ψ would be positive or negative unity, so that the more general symbol should be retained because of the non-commutative nature of ψ in spaces of even dimensions.

In particular, in two dimensions, $ij = \sqrt{-1}$. Hence we get $j = -i\sqrt{-1} = \sqrt{-1}i$, and $i = -\sqrt{-1}j$. If $\alpha = xi + yj$, the operator $\sqrt{-1}(\equiv \psi_2)$ gives $\sqrt{-1}\alpha = xj - yi$. In this case there is no need to retain the symbols i, j; for $\alpha = xi + yj = (x + y\sqrt{-1})i$, and i denotes a given direction, so that α may be completely denoted by $x + y\sqrt{-1}$. It appears, therefore, that complex algebra is a special case of this generalised quaternionic system. Ordinary arithmetic may be regarded as the special case $\psi_0 = 1$.

Thus, in respect of generality, as well as of simplicity, the quaternionic method has the advantage.

In four dimensions, from $ijkl = \psi$, we get $ijk = -\psi l$, $jkl = \psi i$, $kli = -\psi j$, $lij = \psi k$. It does not follow that the space is non-symmetrical, or that, as the condition of symmetry, we should have $ijk = -\psi l$, $jkl = -\psi i$, etc. For we have seen that, in the symmetrical two-dimensional space, we have $i = -\sqrt{-1}j$, $j = \sqrt{-1}i$, *not* $j = -\sqrt{-1}i$, as a necessary condition for symmetry.

In any space $V\alpha\beta$ represents a directed area in the plane of α, β. In three dimensions, it happens to be representable by a linear vector.

Fifth Meeting, March 10, 1893.

John Alison, Esq., M.A., F.R.S.E., President, in the Chair.

Early History of the Symmedian Point.

By J. S. Mackay, M.A., LL.D.

In 1873, at the Lyons meeting of the French Association for the Advancement of the Sciences, Monsieur Emile Lemoine called attention to a particular point within a plane triangle which he called the centre of antiparallel medians. Since that time the

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properties of this remarkable point and of the lines and circles connected with it have been investigated by various writers, foremost among whom is Monsieur Lemoine himself. The results obtained by them are so numerous (indeed every month adds to their number) and so widely scattered through the mathematical periodicals of the world that it would be a task of considerable magnitude to make even an undigested collection of them. It is the purpose of the present paper to state those properties of the point which had been discovered previously to 1873. A short sketch of some of them will be found at the end of a memoir read by Monsieur Lemoine at the Grenoble meeting (1885) of the French Association, and in a memoir by Monsieur Emile Vigarié at the Paris meeting (1889) of the same Association. The references given by Dr Emmerich in his *Die Brocardschen Gebilde* (1891) are very valuable. It is a pity they are not more explicit.

If ABC be a triangle, AA′ the median from A, then AR the image of AA′ in the bisector of angle A is called the symmedian from A. It is not difficult to prove that AA′ bisects all parallels to BC, and that AR bisects all antiparallels to BC. Hence Monsieur Lemoine proposed* to call AR an antiparallel median. This name however has been replaced by symmedian (*symédiane* abbreviated from *symétrique de la médiane*) a happy coinage † of Monsieur Maurice d'Ocagne.

Since the three medians and the three symmedians are isogonally conjugate with respect to the three angles of the triangle, those theorems which have been established regarding isogonally conjugate lines in general can at once be applied to the particular case of medians and symmedians.

The point of concurrency of the three symmedians, which it is usual to denote by K, has received various names such as minimum-point, ‡ Grebe's point, § Lemoine's point. ‖ The designation symmedian point, suggested ¶ by Mr Tucker, seems preferable to all of these.

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* *Nouvelles Annales de Mathématiques*, 2nd series, XII. 364 (1873).

† *Nouvelles Annales de Mathématiques*, 3rd series, II. 451 (1883).

‡ Dr E. W. Grebe in Grunert's *Archiv der Mathematik*, IX. 251 (1847).

§ Dr A. Emmerich's *Die Brocardschen Gebilde*, p. 37 (1891).

‖ Prof. J. Neuberg's *Mémoire sur le Tétraèdre*, p. 3 (1884).

¶ *Educational Times*, XXXVII. 211 (1884).

The first mention of the symmedian point that I have found is in Leybourn's *Mathematical Repository*, old series, III. 71, where the following question is proposed* for demonstration by "Yanto."

If K be the point in a triangle from which perpendiculars are drawn to the sides of the triangle so that the sum of their squares is the least possible; twice the area of the triangle is a mean proportional between the sum of the squares of the sides of the triangle and the sum of the squares of the above-mentioned perpendiculars.

The second mention of K is in Leybourn's *Mathematical Repository*, new series, Vol. I. Part I. pp. 26-7.

Question 12, proposed by James Cunliffe, Bolton, is:

It is required to determine the locus of a point, from whence, if perpendiculars be drawn to three straight lines given by position, the sum of the squares of the said perpendiculars may be equal to a given magnitude.

In the solution of this question—the locus is an ellipse—given by Mr J. I. it is shown that if K be taken such that KL, KM, KN (perpendicular to BC, CA, AB) are proportional to BC, CA, AB, then $KL^2 + KM^2 + KN^2$ is a minimum, and that AK produced divides BC into segments which are proportional to AB^2 and AC^2.

Seeing that solutions of the first 30 questions proposed in the *Mathematical Repository* were to be in the hands of the editor by the first day of February 1804, it may be assumed that Mr J. I.'s solution was published in that year. I have some grounds (which need not be stated here) for conjecturing that Mr J. I. was James Ivory, known for his theorem regarding the attractions of ellipsoids on external and internal particles.

Ivory's theorem that the distances of K from the sides are directly proportional to the sides taken along with the well-known theorem that the distances of the centroid G from the sides are inversely proportional to the sides, establishes the theorem that G and K are inverse points with respect to the triangle.

In Leybourn's *Mathematical Repository*, new series, Vol. I. Part II. p. 19 (1806), Ivory proves the theorem:

If P and Q be two points taken on a pair of lines isogonal with

* I am not quite certain at what date, for my copy of Vol. III. is imperfect. But at p. 80 a letter is printed, dated March 1st, 1802, and at p. 83 another dated Sept. 8, 1802. It may therefore be presumed that the question was published in 1803.

respect to angle BAC, the distances of P from AB and AC are inversely proportional to those of Q from AB and AC.

The converse of this theorem, taken with what immediately precedes, might easily suggest that the lines drawn from A to G and K (hitherto known only by its minimum property) were isogonal with respect to angle BAC; but Ivory makes no explicit mention of the fact.

The other theorem given by Ivory, namely, that AK produced divides BC into segments, which are proportional to AB^2 and AC^2, is easily seen to be a particular case of a theorem regarding isogonals which was known to the ancient Greeks.* The theorem is:

If ABC be a triangle, and if AP, AQ be isogonal with respect to A, and meet BC in P and Q, then

$$BP \cdot BQ : CQ \cdot CP = AB^2 : AC^2.$$

It may be worth mentioning that Pappus proves also that if

$$BP \cdot BQ : CQ \cdot CP > AB^2 : AC^2$$

then $$\text{angle } BAP > \text{angle } CAQ.$$

Lhuilier in his *Élémens d'Analyse*, pp. 296-8 (1809), states and proves the theorem of "Yanto," shows that the distances of any point in a symmedian from the adjacent sides are proportional to those sides, that the segments into which a symmedian divides the opposite side are proportional to the squares of the adjacent sides, and adds:

"This doctrine can be extended to any polygons and even to polyhedrons. I shall content myself, for example, with determining that point in space from which, if perpendiculars be let fall on the faces of a tetrahedron, the sum of their squares is a *minimum*, and with determining that minimum."

He then proves that

(1) The perpendiculars drawn from this minimum-point are directly proportional to the faces on which they fall.

(2) The perpendicular on any face is a fourth proportional to the sum of the squares of the four faces, to the square of this face, and to the altitude of the tetrahedron which corresponds to this face.

(3) Thrice the volume of a tetrahedron is a mean proportional

* See Pappus's *Mathematical Collection*, VI., 12. The same theorem differently stated is more than once proved in Book VII, among the lemmas which Pappus gives for Apollonius's treatise on *Determinate Section*.

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between the sum of the squares of the four faces and the sum of the squares of the perpendiculars let fall on them from the minimum point.

In this connection reference may be made to Professor J. Neuberg's *Mémoire sur le Tétraèdre* (1884).

The fourth discoverer of the point K is L. C. Schulz von Strasznicki. C. F. A. Jacobi says that Schulz published a pamphlet in 1827 with the title "Das gradlinige Dreieck und die dreiseitige Pyramide nach allen Analogien dargestellt." This pamphlet I have not seen. About the same time Schulz published in Baumgaertner and D'Ettingshausen's *Zeitschrift für Physik und Mathematik*, I. 396, II. 530, two articles, the first on the plane triangle and the second on the tetrahedron. Probably these two articles and the pamphlet are the same thing. In the first article he proves the following results: *

(1) If K (defined by its minimum property) be joined to the vertices, the fundamental triangle will be divided into three other triangles whose areas will be as the squares of the sides of the fundamental triangle on which they rest.

(2) The straight lines drawn through each vertex and through K will divide the opposite sides into two segments proportional to the squares of the adjacent sides; hence a simple geometrical construction for finding K.

(3) The same straight lines will divide each of the angles of the triangle into two partial angles whose sines will be as the adjacent sides.

(4) If the point K is replaced by the centroid G, the sines of the partial angles will be as the reciprocals of the adjacent sides.

(5) If the point K is replaced by the circumcentre O, the cosines of the partial angles will be directly as the adjacent sides.

(6) If the point K is replaced by the orthocentre H, the cosines of the partial angles will be inversely as the adjacent sides.

(7) Generally, if the angles of a triangle be divided in such a manner that for each of them the sines of the partial angles may be to each other directly or inversely as any powers or functions of the

* This account of Schulz's articles is taken from Férussac's *Bulletin des Sciences Mathématiques*, VIII. 2 (1827).

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adjacent sides the three straight lines will be concurrent; and if each side be divided into segments which are to each other as functions of the adjacent sides, and each point of section be joined to the opposite vertex, the three straight lines will be concurrent.

Steiner in a paper published* in Gergonne's *Annales de Mathématiques* XIX. 37-64 (1828) states and proves some of the fundamental theorems relating to isogonally conjugate points and lines. Thus

(1) The orthogonal projections on the sides of a triangle of two isogonally conjugate points furnish six concyclic points.

(2) If P, Q be isogonally conjugate points with respect to ABC, the sides of the pedal triangle corresponding to P are perpendicular to QA, QB, QC; and the sides of the pedal triangle corresponding to Q are perpendicular to PA, PB, PC.

(3) If three lines drawn from the vertices of a triangle be concurrent, their isogonal conjugates with respect to the angles of the triangle are also concurrent.

(4) Every point in the interior of a triangle may be considered as one of the foci of an ellipse inscribed in the triangle.

(5) The feet of the perpendiculars let fall from the foci of an ellipse on its tangents are all situated on the same circle having the major axis of this ellipse for diameter.

(6) If an angle be circumscribed to an ellipse the straight lines drawn from the two foci to the vertex of that angle are isogonal with respect to it.

(7) The rectangle under the perpendiculars let fall from the two foci of an ellipse on any one of its tangents is constant and consequently equal to the square of the semiaxis minor of the ellipse.

In C. Adams's *Die Lehre von den Transversalen*, pp. 79-80 (1843) the following theorem is proved:

Let D, E, F be the points of contact of the incircle with the sides of ABC, and Γ be the point at which AD, BE, CF are concurrent. If through Γ parallels be drawn to the sides of triangle DEF, these parallels will cut the sides of DEF in six concyclic points.†

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* Republished in Steiner's *Gesammelte Werke*, I. 191-210 (1881).

† See the following paper on *Adams's Hexagons and Circles*.

It is now known that Γ is the symmedian point of DEF; hence this six-point circle of Adams is the first Lemoine circle of DEF, or as Mr Tucker has called it, the triplicate-ratio circle.

Adams shows also that the centre of his six-point circle is the mid point of ΓI, where I is the incentre of ABC and consequently the circumcentre of DEF.

It will conduce to brevity of statement if the following definitions and notation be laid down.

If AR, BS, CT be the symmedians of ABC, then AR′, BS′, CT′ their harmonic conjugates with respect to the sides of ABC may be called the external symmedians,* or the exsymmedians of ABC. The points R, R′ are situated on BC, S, S′ on CA, T, T′ on AB. Let the exsymmedians intersect each other at K_1, K_2, K_3, and let AK_1 meet the circumcircle ABC whose centre is O at D. The mid point of BC is A′.

The following properties occur in C. Adams's *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, pp. 1-5 (1846).

(1) The theorem quoted from Pappus VI., 12.

(2) The corollary $BR : CR = AB^2 : AC^2$.

(3) The tangents to the circumcircle at the vertices coincide with the exsymmedians of the triangle.

(4) The symmedian from any vertex and the exsymmedians from the two other vertices are concurrent.

(5) DR, DR′ are the symmedian and exsymmedian of triangle BCD drawn from D.

(6) BR, BK_1 are the symmedian and exsymmedian of triangle ABD drawn from B.

Similarly for CR, CK, and triangle ACD.

(7) $AR'^2 + BK_1^2 = K_1R'^2$.

(8) OR is perpendicular to K_1R'.

(9) AR′ is a mean proportional between A′R′ and RR′.

In this connection it may be worth mentioning that Pappus in his *Mathematical Collection*, VII., 119, gives the following theorem as a lemma for one of the propositions in Apollonius's *Loci Plani*:

* Monsieur Clément Thiry in *Le Troisième Livre de Géométrie*, p. 42 (1887).

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If $$AB^2 : AC^2 = BR' : CR'$$
then $$BR' \cdot CR' = AR'^2.$$

Dr E. W. Grebe of Cassel in Grunert's *Archiv der Mathematik*, IX, 250-9 (1847) discusses the point K and gives it the name minimum-point. He indicates two constructions for finding K.

(1) On the sides of ABC let squares X, Y, Z be described either all outwardly to the triangle or all inwardly. Produce the sides of the squares Y, Z opposite to AC, AB to meet in A′; the sides of the squares Z, X opposite to BA, BC to meet in B′; the sides of the squares X, Y opposite to CB, CA to meet in C′. Then A′A, B′B, C′C will be concurrent at K which will be the minimum-point not only of ABC but of A′B′C′.

(2) Find the isogonally conjugate point to G the centroid.

Denote by L, M, N the projections of K on BC, CA, AB.

(3) Various expressions for $KL^2 + KM^2 + KN^2$.

(4) Expressions for the segments BL, CL, CM, AM, AN, BN in terms of the sides a, b, c, and in terms of the sides and angles.

(5) Expressions for AK, BK, CK in terms of the sides, and in terms of the sides and the three medians.

(6) Expressions for MN, NL, LM in terms of the sides and area of ABC, and in terms of the sides, area, and medians of ABC.

(7) K is the centroid of LMN.

Grebe shows that if the square on the side AB be described inwardly to the triangle and the other two squares outwardly, an analogous point, K_3, is obtained, and he gives three sets of expressions for its distances from BC, CA, AB.

The next mention of K is in the *Nouvelles Annales*, 1st series, VII. 407-9 and 454 (1848). The theorem is thus stated:

If through each angle of a triangle a straight line is drawn which cuts the opposite sides into two segments proportional to the squares of the adjacent sides the three straight lines are concurrent at a point such that the sum of the squares of its distances from the sides of the triangle is a minimum.

The theorem was communicated by Captain Hossard to M. Poudra who gave a geometrical solution in the course of which it is seen that the perpendiculars from K on the sides are proportional to those sides and that K is the centroid of the triangle LMN. At the end of Captain Hossard's analytical solution it is added that the square of the distance AK is

$$\frac{b^2c^2(b^2+c^2+2bc\cos A)}{(a^2+b^2+c^2)^2}$$

an expression almost identical with that given by Grebe.

C. F. A. Jacobi in his *Die Entfernungsörter geradliniger Dreiecke*, pp. 12-13 (1851) draws attention to isogonal points (*Gegenpunkte* he calls them), and proves that if K be the point isogonal to G then K is the centroid of the triangle whose vertices are the projections of K on the sides of ABC, and the sum of the squares of the distances of K from the sides of ABC is a minimum. He adds that a Viennese mathematician L. C. Schulz von Strasznicki gave another proof by the help of co-ordinate geometry and the differential calculus.

Monsieur Catalan in Lafremoire's *Théorèmes et Problèmes de Géométrie Élémentaire*, 2nd ed., p. 161 (1852) proves that if K be the minimum point of ABC it is the centroid of the triangle LMN.

In Schlömilch's *Uebungsbuch zum Studium der höheren Analysis*, I. § 33 (1860) there is enunciated the theorem

The three straight lines which join the mid points of the sides of a triangle to the mid points of the perpendiculars on them from the vertices are concurrent.

Dr Emmerich says that the identity of this point of concurrency with the symmedian point was made evident by Wetzig.

Dr Franz Wetzig in Crelle's *Journal* LXII. 349-361 (1863) gives five or six properties of the symmedian point, but adds nothing to what had previously been known. The symmedians he calls minimum-axes, and remarks that they are analogous to the medians. He returns however to the subject four years later.

In *Mathematical Questions from the Educational Times*, III. 30-1 (1865) Mr W. J. Miller points out that the straight lines joining the three excentres I_1, I_2, I_3, of a triangle to the mid points of the sides are concurrent at a point such that the sum of the squares of the perpendiculars drawn therefrom on the sides of the triangle $I_1I_2I_3$ is a minimum, and these perpendiculars are, moreover, proportional to the sides on which they fall.

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In the *Lady's and Gentleman's Diary* for 1865, pp. 89–90, Mr Stephen Watson proposes two questions for solution. The first is:

Show that three rectangles can be inscribed in any triangle, so that they may severally have a side coincident in direction with the respective sides of the triangle, and their diagonals all intersecting in the same point. Also show that one circle will circumscribe a'l the three rectangles, and find its radius.

The common centre of these three rectangles is the symmedian point, and the circle circumscribing them is Lemoine's second circle.

The radius of the circle, given in Mr Watson's solution published the year following, is equal to

$$\frac{abc}{a^2+b^2+c^2}$$

The second is:

Through each two of the angles of a triangle ABC any circles are described cutting the sides again in D, E; F, G; H, I; and at each of those pairs of points tangents are drawn to the circles, meeting in P, Q, R. Show that the loci of P, Q, R are conics passing respectively through the angles of the triangle, and intersecting the two contiguous sides, in each case, in two points D′, E′; F′, G′; H′, I′. Also show that the tangents to those conics at the angles, and the lines D′E′, F′G′, H′I′ all pass through one point.

This point is the symmedian point, and is identified by Mr Watson with the centre of the three rectangles in the previous question.

In the *Nouvelles Annales de Mathématiques*, 2nd series, IV. 403-4 (1865) Monsieur J. J. A. Mathieu mentions as inverse points with respect to triangle ABC the centroid G and the point of intersection of AK_1, BK_2, CK_3. This point of intersection, he states, has for polar the straight line which passes through the points of intersection of each side with the tangent to the circumcircle drawn through the opposite vertex.

Let I, I_1, I_2, I_3 be the incentre and excentres of ABC,
Γ, Γ_1, Γ_2, Γ_3 the Gergonne points,
and Γ', Γ_1', Γ_2', Γ_3' the points complementary to Γ, etc.;
then $\Gamma\Gamma'$, $\Gamma_1\Gamma_1'$, $\Gamma_2\Gamma_2'$, $\Gamma_3\Gamma_3'$
are concurrent at G the centroid of ABC,
and $I\Gamma'$, $I_1\Gamma_1'$, $I_2\Gamma_2'$, $I_3\Gamma_3'$
are concurrent at K the symmedian point of ABC.

The preceding theorem was enunciated by Mr William Godward in the *Lady's and Gentleman's Diary* for 1866, p. 72, and a solution by trilinear coordinates appeared in the same periodical the following year. In connection with this subject it may be worth while to compare *Lady's and Gentleman's Diary* for 1865, pp. 63-5, and *Mathematical Questions from the Educational Times*, II. 86-8 (1865).

In the *Diary* for 1867, p. 71, Mr Thomas Milbourn enunciates the theorem,

If δ be the diameter of the circle remarked by Mr Stephen Watson, that is, the second Lemoine circle, and d the diameter of the circumcircle, then

$$\frac{1}{\delta^2}+\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

In Schlömilch's *Zeitschrift für Mathematik*, XII. 281-301 (1867) Dr Wetzig communicates a considerable number of properties relating not only to K but to K_1, K_2, K_3 which he calls harmonically associated (*harmonisch zugeordneten*) to K with respect to ABC. Thus

(1) If XYZ be the orthic triangle of ABC its sides are parallel to those of $K_1K_2K_3$.

(2) AK_1, BK_2, CK_3 meet at K and bisect the sides of XYZ.

(3) K is the centre of a conic which touches the sides of ABC at X, Y, Z.

(4) On the medians of ABC are situated the symmedian points of the triangles AYZ, BZX, CXY.

(5) AK_1, BK_2, CK_3 meet BC, CA, AB, at R, S, T. Perpendiculars RR′, SS′, TT′ to BC, CA, AB divide the sides of XYZ in the same proportions as the sides of ABC.

(6) A, K, R, K_1 is a harmonic range.

(7) XR′ goes through K.

(8) The symmedian point of AYZ is situated on the perpendicular from K to BC.

(9) $KYZ : KZX : KXY = KBC : KCA : KAB$
$= BC^2 : CA^2 : AB^2.$

(10) In the point systems A, B, C, ..., X, Y, Z, ...
K corresponds to itself and the circumcentre of the first system corresponds to the orthocentre in the second.

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(11) The points K_1, K_2, K_3, of the first system lie with the corresponding points of the second on the perpendicular bisectors of BC, CA, AB and are equally distant from BC, CA, AB.

(12) $K_2A : K_3A = CX : BX$, etc.

(13) $K_2K_3 \cdot K_3K_1 \cdot K_1K_2 : BC \cdot CA \cdot AB = 2$ circle $K_1K_2K_3$: circle ABC

(14) $AA' \cdot BB' \cdot CC' \quad : AK_1 \cdot BK_2 \cdot CK_3 = R : 4$ radius of circle $K_1K_2K_3$
$= AK \cdot BK \cdot CK \ : \ KK_1 \cdot KK_2 \cdot KK_3$

(15) $a \cdot AK : b \cdot BK : c \cdot CK = AA' : BB' : CC'$

(16) $$a^2 \frac{AK_1}{KK_1} = b^2 \frac{BK_2}{KK_2} = c^2 \frac{CK_3}{KK_3} = \frac{a^2+b^2+c^2}{2}$$

(17) $$\frac{KK_1}{AK_1} : \frac{KK_2}{BK_2} : \frac{KK_3}{CK_3} = a^2 : b^2 : c^2$$

(18) Then follow expressions for the distances from BC, CA, AB of K, K_1, K_2, K_3.

If Σ, Σ_1, Σ_2, Σ_3, denote the sum of the squares of these respective distances

$$\frac{1}{\Sigma} = \frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3}$$

(19) If k_1', k_1'', k_1''' denote the distances of K_1 from BC, CA, AB
$k_1'k_1''k_1''' : k_2'k_2''k_2''' : k_3'k_3''k_3''' = a^3 : b^3 : c^3$.

(20) K is the centroid of triangle LMN, and the sides of LMN are proportional to the medians of ABC.

(21) MN is perpendicular to AA′, and the angles of LMN are equal to the angles which the corresponding medians make with one another.

(22) If k_1, k_2, k_3 denote the distances of K from BC, CA, AB
$\frac{1}{3}LMN : ABC = k_1^2 + k_2^2 + k_3^2 : a^2 + b^2 + c^2$

(23) Corresponding property for triangle $L_1M_1N_1$

(24) $$\frac{\frac{1}{3}LMN}{\Sigma} = \frac{L_1M_1N_1}{\Sigma_1} - \text{etc.} = \frac{\Delta}{a^2+b^2+c^2}.$$

A construction for determining K is given in Schömilch's *Zeitschrift*, XVI. 168 (1871) from a communication by Const. Harkema in Petersburg.