}.$$

Note that the sequence of lines in (20.28) does not have to be ordered in time.

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Definition 20.37 *We say that a Friedrichs diagram* B has no external legs *if* $\mathrm{Lg}^+(B) = \mathrm{Lg}^-(B) = \emptyset$.

If $A \subset \mathrm{FDg}(W_n, \dots, W_1)$, *then we set*

$$\begin{aligned} A_{\mathrm{nl}} &:= \{B \in A \ : \ B \ \textit{ has no external legs}\}, \\ A_{\mathrm{cnl}} &:= (A_{\mathrm{nl}})_{\mathrm{con}}, \\ A_{\mathrm{link}} &= (A \backslash A_{\mathrm{nl}})_{\mathrm{con}}. \end{aligned}$$

20.3.4 One-particle spaces

Definition 20.38 *With each* $p \in \mathrm{Pr}$ *we associate a complex Hilbert space* $\mathcal{Z}_p$. $\mathcal{Z}_p$ *is called the* one-particle space of p. *We set*

$$\mathcal{Z} := \bigoplus_{p \in \mathrm{Pr}} \mathcal{Z}_p, \quad \epsilon = \bigoplus_{p \in \mathrm{Pr}} \epsilon_p \mathbb{1}_{\mathcal{Z}_p}.$$

We treat $(\mathcal{Z}, \epsilon)$ as a super-space.

Let $Q \in B^{\mathrm{fin}}(\mathcal{Z})$. Its Wick quantization, denoted as usual $\mathrm{Op}^{a^*, a}(Q)$, is an operator on the Fock space

$$\bigotimes_{p \in \mathrm{Pr}} \Gamma_{\epsilon_p}(\mathcal{Z}_p) \simeq \Gamma(\mathcal{Z}).$$

20.3.5 Incoming and outgoing diagram spaces

Let m be a multi-degree.

Definition 20.39 *We set*

$$\otimes^m \mathcal{Z} := \bigotimes_{p \in \mathrm{Pr}} \otimes^{m_p} \mathcal{Z}_p, \tag{20.29}$$

$$\Gamma^m(\mathcal{Z}) := \bigotimes_{p \in \mathrm{Pr}} \Gamma^{m_p}_{\epsilon_p}(\mathcal{Z}_p), \tag{20.30}$$

$$\mathrm{Pol}^m(\mathcal{Z}) := \bigotimes_{p \in \mathrm{Pr}} \mathrm{Pol}^{m_p}_{\epsilon_p}(\mathcal{Z}_p). \tag{20.31}$$

$\Gamma^m(\mathcal{Z})$ *is called the* m-particle space. *Let* Θ^m *denote the usual projection from* $\otimes^m \mathcal{Z}$ *onto* $\Gamma^m(\mathcal{Z})$.

Let (m^+, m^-) be a pair of multi-degrees. An important role will be played by

$$B\big(\Gamma^{m^-}(\mathcal{Z}), \Gamma^{m^+}(\mathcal{Z})\big). \tag{20.32}$$

(20.32) will sometimes be interpreted as a polynomial in

$$\mathrm{Pol}^{m^-}(\overline{\mathcal{Z}}) \otimes \mathrm{Pol}^{m^+}(\mathcal{Z}).$$

More precisely, with W that belongs to (20.32) we associate

$$\begin{aligned} &W\Big(\{\overline{z}_{i,+,p}\}_{i=1,\dots,m^+_p, p \in \mathrm{Pr}}, \{z_{j,-,q}\}_{j=1,\dots,m^-_q, q \in \mathrm{Pr}}\Big) \\ &= \Big(\bigotimes_{p \in \mathrm{Pr}} \bigotimes_{i=1}^{m^+_p} z_{i,+,p} \Big| W \bigotimes_{q \in \mathrm{Pr}} \bigotimes_{j=1}^{m^-_p} z_{j,-,q} \Big). \end{aligned}$$

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We will sometimes view (20.32) as a subspace of $B(\otimes^{m^-}\mathcal{Z}, \otimes^{m^+}\mathcal{Z})$: an element of W of (20.32) is extended to an operator on $\otimes^{m^-}\mathcal{Z}$ by setting 0 on the orthogonal complement of $\Gamma^{m^-}(\mathcal{Z})$.

Let W be a vertex of multi-degrees (m^+, m^-). With every leg of the vertex we associate a space

$$\mathcal{Z}_{\mathrm{l}} \simeq \mathcal{Z}_p, \quad \mathrm{l} \in \mathrm{Lg}_p^{\pm}(W)\ , \quad p \in \mathrm{Pr}. \tag{20.33}$$

Within each family $\mathrm{Lg}_p^{\pm}(W)$ we order the legs by consecutive integers. For fermionic particles we assume that the numbering is admissible. Note that apart from this condition, the numbering is arbitrary and plays only an auxiliary role. Thanks to this numbering, we have a natural bijection between $\mathrm{Lg}^{\pm}(W)$ and the factors of $\otimes^{m^{\pm}}\mathcal{Z}$. Thus $\otimes^{m^{\pm}}\mathcal{Z}$ can be identified with

$$\bigotimes_{\mathrm{l}\in\mathrm{Lg}^{\pm}(W)} \mathcal{Z}_{\mathrm{l}}. \tag{20.34}$$

Consequently, $B(\otimes^{m^-}\mathcal{Z}, \otimes^{m^+}\mathcal{Z})$ can be identified with

$$B\Big(\bigotimes_{\mathrm{l}\in\mathrm{Lg}^{+}(W)} \mathcal{Z}_{\mathrm{l}}, \bigotimes_{\mathrm{k}\in\mathrm{Lg}^{-}(W)} \mathcal{Z}_{\mathrm{k}}\Big). \tag{20.35}$$

Elements of (20.35) can be viewed as multi-linear functions on

$$\prod_{\mathrm{l}\in\mathrm{Lg}^{+}(W)} \overline{\mathcal{Z}}_{\mathrm{l}} \times \prod_{\mathrm{k}\in\mathrm{Lg}^{-}(W)} \mathcal{Z}_{\mathrm{k}}. \tag{20.36}$$

Indeed, consider an element of (20.35), denoted also by W. We associate with it a function

$$W\Big(\{\overline{z}_{\mathrm{l}}\}_{\mathrm{l}\in\mathrm{Lg}^{+}(W)}, \{z_{\mathrm{k}}\}_{\mathrm{k}\in\mathrm{Lg}^{-}(W)}\Big) := \Big(\bigotimes_{\mathrm{l}\in\mathrm{Lg}^{+}(W)} z_{\mathrm{l}}\Big|W\bigotimes_{\mathrm{k}\in\mathrm{Lg}^{-}(W)} z_{\mathrm{k}}\Big), \tag{20.37}$$

where $z_{\mathrm{l}} \in \mathcal{Z}_{\mathrm{l}^+}$, $z_{\mathrm{k}} \in \mathcal{Z}_{\mathrm{l}^-}$.

20.3.6 Evaluation of a Friedrichs diagram

Let $W_i \in B\big(\Gamma^{m_i^-}(\mathcal{Z}), \Gamma^{m_i^+}(\mathcal{Z})\big)$, $i = n, \dots, 1$, be a sequence of Wick monomials. Let B be a Friedrichs diagram over $W_n \cdots W_1$ with $m^{\pm} = m^{\pm}(B)$.

Definition 20.40 *The* evaluation of the Friedrichs diagram B, *usually denoted by the same letter B, is defined by*

$$B := \overline{\Theta^{m^+}}^{\#}\,\Theta^{m^-\#} \prod_{\{\mathrm{l}^+,\mathrm{l}^-\}\in\mathrm{Ln}(B)} \nabla_{\overline{z}_{\mathrm{l}^+}}\cdot\nabla_{z_{\mathrm{l}^-}} \prod_{j=n}^{1} W_j. \tag{20.38}$$

The above definition uses the polynomial interpretation of a Friedrichs vertex. W_j are treated as polynomials, as in (20.37).

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(20.38) thus defines a multi-linear function with variables $\overline{z}_{\mathrm{l}} \in \overline{\mathcal{Z}}_{\mathrm{l}}$, $\mathrm{l} \in \mathrm{Lg}^{+}(W_n, \dots, W_1)$, $\overline{z}_{\mathrm{l}} \in \overline{\mathcal{Z}}_{\mathrm{l}}$, $\mathrm{l} \in \mathrm{Lg}^{-}(W_n, \dots, W_1)$. The differential operator kills some of the variables; only those in $\mathrm{Lg}^{\pm}(B)$ survive. Then we apply the symmetrization/anti-symmetrization operators.

It is also possible to give an equivalent definition that uses a purely operator language.

Definition 20.41 *For $j = 1, \dots, n$, we set*

$$\begin{aligned}\mathrm{L}^{j}(B) = &\big\{\mathrm{l} \in \mathrm{Lg}^{+}(B) \ : \ j > \mathrm{nr}(\mathrm{l})\big\}\\ &\cup\big\{\mathrm{l} \in \mathrm{Lg}^{-}(B) \ : \ \mathrm{nr}(\mathrm{l}) > j\big\}\\ &\cup\big\{(\mathrm{l}_{+}, \mathrm{l}_{-}) \in \mathrm{Ln}(B) \ : \ \mathrm{nr}(\mathrm{l}_{+}) > j > \mathrm{nr}(\mathrm{l}_{-})\big\}.\end{aligned}$$

$\mathrm{L}^{j}(B)$ *is called the* set of lines bypassing the vertex W_j.

Note that

$$\begin{aligned}\mathrm{Lg}^{-}(B) &= \mathrm{L}^{1}(B) \cup \mathrm{Lg}^{-}(W_1),\\ \mathrm{Lg}^{+}(B) &= \mathrm{L}^{n}(B) \cup \mathrm{Lg}^{+}(W_n).\end{aligned}$$

Definition 20.42 *For each $\ell = \{\mathrm{l}_{+}, \mathrm{l}_{-}\} \in \mathrm{Ln}(B)$ with $\mathrm{nr}(\mathrm{l}_{+}) > \mathrm{nr}(\mathrm{l}_{-})$, let $\mathcal{Z}_{\ell}$ denote the space $\mathcal{Z}_{\mathrm{l}_{-}} \simeq \mathcal{Z}_{\mathrm{l}_{+}}$. Let $\mathbb{1}_{B}^{j}$ denote the identity on*

$$\bigotimes_{\ell \in \mathrm{L}^{j}(B)} \mathcal{Z}_{\ell}.$$

Let W_j be interpreted as operators in

$$B\Big(\bigotimes_{\mathrm{l} \in \mathrm{Lg}^{-}(W_i)} \mathcal{Z}_{\mathrm{l}}, \bigotimes_{\mathrm{l} \in \mathrm{Lg}^{+}(W_i)} \mathcal{Z}_{\mathrm{l}}\Big).$$

In the operator language, the diagram B can be computed as

$$B = \Theta^{m^{+}} \left(W_n \otimes \mathbb{1}_{B}^{n}\right) \cdots \left(W_1 \otimes \mathbb{1}_{B}^{1}\right) \Theta^{m^{-}}.$$

20.3.7 Products of operators

Theorem 20.43

$$\begin{aligned}\mathrm{Op}^{a^{*},a}(W_n) \cdots \mathrm{Op}^{a^{*},a}(W_1) &= \sum_{B \in \mathrm{FDg}(W_n, \dots, W_1)} \mathrm{Op}^{a^{*},a}(B),\\ \big(\Omega \big| \mathrm{Op}^{a^{*},a}(W_n) \cdots \mathrm{Op}^{a^{*},a}(W_1) \Omega\big) &= \sum_{B \in \mathrm{FDg}(W_n, \dots, W_1)_{\mathrm{n1}}} B.\end{aligned}$$

Proof This is essentially a restatement of Thm. 9.36. □

This theorem describes a method of computing the Wick symbol of a product of operators. We first draw the Friedrichs vertices in the appropriate order. Then

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we draw all possible diagrams by joining the legs. Next we evaluate the diagrams, pairing the external legs with creation and annihilation operators. Finally, we sum up all the contributions.

To compute the vacuum expectation value, we do the same steps, except that we consider only diagrams without external legs.

20.4 Friedrichs diagrams and the scattering operator

In this section we describe how to use the formalism of Friedrichs diagrams to compute two quantities useful for many-body quantum theory and quantum field theory: the scattering operator and the energy shift. From the point of view of diagrams, the new feature is the time label that will appear on each vertex. We will always demand that the order of vertices is consistent with the order of time labels.

Throughout the section we keep the terminology and notation of the previous section.

20.4.1 Multiplication of Friedrichs diagrams

Definition 20.44 *A pair consisting of a Friedrichs vertex and $t \in \mathbb{R}$ will be called a* time-labeled Friedrichs vertex. *It will be typically denoted $W(t)$. A sequence $\big(W_n(t_n), \dots, W_1(t_1)\big)$ of time-labeled Friedrichs vertices is* time-ordered *if $t_n > \dots > t_1$.*

We will consider only time-ordered sequences of time-labeled Friedrichs vertices.

Definition 20.45 *Consider two sequences of time-labeled Friedrichs vertices $\big(W_n(t_n), \dots, W_1(t_1)\big)$ and $\big(W'_m(t'_m), \dots, W'_1(t'_1)\big)$. Assume that none of $t_n, \dots, t_1$ coincides with $t'_m, \dots, t'_1$. Let $(s_{m+n}, \dots, s_1)$ be the union of*

$$\{t_n, \dots, t_1\}, \quad \{t'_m, \dots, t'_1\}$$

in decreasing order. Let $(Q_{n+m}(s_{m+n}), \dots, Q_1(s_1))$ be the time-ordered union of

$$\big(W_n(t_n), \dots, W_1(t_1)\big), \quad \big(W'_m(t'_m), \dots, W'_1(t'_1)\big).$$

Note that we obtain an identification of $\mathrm{Lg}^{\pm}\big(W_n(t_n), \dots, W_1(t_1)\big)$ and $\mathrm{Lg}^{\pm}\big(W'_m(t'_m), \dots, W'_1(t'_1)\big)$ with complementary subsets of $\mathrm{Lg}^{\pm}\big(Q_{n+m}(s_{m+n}), \dots, Q_1(s_1)\big)$.

Consider two Friedrichs diagrams

$$B \in \mathrm{FDg}\big(W_n(t_n), \dots, W_1(t_1)\big), \quad B' \in \mathrm{FDg}\big(W'_m(t'_m), \dots, W'_1(t'_1)\big).$$

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$BB' = B'B$ is defined as the Friedrichs diagram in $\mathrm{FDg}\big(Q_{n+m}(s_{m+n}), \dots, Q_1(s_1)\big)$ *such that*

$$\mathrm{Lg}^{\pm}(BB') = \mathrm{Lg}^{\pm}(B) \cup \mathrm{Lg}^{\pm}(B'), \quad \mathrm{Ln}(BB') = \mathrm{Ln}(B) \cup \mathrm{Ln}(B').$$

20.4.2 One-particle dynamics

Let h_p be a self-adjoint operator on $\mathcal{Z}_p$. Set $h := \bigoplus_{p \in \mathrm{Pr}} h_p$ as an operator on $\mathcal{Z}$. Let

$$H_0 = \mathrm{d}\Gamma(h). \tag{20.39}$$

Note that if W is a Wick monomial, then

$$\mathrm{e}^{\mathrm{i}tH_0}\,\mathrm{Op}^{a^*,a}(W)\mathrm{e}^{-\mathrm{i}tH_0} = \mathrm{Op}^{a^*,a}\left(\mathrm{e}^{\mathrm{i}tH_0} W \mathrm{e}^{-\mathrm{i}tH_0}\right), \tag{20.40}$$

where on the right we interpret W as a Wick operator.

Let B be a Friedrichs diagram and $\mathrm{l} \in \mathrm{Lg}^{\pm}(B)$. Then h_{l} will denote $h_{\mathrm{pr(l)}}$, understood as an operator on $\mathcal{Z}_{\mathrm{l}}$. Similarly, if $\ell = (\mathrm{l}^{(+)}, \mathrm{l}^{(-)}) \in \mathrm{Ln}(B)$, and $p = \mathrm{pr}(\mathrm{l}^{(+)}) = \mathrm{pr}(\mathrm{l}^{(-)})$, then h_ℓ denotes h_p understood as an operator from $\mathcal{Z}_{\mathrm{l}^{(-)}}$ to $\mathcal{Z}_{\mathrm{l}^{(+)}}$.

We will sometimes use the symbol H_0 in a meaning slightly different from (20.39).

Definition 20.46 *Let L be a subset of $\mathrm{Lg}^{+}(B) \cup \mathrm{Lg}^{-}(B) \cup \mathrm{Ln}(B)$. Then H_0, understood as an operator on the space $\bigotimes_{\ell \in L} \mathcal{Z}_\ell$, will denote the operator $\sum_{\ell \in L} h_\ell$.*

20.4.3 Time-dependent Wick monomials

Suppose that $\mathbb{R} \ni t \mapsto W_j(t)$, $j = 1, \dots, r$, are fermion-even Wick monomials depending on time, each with a fixed multi-degree. We represent each W_j with a vertex, independent of the time t but distinct for distinct indices j.

We modify the prescription (20.38).

Definition 20.47 *The* evaluation of the diagram B at times $t_n, \dots, t_1 \in \mathbb{R}$ *is*

$$\begin{aligned} B(t_n, \dots, t_1) = &\prod_{\mathrm{l} \in \mathrm{Lg}_+(B)} \mathrm{e}^{\mathrm{i}t_{\mathrm{nr(l)}} h_{\mathrm{l}}} \prod_{\mathrm{l} \in \mathrm{Lg}_-(B)} \mathrm{e}^{-\mathrm{i}t_{\mathrm{nr(l)}} \overline{h}_{\mathrm{l}}} \\ &\times \prod_{\ell = \{\mathrm{l}_+, \mathrm{l}_-\} \in \mathrm{Ln}(B)} \nabla_{\overline{z}_{\mathrm{l}_+}} \cdot \mathrm{e}^{\mathrm{i}(t_{\mathrm{nr}(\mathrm{l}_+)} - t_{\mathrm{nr}(\mathrm{l}_-)})h_\ell} \nabla_{z_{\mathrm{l}_-}} \prod_{j=n}^{1} W_j(t_j). \end{aligned}$$

In the operator language we have

$$\begin{aligned} B(t_n, \dots, t_1) = \Theta_B^+ \mathrm{e}^{\mathrm{i}t_n H_0} \left(W_n(t_n) \otimes \mathbb{1}_B^n\right) \mathrm{e}^{-\mathrm{i}(t_n - t_{n-1})H_0} \dots \\ \times \mathrm{e}^{-\mathrm{i}(t_2 - t_1)H_0} \left(W_1(t_1) \otimes \mathbb{1}_B^1\right) \mathrm{e}^{-\mathrm{i}t_1 H_0} \Theta_B^-. \end{aligned}$$

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The following identity follows immediately from Thm. 20.43 and (20.40).

Theorem 20.48

$$\mathrm{e}^{\mathrm{i}t_n H_0}\,\mathrm{Op}^{a^*,a}\big(W_n(t_n)\big)\mathrm{e}^{-\mathrm{i}(t_n-t_{n-1})H_0}\cdots\mathrm{e}^{-\mathrm{i}(t_2-t_1)H_0}\,\mathrm{Op}^{a^*,a}\big(W_1(t_1)\big)\mathrm{e}^{-\mathrm{i}t_1 H_0}$$
$$=\sum_{B\in\mathrm{FDg}(W_n,\dots,W_1)}\mathrm{Op}^{a^*,a}\big(B(t_n,\dots,t_1)\big).$$

20.4.4 Diagrams for the scattering operator

Set

$$Q(t):=W_1(t)+\cdots+W_r(t),$$
$$H(t):=H_0+\lambda\mathrm{Op}^{a^*,a}\big(Q(t)\big),$$

where $W_j(t)$ are self-adjoint Wick monomials. Our main goal is to describe a method of computing the scattering operator S for H_0 and $\{H(t)\}_{t\in\mathbb{R}}$ (see Def. 20.25). Recall that

$$S=\mathrm{Texp}\Big(-\mathrm{i}\lambda\int\mathrm{Op}^{a^*,a}\big(Q(t)\big)_{\mathrm{I}}\mathrm{d}t\Big),\tag{20.41}$$

where

$$\mathrm{Op}^{a^*,a}\big(Q(t)\big)_{\mathrm{I}}:=\mathrm{e}^{\mathrm{i}tH_0}\,\mathrm{Op}^{a^*,a}\big(Q(t)\big)\mathrm{e}^{-\mathrm{i}tH_0}=\mathrm{Op}^{a^*,a}(Q_{\mathrm{I}}(t)),$$

with $Q_{\mathrm{I}}(t)=\Gamma(\mathrm{e}^{\mathrm{i}th})Q(t)\Gamma(\mathrm{e}^{-\mathrm{i}th})$.

We denote by W_j the Friedrichs vertices in the sense of Def. 20.33 corresponding to the Wick monomials $W_j(t)$. We also often need to use the corresponding time-labeled vertices, which we denote $W_j(t)$.

For brevity, we will denote by $\mathfrak{W}$ the set of vertices $\{W_1,\dots,W_r\}$.

Definition 20.49 *We introduce the notation*

$$\mathrm{FDg}_n\{\mathfrak{W}\}:=\bigcup_{(j_n,\dots,j_1)\in\{1,\dots,r\}^n}\mathrm{FDg}(W_{j_n}\cdots W_{j_1}).\tag{20.42}$$

Note that $\mathrm{FDg}(W_{j_n}\cdots W_{j_1})$ are disjoint for distinct sequences $(j_n,\dots,j_1)\in\{1,\dots,r\}^n$. Therefore, the union in (20.42) involves disjoint sets.

Note also that when we evaluate a diagram in (20.42) on the monomials $W_i(t_i)$, we obtain a function that depends on $t_n,\dots,t_1\in\mathbb{R}$.

The following theorem follows easily from (20.41) and Thm. 20.43:

Theorem 20.50

$$S=\sum_{n=0}^{\infty}\sum_{B\in\mathrm{FDg}_n\{\mathfrak{W}\}}(-\mathrm{i}\lambda)^n\int_{t_n>\cdots>t_1}\cdots\int\mathrm{Op}^{a^*,a}\big(B(t_n,\dots,t_1)\big)\mathrm{d}t_n\cdots\mathrm{d}t_1.$$

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The above theorem is an analog of Thm. 20.20 about the Gaussian integration. Recall that Thm. 20.20 implies Thm. 20.21, the linked cluster theorem for the Gaussian integration. Thm. 20.50 has an analogous consequence:

Theorem 20.51 (Linked cluster theorem for Friedrichs diagrams)

$$S = \mathrm{Op}^{a^*,a}\left(\exp\left(\sum_{n=0}^{\infty}\sum_{B\in \mathrm{FDg}_n\{\mathfrak{W}\}_{\mathrm{con}}}(-\mathrm{i}\lambda)^n \times \int_{t_n>\cdots>t_1}\cdots\int B(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\right)\right)$$

$$\log(\Omega|S\Omega) = \sum_{n=0}^{\infty}\sum_{B\in \mathrm{FDg}_n\{\mathfrak{W}\}_{\mathrm{cnl}}}(-\mathrm{i}\lambda)^n \int_{t_n>\cdots>t_1}\cdots\int B(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1,$$

$$\frac{S}{(\Omega|S\Omega)} = \mathrm{Op}^{a^*,a}\left(\exp\left(\sum_{n=0}^{\infty}\sum_{B\in \mathrm{FDg}_n\{\mathfrak{W}\}_{\mathrm{link}}}(-\mathrm{i}\lambda)^n \times \int_{t_n>\cdots>t_1}\cdots\int B(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\right)\right).$$

20.4.5 Stationary evaluation of a diagram

Let us now assume that the monomials $W_i(t) = W_i$ do not depend on time and $Q = W_1 + \cdots + W_r$. Let $H := H_0 + \lambda \mathrm{Op}^{a^*,a}(Q)$.

Let $\xi \in \mathbb{C}\backslash \mathrm{spec}\,(H_0)$.

Definition 20.52 *For a diagram* $B \in \mathrm{FDg}(W_n,\ldots,W_1)$ *and* $\xi\in\mathbb{C}$ *we define its* stationary evaluation *as*

$$B[\xi] := \Theta_B^+ \left(W_n \otimes \mathbb{1}_B^n\right)(\xi\mathbb{1} - H_0)^{-1}\cdots(\xi\mathbb{1} - H_0)^{-1}\left(W_1\otimes\mathbb{1}_B^1\right)\Theta_B^-.$$

20.4.6 Scattering operator for a time-independent Hamiltonian

The Gell-Mann–Low scattering operator

$$S_{\mathrm{GL}} = \lim_{\epsilon\searrow 0}(\Omega|S_\epsilon^+\Omega)^{-1}S_\epsilon$$

is often used as the starting point for computations in quantum field theory. In the following theorem we give two expressions for this operator: a time-dependent one and a stationary one. Note that the division by $(\Omega|S_\epsilon^+\Omega)$ removes diagrams without external legs, which if non-zero would give a divergent contribution.

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Theorem 20.53

$$S_{\mathrm{GL}} = \mathrm{Op}^{a^*,a}\left(\exp\left(\sum_{n=0}^{\infty}\sum_{B\in \mathrm{FDg}_n\{\mathfrak{W}\}_{\mathrm{link}}}(-\mathrm{i}\lambda)^n\int_{t_n>\cdots>t_1}\cdots\int B(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\right)\right)$$

$$= \mathrm{Op}^{a^*,a}\left(\exp\left(-2\mathrm{i}\pi\sum_{n=0}^{\infty}\sum_{B\in \mathrm{FDg}_n\{\mathfrak{W}\}_{\mathrm{link}}}\lambda^n \times\int\delta(H_0-\xi\mathbb{1})B[\xi+\mathrm{i}0]\delta(H_0-\xi\mathbb{1})\mathrm{d}\xi\right)\right).$$

Proof The first identity follows from Thm. 20.51. Next we compute the integrand using the operator interpretation of $B(t_n,\ldots,t_1)$:

$$\begin{aligned}B(t_n,\ldots,t_1) &= \Theta_B^+\mathrm{e}^{\mathrm{i}t_nH_0}\left(W_n\otimes\mathbb{1}_B^n\right)\mathrm{e}^{-\mathrm{i}(t_n-t_{n-1})H_0}\cdots\\ &\quad\times\mathrm{e}^{-\mathrm{i}(t_2-t_1)H_0}\left(W_1\otimes\mathbb{1}_B^1\right)\mathrm{e}^{-\mathrm{i}t_1H_0}\Theta_B^-\\ &= \int\delta(H_0-\xi\mathbb{1})\mathrm{d}\xi\Theta_B^+\left(W_n\otimes\mathbb{1}_B^n\right)\mathrm{e}^{-\mathrm{i}u_n(H_0-\xi\mathbb{1})}\cdots\\ &\quad\times\mathrm{e}^{-\mathrm{i}u_2(H_0-\xi\mathbb{1})}\left(W_1\otimes\mathbb{1}_B^1\right)\mathrm{e}^{-\mathrm{i}t_1(H_0-\xi\mathbb{1})}\Theta_B^-,\end{aligned}$$

where

$$u_n := t_n - t_{n-1},\ldots,\ u_2 := t_2 - t_1.$$

Now

$$\begin{aligned}&\int_{t_n>\cdots>t_1}\cdots\int B(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\\ &= \int\mathrm{d}\xi\int_0^\infty\mathrm{d}u_n\cdots\int_0^\infty\mathrm{d}u_1\int_{-\infty}^\infty\mathrm{d}t_1\delta(H_0-\xi\mathbb{1})\Theta_B^+\left(W_n\otimes\mathbb{1}_B^n\right)\mathrm{e}^{-\mathrm{i}u_n(H_0-\xi\mathbb{1})}\cdots\\ &\quad\times\mathrm{e}^{-\mathrm{i}u_2(H_0-\xi\mathbb{1})}\left(W_1\otimes\mathbb{1}_B^1\right)\mathrm{e}^{-\mathrm{i}t_1(H_0-\xi\mathbb{1})}\Theta_B^-\\ &= 2\pi(-\mathrm{i})^{n-1}\int\mathrm{d}\xi\delta(H_0-\xi\mathbb{1})\Theta_B^+\left(W_n\otimes\mathbb{1}_B^n\right)\big(H_0-(\xi-\mathrm{i}0)\mathbb{1}\big)^{-1}\cdots\\ &\quad\times\big(H_0-(\xi-\mathrm{i}0)\mathbb{1}\big)^{-1}\left(W_1\otimes\mathbb{1}_B^1\right)\delta(H_0-\xi\mathbb{1})\Theta_B^-,\end{aligned}$$

where we have used the heuristic relations

$$\int_0^{+\infty}\mathrm{e}^{\mathrm{i}u(H_0-\xi\mathbb{1})}\mathrm{d}u = \mathrm{i}\big(H_0-(\xi+\mathrm{i}0)\mathbb{1}\big)^{-1}, \tag{20.43}$$

$$\int_{-\infty}^0\mathrm{e}^{\mathrm{i}u(H_0-\xi\mathbb{1})}\mathrm{d}u = -\mathrm{i}\big(H_0-(\xi-\mathrm{i}0)\mathbb{1}\big)^{-1}, \tag{20.44}$$

$$\int\mathrm{e}^{\mathrm{i}t(H_0-\xi\mathbb{1})}\mathrm{d}t = 2\pi\delta(H_0-\xi\mathbb{1}). \tag{20.45}$$

□

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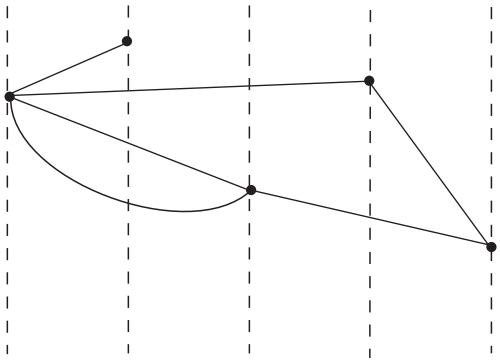

Figure 20.9 Goldstone diagram.

20.4.7 Goldstone theorem

Recall that E denotes the ground-state energy of H, that is, $E := \inf \operatorname{spec} H$. We assume that we can use the heuristic formula for the energy shift

$$E = \lim_{t \to -\infty} \mathrm{i}^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \log(\Omega | \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \Omega), \tag{20.46}$$

which follows from (20.23) if we note that $E_0 = 0$. Then we can derive the following diagrammatic expansion for the energy E:

Theorem 20.54 (Goldstone theorem)

$$E = \sum_{n=0}^{\infty} \sum_{B \in \mathrm{FDg}_n \{\mathfrak{W}\}_{\mathrm{cnl}}} \lambda^n B[0].$$

Proof As explained at the end of Subsect. 20.2.2, $\mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0}$ for $t < 0$ is the scattering operator for the time-dependent perturbation $s \mapsto \lambda \mathbb{1}_{[t,0]}(s) \mathrm{Op}^{a^*,a}(Q)$. Applying Thm. 20.51, we get

$$\begin{aligned} &\log(\Omega | \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \Omega) \\ &\quad = \sum_{n=0}^{\infty} \sum_{B \in \mathrm{FDg}_n \{\mathfrak{W}\}_{\mathrm{cnl}}} (-\mathrm{i}\lambda)^n \int_{0 > t_n > \cdots > t_1 > t} \cdots \int B(t_n, \ldots, t_1) \mathrm{d}t_n \cdots \mathrm{d}t_1. \end{aligned}$$

So

$$\begin{aligned} &\mathrm{i}^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \log(\Omega | \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \Omega) \\ &\quad = \sum_{n=0}^{\infty} \sum_{B \in \mathrm{FDg}_n \{\mathfrak{W}\}_{\mathrm{cnl}}} \mathrm{i}(-\mathrm{i}\lambda)^n \int_{0 > t_n > \cdots > t_2 > t} \cdots \int B(t_n, \ldots, t_2, t) \mathrm{d}t_n \cdots \mathrm{d}t_2. \end{aligned}$$

Now introduce

$$u_2 := t - t_2, \ldots, u_n := t_{n-1} - t_n.$$

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Then $u_2, \dots, u_n \leq 0$, $t \leq u_2 + \cdots + u_n \leq 0$, and

$$\begin{aligned} B(t_n, \dots, t_2, t) &= W_n \mathrm{e}^{-\mathrm{i}(t_n - t_{n-1})H_0} \left(W_{n-1} \otimes \mathbb{1}_B^{n-1}\right) \cdots \\ &\quad \times \left(W_2 \otimes \mathbb{1}_B^2\right) \mathrm{e}^{-\mathrm{i}(t_2 - t)H_0} W_1 \\ &= W_n \mathrm{e}^{\mathrm{i}u_n H_0} \left(W_{n-1} \otimes \mathbb{1}_B^{n-1}\right) \cdots \\ &\quad \times \left(W_2 \otimes \mathbb{1}_B^2\right) \mathrm{e}^{\mathrm{i}u_2 H_0} W_1 . \end{aligned}$$

Then we replace t by $-\infty$ and evaluate the integral using the heuristic relation (20.44). □

Note that an identical expansion can be derived from Sucher's formula,

$$E = \lim_{\epsilon \searrow 0} \frac{\mathrm{i}\epsilon\lambda}{2} \partial_\lambda \log(\Omega | S_\epsilon \Omega). \tag{20.47}$$

20.5 Feynman diagrams and vacuum expectation value

We continue to study diagrammatic expansions of many-body quantum physics and quantum field theory. Until the end of this chapter we will, however, use diagrams different from those of the previous two sections: the so-called *Feynman diagrams*. They are closely related to the diagrams discussed in Sect. 20.1. The main topic of that section was integration w.r.t. a Gaussian measure. This includes in particular the Euclidean quantum field theory. We will see that the formalism of Sect. 20.1 can be adapted to compute scattering operators in many-body quantum theory and quantum field theory. In practice, Feynman diagrams are usually preferred over the Friedrichs diagrams of Sects. 20.3 and 20.4. Their main advantages are a smaller number of diagrams and, in the case of relativistic theories, manifest Lorentz covariance of each diagram.

The main idea in passing from Friedrichs diagrams to Feynman diagrams consists in combining the evolution going forwards and backwards in time in a single line. It is done in a different way for neutral and charged particles. The starting point of the formalism is usually a classical system, neutral or charged, described by its dual phase space $\mathcal{Y}_p$. The one-particle space $\mathcal{Z}_p$ is introduced in the standard way, as explained in Chap. 18. In the case of neutral particles, the lines with both time directions are combined into one unoriented line. In the case of charged lines, one combines particles going forwards and anti-particles going backwards in a single line decorated with an arrow pointing forwards. Similarly, one combines particles going backwards and anti-particles going forwards in a single line oriented backwards.

Our discussion of Feynman diagrams in many-body quantum physics and quantum field theory is divided into two sections. In this section we will show how to compute the vacuum expectation value of scattering operators. This method can be interpreted as a special case of the formalism described in Sect. 20.1 on

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the Gaussian integration. In Sect. 20.6 we will describe how to compute scattering operators. Diagrams needed for scattering operators have some features of Friedrichs diagrams, since their external legs are divided into incoming and outgoing ones.

Throughout this and the next section we will use the terminology and notation of Sect. 20.1. In particular Pr will denote the set of particles, divided into four parts $\mathrm{Pr} = \mathrm{Pr}^{\mathrm{n}}_{\mathrm{s}} \cup \mathrm{Pr}^{\mathrm{c}}_{\mathrm{s}} \cup \mathrm{Pr}^{\mathrm{n}}_{\mathrm{a}} \cup \mathrm{Pr}^{\mathrm{c}}_{\mathrm{a}}$, as in Defs. 20.3 and 20.4. We will often write $\Gamma_1(\mathcal{Z})$ instead of $\Gamma_{\mathrm{s}}(\mathcal{Z})$ and $\Gamma_{-1}(\mathcal{Z})$ instead of $\Gamma_{\mathrm{a}}(\mathcal{Z})$.

Let us first describe the constructions related to the free dynamics. As usual, it is convenient to describe separately the neutral and charged cases.

20.5.1 Free neutral particles

We assume that for every $p \in \mathrm{Pr}^{\mathrm{n}}$ we are given a real dual phase space $\mathcal{Y}_p$ equipped with a dynamics $\{r_{p,t}\}_{t\in\mathbb{R}}$. More precisely,

(1) for $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{s}}$, $\mathcal{Y}_p$ is a symplectic vector space and $\{r_{p,t}\}_{t\in\mathbb{R}}$ is a stable symplectic dynamics on $\mathcal{Y}_p$;
(2) for $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{a}}$, $\mathcal{Y}_p$ is a real Hilbert space and $\{r_{p,t}\}_{t\in\mathbb{R}}$ is a non-degenerate orthogonal dynamics on $\mathcal{Y}_p$.

We use the constructions described in Sect. 18.1. In particular, we write $r_{p,t} = \mathrm{e}^{ta_p}$, and construct the corresponding one-particle spaces $\mathcal{Z}_p$ and the one-particle Hamiltonians $h_p > 0$. Recall that we have a natural decomposition $\mathbb{C}\mathcal{Y}_p = \mathcal{Z}_p \oplus \overline{\mathcal{Z}}_p$, and $a_{p\mathbb{C}} = \mathrm{i}h_p \oplus (-\mathrm{i}\overline{h}_p)$. On the Fock space $\Gamma_{\epsilon_p}(\mathcal{Z}_p)$ we have the Hamiltonian $\mathrm{d}\Gamma(h_p)$ and the fields $\mathcal{Y}_p \ni \zeta \mapsto \phi_p(\zeta)$. We write $\phi_{p,t}(\zeta) := \phi_p(r_{p,-t}\zeta)$.

20.5.2 Free charged particles

We assume that for every $q \in \mathrm{Pr}^{\mathrm{c}}$ we are given a complex dual phase space $\mathcal{Y}_q$ equipped with a dynamics $\{r_{q,t}\}_{t\in\mathbb{R}}$. More precisely,

(1) for $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{s}}$, $\mathcal{Y}_q$ is a charged symplectic vector space and $\{r_{q,t}\}_{t\in\mathbb{R}}$ is a stable charged symplectic dynamics;
(2) for $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{a}}$, $\mathcal{Y}_q$ is a complex Hilbert space and $\{r_{q,t}\}_{t\in\mathbb{R}}$ is a non-degenerate unitary dynamics.

Following Sect 18.2, we write $r_{q,t} = \mathrm{e}^{\mathrm{i}tb_q}$, and construct the corresponding one-particle spaces $\mathcal{Z}_q$ and the one-particle Hamiltonians $h_q > 0$. We have a natural decomposition $\mathcal{Y}_q = \mathcal{Y}_q^{(+)} \oplus \mathcal{Y}_q^{(-)}$, with $b_q = b_q^{(+)} \oplus (-\overline{b_q^{(-)}})$. Then $\mathcal{Z}_q^{(+)} = \mathcal{Y}_q^{(+)}$, $\mathcal{Z}_q^{(-)} = \overline{\mathcal{Y}}_q^{(-)}$, so that $\mathcal{Z}_q = \mathcal{Z}_q^{(+)} \oplus \mathcal{Z}_q^{(-)}$, $h_q = b_q^{(+)} \oplus \overline{b_q^{(-)}}$. On the Fock space $\Gamma_{\epsilon_q}(\mathcal{Z}_q)$ we have the Hamiltonian $\mathrm{d}\Gamma(h_q)$ and the field $\mathcal{Y}_q \ni \zeta \mapsto \psi_q^*(\zeta)$. We set $\psi_{q,t}^*(\zeta) := \psi_q^*(r_{q,-t}\zeta)$.

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20.5.3 Full Hilbert space

Definition 20.55 *Sometimes, for brevity, we will write*

$$\mathcal{Z} = \bigoplus_{p\in \mathrm{Pr}} \mathcal{Z}_p$$

for the total one-particle space.

Clearly, $\mathcal{Z}$ can be treated as a super-space with the grading $\epsilon = \bigoplus_{p\in \mathrm{Pr}} \epsilon_p \mathbb{1}_{\mathcal{Z}_p}$.

The Hilbert space of the system will be

$$\Gamma(\mathcal{Z}) \simeq \bigotimes_{p\in \mathrm{Pr}} \Gamma_{\epsilon_p}(\mathcal{Z}_p).$$

The free Hamiltonian is

$$H_0 = \sum_{p\in \mathrm{Pr}} \mathrm{d}\Gamma(h_p).$$

20.5.4 Wick's time-ordered product

In the presence of fermionic degrees of freedom, it is convenient to modify the definition of the time-ordered product. The so-called Dyson's time-ordered product, defined in Def. 20.22, will be replaced by Wick's time-ordered product, which takes into account the fermionic nature of some operators.

Definition 20.56 *An operator B on $\Gamma(\mathcal{Z})$ is called* bosonic*, resp.* fermionic *if $B = \Gamma(\epsilon)B\Gamma(\epsilon)$, resp. $B = -\Gamma(\epsilon)B\Gamma(\epsilon)$.*

Definition 20.57 *Let $\mathbb{R} \ni t \mapsto B_k(t), \dots, B_1(t)$ be time-dependent operators, each either bosonic or fermionic. Let $t_n, \dots, t_1 \in \mathbb{R}$ be pairwise distinct. We define* Wick's time-ordered product of $B_n(t_n)$,..., $B_1(t_1)$ *by*

$$\mathrm{T}\big(B_n(t_n)\cdots B_1(t_1)\big) := \mathrm{sgn}_{\mathrm{a}}(\sigma) B_{\sigma_n}(t_{\sigma_n})\cdots B_{\sigma_1}(t_{\sigma_1}),$$

where $\sigma_n, \dots, \sigma_1$ is a permutation of $n, \dots, 1$ such that $t_{\sigma_n} \geq \cdots \geq t_{\sigma_1}$, and $\mathrm{sgn}_{\mathrm{a}}(\sigma)$ is the sign of the permutation of the fermionic elements among $B_n(t_n), \dots, B_1(t_1)$.

20.5.5 Feynman 2-point functions: general remarks

An important ingredient of Feynman's diagrammatic approach to quantum field theory is the so-called *Feynman's 2-point functions*. They are given by the vacuum expectation values of time-ordered products of fields. They will be discussed in Subsects. 20.5.6–20.5.9. We will consider separately the neutral and charged cases, which are very similar.

In practice, in the bosonic case one uses two kinds of 2-point functions: the *phase-space* and the *configuration space 2-point functions*. We start with a description of the phase space 2-point functions, since they can be discussed in

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a parallel way for bosons and fermions. However, in the bosonic case, one usually prefers to use 2-point configuration space functions. They will be discussed separately in Subsects. 20.5.8 and 20.5.9. They are used in particular when the interaction depends only on configuration space, which is often the case.

As usual, we will use $t \in \mathbb{R}$ to denote the time variable. The variable $E \in \mathbb{R}$ will have the meaning of energy.

20.5.6 Feynman's phase space 2-point functions for neutral particles

Let us start with neutral particles, bosonic or fermionic.

Definition 20.58 *For $p \in \mathrm{Pr}_{\mathrm{s}}^{\mathrm{n}}$, resp. $p \in \mathrm{Pr}_{\mathrm{a}}^{\mathrm{n}}$ the corresponding* Feynman's phase space 2-point function *is the function with values in operators on $\mathbb{C}\mathcal{Y}_p$ defined as*

$$S_p(t) := \theta(t)\mathrm{e}^{\mathrm{i}th_p}\,\mathbb{1}_{\mathcal{Z}_p} \pm \theta(-t)\mathrm{e}^{-\mathrm{i}t\overline{h}_p}\,\mathbb{1}_{\overline{\mathcal{Z}}_p}\,.$$

Note that if $\zeta_1, \zeta_2 \in \mathcal{Y}_p$, then

$$\zeta_1 \cdot S_p(t)\zeta_2 = \big(\Omega|\mathrm{T}\big(\phi_t(\zeta_1)\phi_0(\zeta_2)\big)\Omega\big)\,.$$

The Fourier transform of S_p is

$$\hat{S}_p(E) = (\mathrm{i}h_p - \mathrm{i}E)^{-1}\,\mathbb{1}_{\mathcal{Z}_p} \mp (-\mathrm{i}\overline{h}_p - \mathrm{i}E)^{-1}\,\mathbb{1}_{\overline{\mathcal{Z}}_p}\,.$$

If $p \in \mathrm{Pr}_{\mathrm{a}}^{\mathrm{n}}$, this simplifies to

$$\hat{S}_p(E) = (a_p - \mathrm{i}E)^{-1}.$$

On the space $C_{\mathrm{c}}^{\infty}(\mathbb{R}, \mathbb{C}\mathcal{Y}_p)$ we obtain a symmetric, resp. anti-symmetric form

$$f_1 \cdot S_p f_2 := \int\int f_1(t_1) \cdot S_p(t_1 - t_2) f_2(t_2)\mathrm{d}t_1\mathrm{d}t_2.$$

20.5.7 Feynman's phase space 2-point functions for charged particles

Next we consider bosonic and fermionic charged particles.

Definition 20.59 *For $q \in \mathrm{Pr}_{\mathrm{s}}^{\mathrm{c}}$, resp. $q \in \mathrm{Pr}_{\mathrm{a}}^{\mathrm{c}}$ the corresponding* Feynman's phase space 2-point function *is the function with values in operators on $\mathcal{Y}_q$ defined as*

$$S_q(t) := \theta(t)\mathrm{e}^{\mathrm{i}tb_q^{(+)}}\,\mathbb{1}_{\mathcal{Y}_q^{(+)}} \pm \theta(-t)\mathrm{e}^{-\mathrm{i}tb_q^{(-)}}\,\mathbb{1}_{\mathcal{Y}_q^{-}}.$$

Note that if $\zeta_1, \zeta_2 \in \mathcal{Y}_q$, then

$$\overline{\zeta}_1 \cdot S_q(t)\zeta_2 = \big(\Omega|\mathrm{T}\big(\psi_t(\zeta_1)\psi_0^*(\zeta_2)\big)\Omega\big)\,.$$

The Fourier transform of S_q is

$$\hat{S}_q(E) = \mathrm{i}^{-1}(b_q^{(+)} - E)^{-1}\,\mathbb{1}_{\mathcal{Y}_q^{(+)}} \mp \mathrm{i}^{-1}(-b_q^{(-)} - E)^{-1}\,\mathbb{1}_{\mathcal{Y}_q^{(-)}}\,.$$

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If $q \in \mathrm{Pr}_{\mathrm{a}}^{\mathrm{c}}$, this simplifies to

$$\hat{S}_q(E) = \mathrm{i}^{-1}(b_q - E)^{-1}.$$

On the space $C_{\mathrm{c}}^{\infty}(\mathbb{R}, \mathcal{Y}_q)$ we obtain a Hermitian, resp. anti-Hermitian form

$$\overline{f}_1 \cdot S_q f_2 := \int\int \overline{f_1(t_1)} \cdot S_q(t_1 - t_2) f_2(t_2) \mathrm{d}t_1 \mathrm{d}t_2.$$

20.5.8 Feynman's configuration space 2-point functions for neutral bosons

Consider a neutral boson whose phase space is split into a configuration and momentum space.

More precisely, suppose that $p \in \mathrm{Pr}_{\mathrm{s}}^{\mathrm{n}}$ and $\tau_p \in L(\mathcal{Y}_p)$ satisfies

$$\begin{aligned} (\tau_p y_1) \cdot \omega_p \tau_p y_2 &= -y_1 \cdot \omega_p y_2, \quad y_1, y_2 \in \mathcal{Y}, \\ \tau_p a_p &= -\tau_p a_p, \quad \tau_p^2 = \mathbb{1}_{\mathcal{Y}_p}. \end{aligned}$$

Set

$$\begin{aligned} \mathcal{X}_p &:= \{y \in \mathcal{Y}_p \ : \ \tau_p y = y\}, \\ \Xi_p &:= \{y \in \mathcal{Y}_p \ : \ \tau_p y = -y\}, \end{aligned}$$

and $\mathbb{1}_{\mathcal{X}_p} := \frac{1}{2}(\mathbb{1}_{\mathcal{Y}} + \tau_p)$.

In other words, τ_p is a *time reversal* in the terminology of Def. 18.13, the dynamics is time reversal invariant and $\mathcal{X}_p$, resp. Ξ_p is the corresponding *configuration*, resp. *momentum space* according to Subsect. 18.3.1, and $\mathbb{1}_{\mathcal{X}_p}$ is the projection onto $\mathcal{X}_p$ along Ξ_p. Following our standard notation, $\mathbb{1}_{\mathcal{X}_p,\mathbb{C}}$ denotes the linear extension of $\mathbb{1}_{\mathcal{X}_p}$ to the projection onto $\mathbb{C}\mathcal{X}_p$ along $\mathbb{C}\Xi_p$.

Definition 20.60 *The* configuration space Feynman's 2-point function *is the function with values in operators on $\mathbb{C}\mathcal{X}_p$ defined as*

$$D_p(t) := \mathbb{1}_{\mathcal{X}_p,\mathbb{C}} S_p(t) \mathbb{1}_{\mathcal{X}_p,\mathbb{C}},$$

where $S_p(t)$ was introduced in Def. 20.58.

Define

$$T_p := \mathbb{1}_{\mathcal{Z}_p} \mathbb{1}_{\mathcal{X}_p,\mathbb{C}}$$

as a map $T_p : \mathbb{C}\mathcal{X}_p \to \mathcal{Z}_p$. Note that $\tau_{p,\mathbb{C}}$ is a unitary map transforming $\mathcal{Z}_p$ onto $\overline{\mathcal{Z}}_p$ and such that $\tau_{p,\mathbb{C}} h_p \tau_{p,\mathbb{C}}^{-1} = \overline{h}_p$. Therefore,

$$D_p(t) = T_p^* \mathrm{e}^{\mathrm{i}|t|h_p} T_p.$$

The Fourier transform of D_p is

$$\hat{D}_p(E) = T_p^* \frac{2h_p}{E^2 - h_p^2} T_p.$$

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On the space $C_{\mathrm{c}}^{\infty}(\mathbb{R}, \mathbb{C}\mathcal{X}_p)$ we obtain a symmetric form,

$$g_1 \cdot D_p g_2 := \int \int g_1(t_1) \cdot D_p(t_1 - t_2) g_2(t_2) \mathrm{d}t_1 \mathrm{d}t_2 .$$

20.5.9 Feynman's configuration space 2-point functions for charged bosons

Consider now a charged boson whose phase space is split into a configuration and momentum space.

More precisely, let $q \in \mathrm{Pr}_{\mathrm{s}}^{\mathrm{c}}$ and suppose that κ_q is a linear map on $\mathcal{Y}_q$ satisfying

$$\overline{\kappa_q y_1} \cdot \omega_q \kappa_q y_2 = -\overline{y}_1 \cdot \omega_q y_2, \quad y_1, y_2 \in \mathcal{Y},$$
$$\kappa_q b_q = -\kappa_q b_q, \quad \kappa_q^2 = \mathbb{1}_{\mathcal{Y}_q} .$$

Set

$$\mathcal{X}_p := \{y \in \mathcal{Y}_p \ : \ \kappa_p y = y\},$$
$$\Xi_p := \{y \in \mathcal{Y}_p \ : \ \kappa_p y = -y\},$$

and $\mathbb{1}_{\mathcal{X}_p} := \frac{1}{2}(\mathbb{1}_{\mathcal{Y}} + \kappa_p)$. In other words, κ_q is an involutive *Racah time reversal* in the terminology of Def. 18.38, the dynamics is Racah time reversal invariant and $\mathcal{X}_p$, resp. Ξ_p is the *configuration*, resp. *momentum space* in the terminology of Subsect. 18.3.4, and $\mathbb{1}_{\mathcal{X}_p}$ is the projection onto $\mathcal{X}_p$ along Ξ_p.

Definition 20.61 *The* configuration space Feynman's 2-point function *is the function with values in operators on $\mathcal{X}_q$ defined as*

$$D_q(t) := \mathbb{1}_{\mathcal{X}_q} S_q(t) \mathbb{1}_{\mathcal{X}_q} ,$$

where $S_q(t)$ was introduced in Def. 20.58.

Define

$$T_q := \mathbb{1}_{\mathcal{Y}_q^{(+)}} \mathbb{1}_{\mathcal{X}_q}$$

as a map $T_q : \mathcal{X}_q \to \mathcal{Y}_q^{(+)}$. Note that κ_q is a unitary map transforming $\mathcal{Y}_q^{(+)}$ onto $\mathcal{Y}_q^{(-)}$ and such that $\kappa_q b_q^{(+)} \kappa_q^{-1} = b_q^{(-)}$. Therefore,

$$D_q(t) = T_q^* \mathrm{e}^{\mathrm{i}|t| b_q^{(+)}} T_q .$$

The Fourier transform of D_q is

$$\hat{D}_q(E) = T_q^* \frac{2 b_q^{(+)}}{E^2 - (b_q^{(+)})^2} T_q .$$

On the space $C_{\mathrm{c}}^{\infty}(\mathbb{R}, \mathcal{X}_q)$ we obtain a Hermitian form,

$$\overline{g}_1 \cdot D_q g_2 := \int \int \overline{g_1(t_1)} \cdot D_q(t_1 - t_2) g_2(t_2) \mathrm{d}t_1 \mathrm{d}t_2 .$$

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20.5.10 Wick quantization of Feynman polynomials

In the Feynman formalism, perturbations are described by polynomials on the phase space.

Definition 20.62 *We set*

$$\mathcal{Y}^{\mathrm{n}} := \bigoplus_{p\in \mathrm{Pr}^{\mathrm{n}}} \mathcal{Y}_p, \quad \mathcal{Y}^{\mathrm{c}} := \bigoplus_{q\in \mathrm{Pr}^{\mathrm{c}}} \mathcal{Y}_q.$$

A polynomial on $\mathcal{Y} := \mathbb{C}\mathcal{Y}^{\mathrm{n}} \oplus \mathcal{Y}^{\mathrm{c}} \oplus \overline{\mathcal{Y}}^{\mathrm{c}}$ *will be called a* Feynman polynomial.

Definition 20.63 *We set*

$$\mathcal{Z}^{\mathrm{n}} := \bigoplus_{p\in \mathrm{Pr}^{\mathrm{n}}} \mathcal{Z}_p, \quad \mathcal{Z}^{(\pm)} := \bigoplus_{q\in \mathrm{Pr}^{\mathrm{c}}} \mathcal{Z}_q^{(\pm)}.$$

Clearly, $\mathbb{C}\mathcal{Y}^{\mathrm{n}} = \mathcal{Z}^{\mathrm{n}} \oplus \overline{\mathcal{Z}}^{\mathrm{n}}$, $\overline{\mathcal{Y}}^{\mathrm{c}} = \overline{\mathcal{Z}}^{(+)} \oplus \mathcal{Z}^{(-)}$, $\mathcal{Y}^{\mathrm{c}} = \mathcal{Z}^{(+)} \oplus \overline{\mathcal{Z}}^{(-)}$. Therefore, we can identify $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ with $\mathcal{Y}$, where $\mathcal{Z}$ is defined in Def. 20.55. It is convenient to introduce a special notation for this identification.

Definition 20.64 $\rho : \mathcal{Z} \oplus \overline{\mathcal{Z}} \to \mathcal{Y}$ *denotes the map*

$$\rho(z_1^{\mathrm{n}}, z_1^{(+)}, z_1^{(-)}, \overline{z}_2^{\mathrm{n}}, \overline{z}_2^{(+)}, \overline{z}_2^{(-)}) := (z_1^{\mathrm{n}} \oplus \overline{z}_2^{\mathrm{n}}, z_1^{(+)} \oplus \overline{z}_2^{(-)}, \overline{z}_2^{(+)} \oplus z_1^{(-)}).$$

Definition 20.65 *Given a Feynman polynomial* $G \in \mathrm{Pol}(\mathcal{Y})$*, we will write* $G \circ \rho := \Gamma(\rho^{\#})G$*, which is a polynomial in* $\mathrm{Pol}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$*. Its Wick quantization, which is an operator on* $\Gamma(\mathcal{Z})$*, will have a special notation:*

$$:G(\phi, \psi^*, \psi): := \mathrm{Op}^{a^*,a}(G \circ \rho). \tag{20.48}$$

We will use the concept of the multi-degree introduced in Def. 20.5. The following definition is parallel to Def. 20.15:

Definition 20.66 *Given a multi-degree* m*, we define*

$$\begin{aligned}
\mathrm{Pol}^m(\mathcal{Z}) := &\Big(\bigotimes_{p\in \mathrm{Pr}^{\mathrm{n}}} \mathrm{Pol}_{\epsilon_p}^{m_p}(\mathcal{Z}_p)\Big) \\
&\otimes\Big(\bigotimes_{q\in \mathrm{Pr}^{\mathrm{c}}} \mathrm{Pol}_{\epsilon_q}^{m_q^{(+)}}(\mathcal{Z}_q^{(+)})\Big) \otimes \Big(\bigotimes_{q'\in \mathrm{Pr}^{\mathrm{c}}} \mathrm{Pol}_{\epsilon_{q'}}^{m_{q'}^{(-)}}(\mathcal{Z}_{q'}^{(-)})\Big), \\
\Gamma^m(\mathcal{Z}) := &\Big(\bigotimes_{p\in \mathrm{Pr}^{\mathrm{n}}} \Gamma_{\epsilon_p}^{m_p}(\mathcal{Z}_p)\Big) \\
&\otimes\Big(\bigotimes_{q\in \mathrm{Pr}^{\mathrm{c}}} \Gamma_{\epsilon_q}^{m_q^{(+)}}(\mathcal{Z}_q^{(+)})\Big) \otimes \Big(\bigotimes_{q'\in \mathrm{Pr}^{\mathrm{c}}} \Gamma_{\epsilon_{q'}}^{m_{q'}^{(-)}}(\mathcal{Z}_{q'}^{(-)})\Big).
\end{aligned}$$

We also define

$$\begin{aligned}
H_m &:= \sum_{p\in \mathrm{Pr}^{\mathrm{n}}} \mathrm{d}\Gamma^{m_p}(h_p) + \sum_{q\in \mathrm{Pr}^{\mathrm{c}}} \mathrm{d}\Gamma^{m_q^{(+)}}(h_q^{(+)}) + \sum_{q'\in \mathrm{Pr}^{\mathrm{c}}} \mathrm{d}\Gamma^{m_{q'}^{(-)}}(h_{q'}^{(-)}), \\
\Theta^m &:= \bigotimes_{p\in \mathrm{Pr}^{\mathrm{n}}} \Theta^{m_p} \otimes \bigotimes_{q\in \mathrm{Pr}^{\mathrm{c}}} \Theta^{m_q^{(+)}} \otimes \bigotimes_{q'\in \mathrm{Pr}^{\mathrm{c}}} \Theta^{m_{q'}^{(-)}}
\end{aligned}$$

as operators on $\Gamma^m(\mathcal{Z})$*.*

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20.5.11 Evaluation of Feynman diagrams with no external legs

Let $\mathbb{R} \ni t \mapsto F_1(t), \dots, F_r(t)$ be time-dependent Feynman monomials, each of a constant multi-degree. Set

$$G(t) := F_1(t) + \cdots + F_r(t),$$
$$H(t) := H_0 + \lambda{:}G(t, \phi, \psi^*, \psi){:}.$$

Our aim is to compute the scattering operator

$$S = \mathrm{Texp}\Big(-\mathrm{i}\lambda \int \mathrm{e}^{\mathrm{i}tH_0}{:}G(t, \phi, \psi^*, \psi){:}\mathrm{e}^{-\mathrm{i}tH_0}\,\mathrm{d}t\Big). \tag{20.49}$$

We use the terminology of Section 20.1. We will denote by $F_1, \dots, F_r$ distinct vertices of the same multi-degree as the Feynman monomials $F_1(t), \dots, F_r(t)$. For brevity, we will write $\mathfrak{V}$ for the set $\{F_1, \dots, F_r\}$.

Let $(F_{j_n}, \dots, F_{j_1})$ be a sequence of Feynman vertices in $\mathfrak{V}$ and let D be a Feynman diagram over $\prod_{i=n}^{1} F_{j_i}$ with no external legs.

Definition 20.67 *The* evaluation of the diagram D at times $t_n, \dots, t_1 \in \mathbb{R}$ *is*

$$\begin{aligned} &D(t_n, \dots, t_1) \\ &\quad := \prod_{\ell = \{\mathrm{l}, \mathrm{l}'\} \in \mathrm{Ln_n}(D)} \nabla_{y_\mathrm{l}} S_\ell(t_{\mathrm{nr(l)}} - t_{\mathrm{nr(l')}}) \nabla_{y_{\mathrm{l}'}} \\ &\qquad \times \prod_{\kappa = \{\mathrm{k}^{(+)}, \mathrm{k}^{(-)}\} \in \mathrm{Ln_c}(D)} \nabla_{\overline{y}_{\mathrm{k}^{(-)}}} S_\kappa(t_{\mathrm{nr}(\mathrm{k}^{(+)})} - t_{\mathrm{nr}(\mathrm{k}^{(-)})}) \nabla_{y_{\mathrm{k}^{(+)}}} \prod_{i=n}^{1} F_{j_i}(t_i). \end{aligned}$$

Remark 20.68 *If for some particle $p \in \mathrm{Pr_s}$ the polynomials F_i depend only on the configuration space, and not on the momentum space, which is often the case for bosons, we can replace the phase space 2-point function S_p by the configuration space 2-point function D_p.*

20.5.12 Vacuum expectation value of the scattering operator

Feynman diagrams with no external legs can be used to compute the vacuum expectation value of scattering operators. The following theorem is closely related to Thm. 20.21.

Theorem 20.69

$$\log(\Omega|S\Omega) = \sum_{n=0}^{\infty} \sum_{D \in \mathrm{Dg}_n \{\mathfrak{V}\}_{\mathrm{cnl}}} (-\mathrm{i}\lambda)^n \int \cdots \int D(t_n, \dots, t_1)\mathrm{d}t_n \cdots \mathrm{d}t_1. \tag{20.50}$$

Proof We first obtain

$$(\Omega|S\Omega) = \sum_{n=0}^{\infty} \sum_{D \in \mathrm{Dg}_n \{\mathfrak{V}\}_{\mathrm{nl}}} (-\mathrm{i}\lambda)^n \int \cdots \int D(t_n, \dots, t_1)\mathrm{d}t_n \cdots \mathrm{d}t_1. \tag{20.51}$$

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Following the arguments of Thm. 20.21, (20.51) equals

$$\exp\left(\sum_{n=0}^{\infty}\sum_{D\in \mathrm{Dg}_n\{\mathfrak{V}\}_{\mathrm{cnl}}}(-\mathrm{i}\lambda)^n\int\cdots\int D(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\right) \qquad \square$$

Here is a reformulation of (20.50) in terms of the Fourier transforms of the diagrams:

$$\log(\Omega|S\Omega)=\sum_{n=0}^{\infty}\sum_{D\in \mathrm{Dg}_n\{\mathfrak{V}\}_{\mathrm{cnl}}}(-\mathrm{i}\lambda)^n\hat{D}(0,\ldots,0).$$

20.5.13 Energy shift

Assume now that $F_i(t)=F_i$ do not depend on time. Then the Fourier transform $\hat{D}(\tau_n,\ldots,\tau_1)$ is supported in $\tau_n+\cdots+\tau_1=0$ and one can write

$$\hat{D}(\tau_n,\ldots,\tau_1)=2\pi\delta(\tau_n+\cdots+\tau_1)\hat{D}[\tau_n,\ldots,\tau_1],$$

where $\hat{D}[\tau_n,\ldots,\tau_1]$ is defined on $\tau_n+\cdots+\tau_1=0$.

Let E be the ground-state energy of H. The (partly heuristic) arguments that gave Thm. 20.54 can be used to give a formula for the energy shift in terms of Feynman diagrams:

Theorem 20.70

$$E=\sum_{n=0}^{\infty}\mathrm{i}(-\mathrm{i}\lambda)^n\sum_{D\in \mathrm{Dg}_n\{\mathfrak{V}\}_{\mathrm{cnl}}}\hat{D}[0,\ldots,0].$$

Proof The function $D(t_n,\ldots,t_1)$ is translation invariant; therefore it can be written as

$$D(t_n,\ldots,t_1)=d(t_n-t_1,\ldots,t_2-t_1),$$

for some function d. We compute

$$\begin{aligned}
&\hat{D}(\tau_n,\ldots,\tau_1)\\
&\quad=\int\cdots\int \mathrm{e}^{-\mathrm{i}t_n\tau_n-\cdots-\mathrm{i}t_1\tau_1}D(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\\
&\quad=\int\cdots\int \mathrm{e}^{\mathrm{i}s_n\tau_n-\cdots-\mathrm{i}s_2\tau_2-\mathrm{i}s_1(\tau_1+\cdots+\tau_n)}d(s_n,\ldots,s_2)\mathrm{d}s_n\cdots\mathrm{d}s_1\\
&\quad=2\pi\delta(\tau_n+\cdots+\tau_1)\hat{d}(\tau_n,\ldots,\tau_2),
\end{aligned}$$

where we have used the substitution

$$s_j=t_j-t_1,\ n\geq j\geq 2,\qquad s_1=t_1.$$

Thus, with $\tau_1=-\tau_n-\cdots-\tau_2$,

$$\hat{D}(\tau_n,\ldots,\tau_1)=\hat{d}(\tau_n,\ldots,\tau_2).$$

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Now we would like to use Sucher's formula, (20.26). Terms in the expansion for $\frac{1}{2}\mathrm{i}\epsilon\lambda\partial_\lambda \log(\Omega|S_\epsilon\Omega)$ are of the form $\mathrm{i}(-\mathrm{i}\lambda)^n$ times

$$\begin{aligned}&\frac{1}{2}n\epsilon\int\cdots\int \mathrm{e}^{-\epsilon(|t_n|+\cdots|t_1|)}D(t_n,\dots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\\&\quad=\frac{1}{2}n\epsilon\int \mathrm{e}^{-\epsilon(|s_1+s_n|+\cdots+|s_1+s_2|+|s_1|)}d(s_n,\dots,s_2)\mathrm{d}s_n\cdots\mathrm{d}s_2.\end{aligned}$$

We perform the integral in s_1:

$$\begin{aligned}&\frac{1}{2}n\epsilon\int \mathrm{e}^{-\epsilon(|s_1+s_n|+\cdots+|s_1+s_2|+|s_1|)}\mathrm{d}s_1\\&\quad=\frac{1}{2}n\int \mathrm{e}^{-(|u+\epsilon s_n|+\cdots+|u+\epsilon s_2|+|u|)}\mathrm{d}u\\&\quad\to\frac{1}{2}n\int \mathrm{e}^{-n|u|}\mathrm{d}u=1.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{\epsilon\searrow 0}\frac{1}{2}n\epsilon\int \mathrm{e}^{-\epsilon(|t_n|+\cdots|t_1|)}D(t_n,\dots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1\\&\quad=\int d(s_n,\dots,s_2)\mathrm{d}s_n\cdots\mathrm{d}s_2\\&\quad=\hat{d}(0,\dots,0)\;=\;\hat{D}(0,\dots,0).\end{aligned}$$

□

20.5.14 Polynomials on path spaces

The formalism of Feynman diagrams can be interpreted to some extent as a special case of the formalism described in Sect. 20.1. With this interpretation we say that to obtain the vacuum expectation values of scattering operators we need to integrate over various paths (trajectories).

Paths are functions of time with values in the phase space or the configuration space. We equip path spaces with an appropriate (bilinear or sesquilinear) form defined with the help of the Feynman propagator. In this way we obtain one of the basic ingredients of the formalism of Gaussian integration described in Sect. 20.1: the family of spaces $\mathcal{V}_p$ equipped with a form σ_p.

We can distinguish two kinds of paths: *phase space paths* and *configuration space paths*. Their names are quite awkward; therefore we will abbreviate them: *ph-paths* for the former and *c-paths* for the latter.

Definition 20.71 (1) *Let $p\in\mathrm{Pr}^{\mathrm{n}}$ (p is a neutral particle). For the space of corresponding* smooth ph-paths *we can take $C^\infty_{\mathrm{c}}(\mathbb{R},\mathcal{Y}_p)$. It is equipped with the form*

$$f\cdot S_p f'=\int f(t)\cdot S_p(t-t')f'(t')\mathrm{d}t\mathrm{d}t'.$$

Note that S_p is symmetric, resp. anti-symmetric for $p\in\mathrm{Pr}^{\mathrm{n}}_{\mathrm{s}}$, resp. $p\in\mathrm{Pr}^{\mathrm{n}}_{\mathrm{a}}$.

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(2) *Let $q \in \mathrm{Pr}^{\mathrm{c}}$ (q is a charged particle). For the space of corresponding* ph-paths *we can take $C^\infty_{\mathrm{c}}(\mathbb{R}, \mathcal{Y}_q)$. It is equipped with the form*

$$\overline{g} \cdot S_q g' = \int \overline{g(t)} \cdot S_q(t - t') g'(t') \mathrm{d}t \mathrm{d}t'.$$

Note that S_q is Hermitian, resp. anti-Hermitian for $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{s}}$, resp. $p \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{a}}$.

In the bosonic case, one often prefers to use paths with values in the configuration space $\mathcal{X}_p$, rather than in the dual phase space $\mathcal{Y}_p$.

Definition 20.72 (1) *Let $p \in \mathrm{Pr}^{\mathrm{n}}_{\mathrm{s}}$ (p is a neutral boson). For the space of corresponding* c-paths *we can take $C^\infty_{\mathrm{c}}(\mathbb{R}, \mathcal{X}_p)$. It is equipped with the form*

$$f \cdot D_p f' = \int f(t) \cdot D_p(t - t') f'(t') \mathrm{d}t \mathrm{d}t'.$$

Note that D_p is symmetric.

(2) *Let $q \in \mathrm{Pr}^{\mathrm{c}}_{\mathrm{s}}$ (q is a charged boson). For the space of corresponding* c-paths *we can take $C^\infty_{\mathrm{c}}(\mathbb{R}, \mathcal{X}_q)$. It is equipped with the form*

$$\overline{g} \cdot D_q g' = \int \overline{g(t)} \cdot D_q(t - t') g'(t') \mathrm{d}t \mathrm{d}t'.$$

Note that D_q is Hermitian.

Remark 20.73 *Note that most textbooks start their exposition of the path integration formalism from what we call configuration space paths for neutral bosons.*

For a neutral particle p, the spaces $(C^\infty_{\mathrm{c}}(\mathbb{R}, \mathcal{Y}_p), S_p)$ or $(C^\infty_{\mathrm{c}}(\mathbb{R}, \mathcal{X}_p), D_p)$ can be treated as $(\mathcal{V}_p, p)$ of Def. 20.12 (1). A similar remark applies to charged particles. We introduce the space $\mathcal{V}$ as in Def. 20.12 (3) and note that $\mathcal{V} = C^\infty_{\mathrm{c}}(\mathbb{R}, \mathcal{Y})$, where $\mathcal{Y}$ is defined as in Def. 20.62. We introduce the Wick transform, denoted by double dots, the Gaussian integral, etc.

As discussed in Sect. 20.1, we would like to integrate "monomials of degree m", that is, m-linear symmetric or anti-symmetric functions on $\mathcal{V}^m$. The space of such monomials was denoted by $\mathrm{Pol}^m(\mathcal{V})$. In Sect. 20.1 we assumed that the spaces $\mathcal{V}$ are finite-dimensional, which allowed us to ignore questions about their topology. Path spaces are necessarily infinite-dimensional and difficulties arising from various possible topologies show up. We will keep the notation $\mathrm{Pol}^m(\mathcal{V})$ for monomials of degree m, but we need to make precise what we mean by this. To reduce the complexity of notation, let us assume that we have a single species of particles, which are neutral. They can be bosonic or fermionic. For definiteness, we will use phase space paths.

A reasonable and sufficiently broad definition of $\mathrm{Pol}^m(\mathcal{V})$ is the following. We say that $P \in \mathrm{Pol}^m(\mathcal{V})$ if it is given by a family of distributions

$$P(\cdot, \dots, \cdot) \in \mathcal{D}'\big(\mathbb{R}^m, (\overset{\mathrm{a}1}{\otimes}{}^m \mathcal{Y})^\#\big)$$

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with the appropriate symmetry or anti-symmetry properties, and its action on $f_i \in C_c^\infty(\mathbb{R}, \mathcal{Y})$, $i = 1, \dots, m$, is

$$\langle P | f_1, \dots, f_m \rangle = \int \cdots \int \langle P(t_m, \dots, t_1) | f_m(t_m) \cdots f_1(t_1) \rangle \mathrm{d}t_m \cdots \mathrm{d}t_1 .$$

We will see that $\mathrm{Pol}^m(\mathcal{V})$ is large enough to contain objects that we need to integrate when computing the scattering operator.

Note that our choice of spaces of the form $C_c^\infty(\mathbb{R}, \mathcal{Y})$ for path spaces is to some extent arbitrary. One could try to replace C_c^∞ by some other class of functions. Nevertheless, one really needs $\mathrm{Pol}^m(\mathcal{V})$ to be quite large, which is made possible with this choice.

When we compute the scattering operator, a special role is played by the time variable. In fact, in this context we often deal with monomials whose associated functions depend on a single time variable, as explained in the following definition.

Definition 20.74 *Let*

$$\mathbb{R} \ni t \mapsto F(t) \in \mathrm{Pol}^m(\mathcal{Y})$$

be a function. We will still denote by F the element of $\mathrm{Pol}^m(\mathcal{V})$ whose associated distribution $F(t_1, \dots, t_m)$ is

$$F(t_1, \dots, t_m) = \int F(t) \delta(t_1 - t) \cdots \delta(t_m - t) \mathrm{d}t,$$

that is,

$$\langle F | f_1, \dots, f_m \rangle = \int F(t) f_1(t), \dots, f_m(t) \mathrm{d}t.$$

20.5.15 Feynman formalism and Gaussian integration

Let $F_1(t), \dots, F_r(t)$, $G(t)$ be as in Subsect. 20.5.11. Note that the function $t \mapsto G(t)$ takes values in $\mathrm{Pol}(\mathcal{Y})$. We will denote by G its interpretation as an element of $\mathrm{Pol}(\mathcal{V})$, using the convention in Def. 20.74.

We define the scattering operator S as in (20.49). The following theorem shows that one can reduce computations in quantum field theory to Gaussian integrals on appropriate path spaces. The theorem follows from a comparison of the formulas for the evaluation of diagrams in Defs. 20.17, 20.67, using the covariance of Defs. 20.71 or 20.72.

Theorem 20.75 *We have the following identity:*

$$(\Omega | S \Omega) = \int \mathrm{e}^{\lambda : G :}, \tag{20.52}$$

where the right hand side is given by the formalism of Sect. 20.1.

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20.6 Feynman diagrams and the scattering operator

In this section we describe how to modify the formalism of the previous section to compute the scattering operator. Diagrams for the scattering operator will have external legs of two kinds: incoming and outgoing, similarly to Friedrichs diagrams. Vertices, however, will be typical for Feynman diagrams – diagrams with a different order of time labels will not be distinguished.

Throughout this section we keep the terminology and notation of the previous section. In particular, let $\mathbb{R} \ni t \mapsto F_1(t), \dots, F_r(t)$ be time-dependent Feynman monomials, each of a constant multi-degree with the corresponding vertices denoted by $F_1, \dots, F_r$. $\mathfrak{V}$ denotes the set $\{F_1, \dots, F_r\}$. We set

$$G(t) := F_1(t) + \cdots + F_r(t),$$

and perturb H_0 by $:G(t, \phi, \psi^*, \psi):$. Our aim is to compute the scattering operator

$$S = \mathrm{Texp}\Big(-\mathrm{i}\lambda \int \mathrm{e}^{\mathrm{i}tH_0} :G(t, \phi, \psi^*, \psi): \mathrm{e}^{-\mathrm{i}tH_0}\, \mathrm{d}t\Big). \tag{20.53}$$

20.6.1 Feynman diagrams with external legs

We assume that F_{j_i}, $i = n, \dots, 1$, is a sequence in $\mathfrak{V}$. Let D be a Feynman diagram over $\prod_{i=n}^{1} F_{j_i}$.

Definition 20.76 *Let $i = n, \dots, 1$. The* multi-degree of D at the ith vertex, *denoted $m_i(D)$, is defined as*

$$m^{\mathrm{n}}_{p,i}(D) := \#\mathrm{Lg}_p(D) \cap \mathrm{Lg}_p(F_{j_i}), \quad p \in \mathrm{Pr}_{\mathrm{n}},$$
$$m^{(\pm)}_{q,i}(D) := \#\mathrm{Lg}^{(\pm)}_q(D) \cap \mathrm{Lg}^{(\pm)}_q(F_{j_i}), \quad q \in \mathrm{Pr}_{\mathrm{c}}.$$

The detailed multi-degree of D *is the sequence* $\underline{m}(D) = (m_n(D), \dots, m_1(D))$.

Note that the diagram has no legs iff all entries of $\underline{m}(D)$ are zero.

Recall that in Def. 20.67 we defined the evaluation of a Feynman diagram without external legs. We would like to generalize this definition to all Feynman diagrams. In the literature one can find two conventions for evaluation of such diagrams: either one includes the propagators for external legs or not. In the definition below we adopt the latter convention.

Definition 20.77 *The* amputated evaluation of the diagram D at times $t_n, \dots, t_1 \in \mathbb{R}$ *is an element of* $\overset{1}{\underset{i=n}{\otimes}} \Gamma^{m_i(D)}(\mathcal{Y})$ *given by*

$$\begin{aligned} &D^{\mathrm{amp}}(t_n, \dots, t_1) \\ &:= (-\mathrm{i}\lambda)^n \overset{1}{\underset{i=n}{\otimes}} \Theta^{m_i(D)\#} \prod_{\ell = \{\mathrm{l}, \mathrm{l}'\} \in \mathrm{Ln}^{\mathrm{n}}(D)} \nabla_{y_{\mathrm{l}}} S_\ell(t_{\mathrm{nr(l)}} - t_{\mathrm{nr(l')}}) \nabla_{y_{\mathrm{l}'}} \\ &\quad \times \prod_{\kappa = \{\mathrm{k}^{(+)}, \mathrm{k}^{(-)}\} \in \mathrm{Ln}^{\mathrm{c}}(D)} \nabla_{\overline{y}_{\mathrm{k}^{(-)}}} S_\kappa(t_{\mathrm{nr}(\mathrm{k}^{(+)})} - t_{\mathrm{nr}(\mathrm{k}^{(-)})}) \nabla_{y_{\mathrm{k}^{(+)}}} \prod_{i=n}^{1} F_{j_i}(t_i). \end{aligned} \tag{20.54}$$

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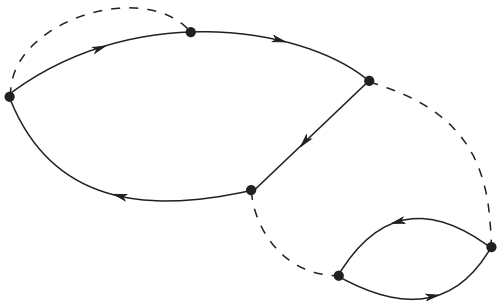

Figure 20.10 Feynman diagram without external lines.

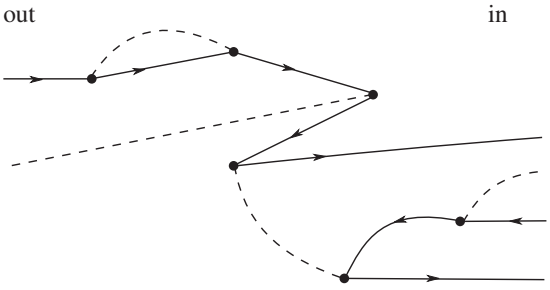

Figure 20.11 Scattering-type Feynman diagram.

Note that if D has no legs, then (20.54) coincides with $D(t_n, \dots, t_1)$ defined in Def. 20.67.

The precise interpretation of (20.54) is similar to that of (20.12); see the discussion after Def. 20.17. The main difference is that we symmetrize or antisymmetrize only within each vertex.

20.6.2 Feynman diagrams with incoming and outgoing external legs

Let m be a multi-degree. Recall the identification $\rho : \overline{\mathcal{Z}} \oplus \mathcal{Z} \to \mathcal{Y}$ defined in Subsect. 20.5.10. Clearly, we have the identification

$$\begin{aligned}\Gamma(\rho)^{\#} : \mathrm{Pol}^m(\mathcal{Y}) &\to \mathrm{Pol}^m(\overline{\mathcal{Z}} \oplus \mathcal{Z}) \\ &\simeq \bigoplus_{m^+ + m^- = m} \mathrm{Pol}^{m^+}(\overline{\mathcal{Z}}) \otimes \mathrm{Pol}^{m^-}(\mathcal{Z}).\end{aligned}$$

If $\underline{m} = (m_1, \dots, m_n)$ is a sequence of multi-degrees, then this yields

$$\begin{aligned}\bigotimes_{i=1}^{n} \Gamma(\rho)^{\#} : \bigotimes_{i=1}^{n} \mathrm{Pol}^{m_i}(\mathcal{Y}) &\to \bigotimes_{i=1}^{n} \mathrm{Pol}^{m_i}(\overline{\mathcal{Z}} \oplus \mathcal{Z}) \\ &\simeq \bigoplus_{\underline{m}^+ + \underline{m}^- = \underline{m}} \bigotimes_{i=n}^{1} \mathrm{Pol}^{m_i^+}(\overline{\mathcal{Z}}) \otimes \mathrm{Pol}^{m_i^-}(\mathcal{Z}). \quad (20.55)\end{aligned}$$

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Let $D^{\mathrm{amp}}(t_n,\dots,t_1)$ be the evaluation of a Feynman diagram with external legs, which, following Def. 20.65, is an element of $\overset{n}{\underset{i=1}{\otimes}} \mathrm{Pol}^{m_i}(\mathcal{Y})$. By (20.55), we have a unique decomposition

$$\overset{n}{\underset{i=1}{\otimes}} \Gamma(\rho)^{\#}\, D^{\mathrm{amp}}(t_n,\dots,t_1) = \sum_{\underline{m}^+ + \underline{m}^- = \underline{m}(D)} D^{\mathrm{amp}}_{\underline{m}^+,\underline{m}^-}(t_n,\dots,t_1). \quad (20.56)$$

Here, $D^{\mathrm{amp}}_{m^+,m^-}(t_n,\dots,t_1)$ are elements of

$$\overset{1}{\underset{i=n}{\otimes}} \mathrm{Pol}^{m_i^+}(\overline{\mathcal{Z}}) \otimes \mathrm{Pol}^{m_i^-}(\mathcal{Z}),$$

$\underline{m}(D)$ is the detailed multi-degree of D and $\underline{m}^+, \underline{m}^-$ sum up to the detailed multi-degree of D. In other words,

$$m^{\mathrm{n}}_{p,i}(D) = m^{\mathrm{n}+}_{p,i} + m^{\mathrm{n}-}_{p,i}, \quad p \in \mathrm{Pr}^{\mathrm{n}},$$
$$m^{(\pm)}_{q,i}(D) = m^{(\pm)+}_{q,i} + m^{(\pm)-}_{q,i}, \quad q \in \mathrm{Pr}^{\mathrm{c}}.$$

We also set

$$|\underline{m}^{\pm}| := \sum_{i=n}^{1} m_i^{\pm}.$$

Definition 20.78 *The* amputated evaluation of D with $\underline{m}^-$ incoming and $\underline{m}^+$ outgoing legs at times $t_n,\dots,t_1 \in \mathbb{R}$, *denoted* $D^{\mathrm{amp}}_{m^+,m^-}(t_n,\dots,t_1)$, *is defined by (20.56).*

Note that $D^{\mathrm{amp}}_{\underline{m}^+,\underline{m}^-}(t_n,\dots,t_1)$ can be interpreted as an operator in

$$B\Big(\overset{1}{\underset{i=n}{\otimes}} \Gamma^{m_i^-}(\mathcal{Z}), \overset{1}{\underset{i=n}{\otimes}} \Gamma^{m_i^+}(\mathcal{Z})\Big).$$

In what follows we stick to this interpretation.

Definition 20.79 *The* scattering evaluation of D with $\underline{m}^-$ incoming and $\underline{m}^+$ outgoing legs at times $t_n,\dots,t_1 \in \mathbb{R}$ *is an operator in* $B\big(\Gamma^{|\underline{m}^-|}(\mathcal{Z}), \Gamma^{|\underline{m}^+|}(\mathcal{Z})\big)$ *defined by*

$$D^{\mathrm{scat}}_{\underline{m}^+,\underline{m}^-}(t_n,\dots,t_1)$$
$$= \Theta^{|\underline{m}^+|} \overset{1}{\underset{i=n}{\otimes}} \mathrm{e}^{\mathrm{i}t_i H_{m_i^+}}\, D^{\mathrm{amp}}_{\underline{m}^+,\underline{m}^-}(t_n,\dots,t_1) \overset{1}{\underset{i=n}{\otimes}} \mathrm{e}^{-\mathrm{i}t_i H_{m_i^-}}\, \Theta^{|\underline{m}^-|}.$$

Remark 20.80 *The fact that a single Feynman diagram gives rise to many terms in the scattering operator and each of them is an analytic continuation of the others is called the* crossing symmetry.

20.6.3 Scattering operator and Feynman diagrams

Here is the analog of Thm. 20.51 in the formalism of Feynman diagrams:

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Theorem 20.81

$$S = \sum_{n=0}^{\infty} \sum_{\substack{D\in \mathrm{Dg}_n\{\mathfrak{V}\} \\ \underline{m}^+ + \underline{m}^- = \underline{m}(D)}} (-\mathrm{i}\lambda)^n \,\mathrm{Op}^{a^*,a} \times \left(\int \cdots \int D^{\mathrm{scat}}_{\underline{m}^+,\underline{m}^-}(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1 \right)$$

$$= \mathrm{Op}^{a^*,a}\left(\exp\left(\sum_{n=0}^{\infty} \sum_{\substack{D\in \mathrm{Dg}_n\{\mathfrak{V}\}_{\mathrm{con}} \\ \underline{m}^+ + \underline{m}^- = \underline{m}(D)}} (-\mathrm{i}\lambda)^n \times \int \cdots \int D^{\mathrm{scat}}_{\underline{m}^+,\underline{m}^-}(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1 \right)\right)$$

$$\frac{S}{(\Omega|S\Omega)} = \mathrm{Op}^{a^*,a}\left(\exp\left(\sum_{n=0}^{\infty} \sum_{\substack{D\in \mathrm{Dg}_n\{\mathfrak{V}\}_{\mathrm{cnl}} \\ \underline{m}^+ + \underline{m}^- = \underline{m}(D)}} (-\mathrm{i}\lambda)^n \times \int \cdots \int D^{\mathrm{scat}}_{\underline{m}^+,\underline{m}^-}(t_n,\ldots,t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1 \right)\right).$$

Here is a reformulation of the last formula in terms of Fourier transforms of the diagrams:

$$\frac{S}{(\Omega|S\Omega)} = \mathrm{Op}^{a^*,a}\left(\exp\left(\sum_{n=0}^{\infty} \sum_{\substack{D\in \mathrm{Dg}_n\{\mathfrak{V}\}_{\mathrm{link}} \\ \underline{m}^+ + \underline{m}^- = \underline{m}(D)}} (-\mathrm{i}\lambda)^n \int \cdots \int \mathrm{d}\xi_n^+\cdots\mathrm{d}\xi_1^+\mathrm{d}\xi_n^-\cdots\mathrm{d}\xi_1^- \right.\right.$$
$$\times \Theta^{|\underline{m}^+|} \bigotimes_{i=n}^{1} \delta(H_{m_i^+} - \xi_i^+ \mathbb{1}) \hat{D}^{\mathrm{amp}}_{\underline{m}^+,\underline{m}^-}\left(\xi_n^+ - \xi_n^-,\ldots,\xi_1^+ - \xi_1^-\right)$$
$$\left.\left. \times \bigotimes_{n=1}^{1} \delta(H_{m_i^-} - \xi_i^- \mathbb{1}) \Theta^{|\underline{m}^-|} \right)\right).$$

Note that if the ith vertex has no incoming lines, then $H_{m_i^-} = 0$. Therefore, the delta function $\delta(H_{m_i^-} - \xi_i^- \mathbb{1})$ sets $\xi_i^- = 0$. Hence, we can drop the variable ξ_i^- altogether from the formula. A similar remark concerns outgoing lines.

20.6.4 Gell-Mann–Low scattering operator for time-independent perturbations

Assume now that $F_i(t) = F_i$ do not depend on time. Then the Fourier transform of $D^{\mathrm{amp}}(\tau_n,\ldots,\tau_1)$ is supported in $\tau_n + \cdots + \tau_1 = 0$, and one can write

$$\hat{D}^{\mathrm{amp}}(\tau_n,\ldots,\tau_1) = 2\pi\delta(\tau_n + \cdots + \tau_1)\hat{D}^{\mathrm{amp}}[\tau_n,\ldots,\tau_1],$$

where $\hat{D}^{\mathrm{amp}}[\tau_n,\ldots,\tau_1]$ is defined on $\tau_n + \cdots + \tau_1 = 0$.

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Usually, the standard scattering operator does not exist; instead one can define the Gell-Mann–Low scattering operator:

$$S_{\mathrm{GL}} = \mathrm{Op}^{a^*,a}\Bigg(\exp\Bigg(\sum_{n=0}^{\infty}\sum_{\substack{D\in \mathrm{Dg}_n\{\mathfrak{V}\}_{\mathrm{link}}\\ \underline{m}^+ + \underline{m}^- = \underline{m}(D)}} (-\mathrm{i}\lambda)^n$$
$$\times \int\cdots\int \mathrm{d}\xi_n^+\cdots\mathrm{d}\xi_1^+\,\mathrm{d}\xi_n^-\cdots\mathrm{d}\xi_1^- 2\pi\delta(\xi_n^+ + \cdots + \xi_1^+ - \xi_n^+ - \cdots - \xi_1^-)$$
$$\times\Theta^{|\underline{m}^+|}\bigotimes_{i=n}^{1}\delta(H_{m_i^+} - \xi_i^+\mathbb{1})\hat{D}^{\mathrm{amp}}_{\underline{m}^+,\underline{m}^-}\left[\xi_n^+ - \xi_n^-,\ldots,\xi_1^+ - \xi_1^-\right]$$
$$\times\bigotimes_{i=n}^{1}\delta(H_{m_i^-} - \xi_i^-\mathbb{1})\Theta^{|\underline{m}^-|}\Bigg)\Bigg).$$

20.6.5 Friedrichs diagrams as Feynman diagrams

It is possible to interpret Friedrichs diagrams as a kind of Feynman diagrams. In fact, suppose we use the framework of Sects. 20.3 and 20.4. All the particles from Pr we interpret as charged particles, renaming $\mathcal{Z}_q$, $q \in \mathrm{Pr}$, as $\mathcal{Y}_q$. In the bosonic case, we use the charged symplectic form equal to i times the scalar product. In the fermionic case, we just keep the scalar product. Note that there are no anti-particles: $\mathcal{Y}_p = \mathcal{Y}_p^{(+)}$. Therefore, the Feynman propagator is zero for negative times. We write the scattering operator using Wick's chronological product.

Note that the above trick, even if somewhat artificial, can be used to reduce the theory of Friedrichs diagrams to Feynman diagrams, which gives in particular a convenient way to show the linked cluster theorem in the context of Friedrichs diagrams.

20.7 Notes

A description of some elements of the diagram formalism can be found in any textbook on quantum field theory, e.g. Schweber (1962), Weinberg (1995) or Srednicki (2007).

A mathematical exposition of what we call the Friedrichs diagrams is contained in the books by Friedrichs (1963) and by Hepp (1969). Hepp describes and proves the linked cluster theorem.

The diagram formalism is also one of the basic tools of non-relativistic many-body quantum theory. Therefore, its exposition can be found in many textbooks on this subject, such as the monograph of Fetter–Walecka (1971).

A book that specializes in the topic of Feynman diagrams was written by Mattuck (1967).

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