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IRREGULAR CANONICAL DOUBLE SURFACES

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Abstract. We classify minimal irregular surfaces of general type X with K_X ample and $K_X^2 = 6p_g - 14$ such that the canonical map is 2–to–1 onto a canonically embedded surface.

§0. Introduction

Let X be a minimal surface of general type of geometric genus p_g, let $\Sigma \subset \mathbf{P}^{p_g-1}$ be the canonical image of X and let $\phi : X \to \Sigma$ be the canonical map. If Σ is a surface but ϕ is not birational, then by Theorem 3.1 of [Be1] either i) $p_g(\Sigma) = 0$ or ii) Σ is the canonical image of a surface of general type S whose canonical map is birational (and then, of course, $p_g(\Sigma) = p_g$). Recall (cf. [Ho2], Lemma 1.1 or [Be1], Theorem 5.5) that the Castelnuovo inequality $K^2 \geq 3p_g - 7$ holds for surfaces of general type with birational canonical map; so in case ii) S satisfies $K_S^2 \geq 3p_g - 7$, and $K_X^2 \geq 6p_g - 14$, with equality holding if and only if the canonical system of X is base point free and the minimal resolution S of Σ is a minimal surface on the Castelnuovo line $K_S^2 = 3p_g - 7$ (cf. Proposition 2.3 and proof).

Case ii) of the theorem quoted above was thought to be impossible for a long time. In fact only very few examples are known, and all but one series due to Beauville (cf. Section 3, Example 4) have bounded invariants. The examples in this infinite series satisfy: $K_X^2 = 6p_g - 14$, $\deg \phi = 2$ and $q(X) := h^0(X, \Omega_X^1) = 2$.

The main purpose of this paper is to show that these are almost the only examples satisfying $K_X^2 = 6p_g - 14$ and $q(X) \geq 2$. Therefore we make the following:

Assumption 0.1. *Let X be a minimal surface of general type of geometric genus p_g, with $K_X^2 = 6p_g - 14$, $q(X) > 0$ and K_X ample, let*

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$\Sigma \subset \mathbf{P}^{p_g-1}$ be the canonical image of X and let $\phi : X \to \Sigma$ be the canonical map: assume that Σ is a canonical surface and that ϕ is not birational. Moreover, if $p_g = 5, 7$ assume that Σ is isomorphic to a divisor with at most rational double points in a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, such that the fibres F of the projection $\Sigma \to \mathbf{P}^1$ are plane quartics.

If the above assumption is satisfied, then the minimal desingularization S of Σ is a minimal surface satisfying $K_S^2 = 3p_g - 7$. Surfaces with these numerical invariants have been described by Ashikaga and Konno in [AK]: under our assumptions one shows (cf. Proof of Proposition 2.3) that $p_g = 4$ cannot occur and that for $p_g = 6$ or $p_g \geq 8$ Σ is isomorphic to a divisor with at most rational double points in a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, such that the fibres F of the projection $\Sigma \to \mathbf{P}^1$ are plane quartics. (This accounts for the somewhat funny-looking final part of Assumption 0.1.) We divide the surfaces X in types *I* and *II*, according to whether, for a generic fibre F, $\phi^* F$ is connected or not. Surfaces of type *I* are the "general case", and, if $q(X) \geq 2$, they correspond to Beauville's examples; more precisely, we prove the following:

THEOREM 0.2. *Assume that* 0.1 *holds, that $q(X) \geq 2$ and that X is of type I: then $p_g(X) \equiv 1 \pmod 4$, $q(X) = 2$, the Albanese surface A of X has an irreducible principal polarization, and X can be constructed as in Example* 4 *of Section* 3, *with $n = (p_g(X) + 3)/4$.*

Let us remark that we do not know any example of type *I* surfaces with $q(X) = 1$. To establish whether such surfaces exist, and in case of existence whether they have bounded invariants is an interesting problem.

On the other hand, surfaces of type *II* should be regarded as exceptions, and can be described completely:

THEOREM 0.3. *Assume that* 0.1 *holds and that X is of type II, let $X \to B \to \mathbf{P}^1$ be the Stein factorization of the pencil $\phi^*|F|$ and let g be the genus of B: then there exist integers $0 \leq a \leq b \leq c$ with $c \leq g$ and $a + b + c = p_g - 3$ such that Σ is isomorphic to a divisor in $\mathbf{P}_{a,b,c} := \mathrm{Proj}(\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c))$ with the following properties:* i) *Σ is linearly equivalent to $4T - (a + b + c - 2)L$, where T is the tautological hyperplane section and L is the fibre of $\mathbf{P}_{a,b,c}$ (and $F = L|_\Sigma$),* ii) *the pencil $|F|$ on Σ has precisely $2g + 2$ double fibres,* iii) *the only singularities of Σ are nodes and Σ is smooth outside the double fibres of $|F|$. The double fibres of $|F|$ occur at the branch points of B and each contains* 8 *nodes .*

Conversely, given integers $0 \leq a \leq b \leq c$ *and* g*, with* $c \leq g$*, if* $\Sigma \subset \mathbf{P}_{a,b,c}$ *is a divisor satisfying conditions* i), ii), iii) *above, then* Σ *has* $16g+16$ *nodes and there exists a double cover* $\phi : X \to \Sigma$ *branched over the nodes such that* X *is a surface of type II and* ϕ *is the canonical map of* X*.*

The numerical possibilities for the invariants of X *are the following:*

a) $p_g = 3g+3, \quad q = g, \quad a = b = c = g, \quad 0 < g \leq 26;$

b) $p_g = 3g+2, \quad q = g+1, \quad a = g-1,\ b = c = g, \quad 0 < g \leq 17;$

c) $p_g = 3g+1, \quad q = g+2, \quad a = b = g-1,\ c = g$ *or* $a = g-2,\ b = c = g,$ $0 < g \leq 8.$

In [Be1] it is also proven that if the canonical map $\phi : X \to \Sigma$ is not birational and Σ is a canonically embedded surface then $\deg \phi \leq 3$ for $\chi(\mathcal{O}_X) \geq 14$, and if $\deg \phi = 3$ then $q(X) \leq 3$. Theorems 0.2 and 0.3 imply in particular that the irregularity $q(X)$ is also bounded under Assumption 0.1. It would be interesting to know whether $q(X)$ is bounded in general for $\deg \phi = 2$. Another interesting problem is to study *regular* surfaces X such that the canonical map is not birational and the canonical image is a canonically embedded surface: only very few examples are known (cf. Section 3) and, lacking the information given by the Albanese map, their structure is quite mysterious even when the invariants satisfy the "minimal" relation $K_X^2 = 6p_g - 14$.

The paper is organized as follows: in Section 1 we set the notation and recall some facts on double covers that will be used later. In Section 2 we describe the general set-up and establish various facts about X, S and Σ. In particular we study the structure of degenerate fibres of ϕ^*F both for type *I* and type *II*. In Section 3 we describe the construction of all the examples known to us of surfaces of general type with 2-1 canonical map onto a canonical surface. In Section 4 we look at the surfaces of type *I* with $q(X) \geq 2$ and we show, using a fine analysis involving the Albanese map and the Prym variety of $\phi^*F \to F$ for general F, that these are exactly Beauville's examples. In Section 5 we describe the surfaces of type *II* in detail and determine the possible ranges for their invariants. Section 6 contains a computation with Macaulay that shows that Example 3 of Section 3 actually exists.

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§1. Notation and conventions

All varieties are normal projective varieties over the complex numbers. The n-dimensional projective space is denoted by $\mathbf{P}^n$, and its group of automorphisms by $\mathbf{PGL}(n)$. As usual, $\mathcal{O}_Y$ is the structure sheaf of the variety Y, $\mathrm{H}^i(Y,\mathcal{F})$ is the i-th cohomology group of a sheaf $\mathcal{F}$ on Y, and $h^i(Y,\mathcal{F})$ is the dimension of $\mathrm{H}^i(Y,\mathcal{F})$; for a line bundle M on Y, we denote by $|M|$ the complete linear system $\mathbf{P}(\mathrm{H}^0(Y,M))$. When dealing with smooth varieties, we do not distinguish between line bundles and divisors. If Y is smooth, then K_Y denotes a canonical divisor and $\mathrm{Pic}(Y)$ the Picard group of Y. If Y is a surface, then $p_g(Y) = h^0(Y,K_Y)$ is the *geometric genus* and $q(Y) = h^1(Y,\mathcal{O}_Y)$ is the *irregularity*, K_Y^2 is the self-intersection of the canonical divisor; we denote by $\chi(Y) = 1 - q(Y) + p_g(Y)$ the Euler characteristic of $\mathcal{O}_Y$ and by $c_2(Y)$ the second Chern class of the tangent bundle of Y, or, which is the same, the topological Euler characteristic of Y. A surface Y is said to be *irregular* if $q(Y) \neq 0$. The intersection number of two divisors C, D on a smooth surface is denoted simply by CD, linear equivalence is denoted by $\equiv$. A *node* of a surface is a double point of type A_1, namely a hypersurface singularity that in suitable local analytic coordinates is defined by the equation $x^2 + y^2 + z^2 = 0$.

A *double cover* is a finite map $f : X \to Y$ of degree 2 between normal projective varieties; we denote by $i : X \to X$ the involution that interchanges the two points of a generic fibre of f. In this paper we will need to consider only the following two cases: a) both X and Y are smooth, and b) X is a smooth surface, Y is normal and f is unramified in codimension 1.

In case a), f is a flat map and $f_*\mathcal{O}_X$ splits under the action of i as $O_Y \oplus \mathcal{L}^{-1}$, where $\mathcal{L}$ is a line bundle and i acts on $\mathcal{L}^{-1}$ as multiplication by -1. The branch locus of f is a smooth divisor $B \equiv 2\mathcal{L}$, the ramification locus is a divisor $R \equiv f^*\mathcal{L}$ and one has:

$$(1.1)\quad K_X = f^*(K_Y + \mathcal{L}) \quad K_X^2 = 2(K_Y + \mathcal{L})^2 \quad f_*K_X = K_Y \oplus (K_Y + \mathcal{L})$$

$$(1.2)\qquad h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y) + h^i(\mathcal{L}), \quad i = 1, \dots \dim Y$$

(Actually, the above formulas also hold if Y is a surface with rational double points and B is a smooth divisor containing no singulairities of Y). The cover $\phi : X \to Y$ can be reconstructed from Y, $\mathcal{L}$, B as follows. Let $p : \mathcal{L} \to Y$ be the projection, let w be the tautological section of $p^*\mathcal{L}$ and let $\sigma \in \mathrm{H}^0(Y,\mathcal{L}^2)$ be a section vanishing on B: the zero locus in $\mathcal{L}$ of the section $w^2 - p^*\sigma$ of $p^*\mathcal{L}^2$, together with the restriction of the map p, is a double cover of Y isomorphic to $\phi : X \to Y$. Moreover, it is clear that, given a line bundle $\mathcal{L}$ on Y and a divisor B in the linear system $|\mathcal{L}^2|$, the above construction yields a finite degree 2 map $\phi : X \to Y$.

A *linearization* of a line bundle N on X is an involution $i_N : N \to N$ that lifts the involution $i : X \to X$. If N is a linearized line bundle, we say that $\sigma \in \mathrm{H}^0(X, N)$ is *even* if $i_{N*}\sigma = \sigma$ and *odd* if $i_{N*}\sigma = -\sigma$. A divisor defined by an even (odd) section is called *symmetric* (*antisymmetric*). The canonical bundle K_X and the pull-backs of line bundles from Y have natural linearizations: in these cases, unless otherwise stated, we consider the natural linearizations.

Consider now case b): the singularities of Y are nodes, that are the images of the fixed points of i. If ν is the number of nodes of Y, then one has (see [Ba] (0.6)):

$$\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_Y) - \frac{1}{4}\nu \tag{1.3}$$

A set J of nodes on a normal surface Y is said to be *even* if there exists a double cover $\phi : X \to Y$ branched precisely over J.

PROPOSITION 1.1. *Let W be a smooth 3-fold, let $Y \subset W$ be a divisor whose only singularities are nodes; if there exist divisors D, D' in W such that $D \equiv 2D'$ and D restricted to Y is equal to $2C$, where C is a curve passing though all the nodes of Y, then the nodes of Y are an even set.*

Proof. Denote by $\eta : \hat{W} \to W$ the blow-up at the nodes of Y, by $\epsilon : \hat{Y} \to Y$ its restriction to the strict transform $\hat{Y}$ of Y, by $\hat{E}_i$, E_i the exceptional divisors of η and ϵ respectively, and by $\hat{D}$, $\hat{C}$ the strict transforms of D on $\hat{W}$ and of C on $\hat{W}$. One has the following linear equivalence on $\hat{W}$: $2\eta^*D' \equiv \eta^*D = \hat{D} + \sum E_i$, which restricts to $2\epsilon^*D' \equiv \epsilon^*D = 2\hat{C} + \sum E_i$. So $\sum E_i \equiv 2(\epsilon^*D' - \hat{C})$, and there exists a smooth double cover $g : \hat{X} \to \hat{Y}$ branched over $\sum E_i$; the ramification divisor of g is a union of disjoint -1 curves that can be contracted to yield $f : X \to Y$ branched over the nodes of Y. $\square$

§2. The set-up

The notations and the assumptions introduced in this section will be maintained throughout all the paper. We start by making the following:

ASSUMPTION 2.1. *Let X be a minimal surface of general type of geometric genus p_g, with $K_X^2 = 6p_g - 14$, $q(X) > 0$ and K_X ample, let $\Sigma \subset \mathbf{P}^{p_g-1}$ be the canonical image of X and let $\phi : X \to \Sigma$ be the canonical map: assume that Σ is a canonical surface and that ϕ is not birational. Moreover, if $p_g = 5, 7$ assume that Σ is isomorphic to a divisor with at most rational double points in a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$, such that the fibres F of the projection $\Sigma \to \mathbf{P}^1$ are plane quartics.*

Let now A be the Albanese variety of X, let $x_0 \in X$ be a fixed point of i, let $\alpha : X \to A$ be the Albanese map with base point x_0 and let $K = A/ < -1 >$ be the Kummer variety of A. Since Σ is regular, the involution i on X induces on A the multiplication by -1, and there is an induced map $f : \Sigma \to K$. Thus we have the following *basic commutative diagram*, where $q : A \to K$ is the natural projection:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ \phi \downarrow & & \downarrow q \\ \Sigma & \xrightarrow{f} & K \end{array} \tag{2.1}$$

Assuming $K_X^2 = 6p_g - 14$ is equivalent to considering the lowest possible value of K^2 in the above situation, as it appears from the next proposition and its proof. In order to state it we introduce the following

NOTATION - DEFINITION 2.2. Let $0 \le a \le b \le c$ be integers: we write $\mathbf{P}_{a,b,c} = \mathrm{Proj}(\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c))$, and denote by T the tautological hyperplane section and by L the fibre of $\mathbf{P}_{a,b,c}$. We define a Castelnuovo surface of type (a, b, c) to be a divisor Σ in $\mathbf{P}_{a,b,c}$ linearly equivalent to $4T - (a + b + c - 2)L$ with at most rational double points as singularities. Notice that T restricts to the canonical divisor of Σ and that the minimal desingularization S of Σ is a minimal surface satisfying: $K_S^2 = 3(a + b + c) + 2 = 3p_g(S) - 7$, $q(S) = 0$.

PROPOSITION 2.3. *Assume that* 2.1 *holds; then:*

i) *$|K_X|$ is base point free and the only singularities of Σ are $4(1+p_g+q)$ nodes, that are the critical values of ϕ;*

ii) *$p_g \ge 5$, $\deg \phi = 2$, $K_S^2 = 3p_g - 7$;*

iii) *there exist integers $0 \le a \le b \le c$ with $a+b+c = p_g - 3$ such that Σ is a Castelnuovo surface of type (a, b, c) (thus,in particular, one has $a > 0$ for $p_g > 5$).*

Proof. Write $K_X = M + Z$, where M is the moving part and Z is the fixed part of K_X and denote by d the degree of $\Sigma \subset \mathbf{P}^{p_g-1}$. By [Ho2], Lemma 1.1 or [Be1], Theorem 5.5, one has $d \geq 3p_g - 7$ and thus:

$$6p_g - 14 = K_X^2 = K_X M + K_X Z \geq K_X M = M^2 + MZ \geq M^2 \geq (\deg\phi)d \geq (\deg\phi)(3p_g - 7) \geq 6p_g - 14.$$

The first and the second inequality are consequences of the fact that K_X and M are nef. It follows that $\deg\phi$ is equal to 2 and all the above inequalities are equalities. In particular, one has $K_X Z = 0$, $MZ = 0$: by the 1-connectedness of canonical divisors, this implies that Z is empty and $|K_X| = |M|$ is base point free. The surface Σ satisfies $d = 3p_g - 7$: so it is a Castelnuovo variety in the sense of [Ha], p.44 (in the notation of [Ha], here one has $M = 3$, $\epsilon = 1$, $k = 2$ and $n = p_g - 1$). Thus Σ is projectively normal (cf. [Ha], p.66), and therefore it is normal. So $\phi : X \to \Sigma$ is a double cover as defined in Section 1, the only singularities of Σ are nodes, and Σ is the canonical model of S. The map ϕ is necessarily unramified in codimension 1, since otherwise K_X would have a fixed part, and so the number of nodes of Σ can be computed by means of formula 1.3.

Castelnuovo varieties are listed in [Ha], p.65 (apart from a small mistake corrected in [Mi]): taking into account Assumption 2.1 and the above discussion, it follows that there exists (a, b, c) with $a + b + c = p_g - 3$ and a divisor $S' \subset \mathbf{P}_{a,b,c}$ of type $4T - (p_g - 5)F$ such that Σ is the image of S' via the birational map ψ induced by $|T|$ (cf. Notation 2.2). (Remark that this implies that $|K_S|$ is free; this fact can also be shown directly using the same argument as in Lemma 2.1 of [Ko]). If $a > 0$, then ψ is an isomorphism with its image; if $a = 0$ and $p_g > 5$, then ψ contracts a curve $E \subset S'$ (cf. [AK], 2.4) and is an isomorphism outside E.

We wish to show that the latter case cannot occur. By contradiction, assume that that is the case. The map $\epsilon : S \to \Sigma$ factors as $S \to S' \to \Sigma$, where $S' \to \Sigma$ contracts E to a singular point of Σ and is an isomorphism outside E. Denote by $\bar{\phi} : \hat{X} \to S$ the map obtained by resolving the indeterminacy of $X \to \Sigma$, by $\bar{E}$ the -2-curve on S which is the strict transform of E, and by $\bar{F}$ the fibre of the pencil on S induced by the projection $S' \to \mathbf{P}^1$. By the above discussion, $\bar{\phi}$ is a double cover branched over the sum Δ of all the -2-curves of S, and thus Δ is divisible by 2 in $\mathrm{Pic}(X)$. In particular $D\Delta$ is even for any divisor D on $\hat{X}$. On the other hand, $\bar{E}\bar{F} = 1$ and $\bar{F}C = 0$ for any -2-curve C distinct from E, since such

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a C arises from a node of S', and so $\Delta\bar{F} = 1$. Therefore the case $a = 0$ and $p_g > 5$ is excluded.

In order to finish the proof we only need to exclude the case $p_g = 4$. By the previous argument, in that case X would be a double cover of a quintic branched on a set of at least 24 nodes, but such a set does not exist by the proposition on page 209 of [Be2]. $\square$

Denote by $|F|$ the pencil on Σ induced by the projection $\mathbf{P}_{a,b,c} \to \mathbf{P}^1$ and by $|\tilde{F}|$ the pull-back $\phi^*|F|$. Surfaces X as in Assumption 2.1 fall into two types according to the nature of $|\tilde{F}|$:

DEFINITION 2.4. We say that a surface X as in Assumption 2.1 is of type *I* if $|\tilde{F}|$ is irreducible, and of type *II* if $|\tilde{F}|$ is reducible.

Remark that if X is of type *I* then $|\tilde{F}|$ is a linear pencil of genus 5. We will show later (Proposition 5.3) that, if X is of type *II*, then there are only a finite number of numerical possibilities for the invariants of X, so that one should think of surfaces of type *I* as of the "general case".

PROPOSITION 2.5. *Every irreducible component of a fibre of the pencil $|F|$ on Σ has multiplicity ≤ 2, and every double fibre contains 8 singular points of Σ. Moreover, if X is of type I, then $|F|$ and $\phi^*|F|$ have no multiple fibres.*

Proof. The fibres F on Σ are (possibly singular) plane quartics. Since the only singularities of Σ are nodes and the fibres F are the restriction to Σ of smooth Cartier divisors on the smooth threefold $\mathbf{P}_{a,b,c}$, the fibres of $|F|$ have double points at the singular points of Σ. So, if C is a component of a fibre of multiplicity $m > 2$, then C does not contain any singular point of Σ. C is necessarily a line and thus, if C' denotes the strict transform of C on S, one has: $K_S C' = 1$, $C'^2 = -3$, $C'(\epsilon^* F - mC') = 4 - m$. (The last equality is a consequence of the fact that C' contains no singular point of Σ and F is a plane quartic.) So it follows: $0 = C'\epsilon^* F = mC'^2 + C'(\epsilon^* F - mC') = -3m + 4 - m = 4(1 - m)$, a contradiction since $m > 1$. Let now $2C$ be a double fibre on Σ, with C an irreducible plane conic. Let $P_1, \dots P_k$ be the nodes of Σ that lie on C, let $E_1, \dots E_k$ be the corresponding -2-curves on S and let C' be the strict transform of C on S.

The pull-back of the fibre $2C$ to S is $\epsilon^*(2C) = 2C' + E_1 + \dots + E_k$, and $C'E_i = 1$, for $i = 1, \dots k$. If C is irreducible, then we have: $K_S C' = 2$,

$C'^2 = -4$, $0 = C'\epsilon^*F = C'(2C' + E_1 + \ldots + E_k) = -8 + k$, $k = 8$. If C consists of a pair of distinct lines, then a similar computation shows that each line contains 4 nodes of Σ.

Assume now that the surface X is of type I, and suppose that $2C$ is a double fibre on Σ. The pull-back of $2C$ to X is a double fibre $\tilde{F} = 2D$, where D is either a smooth hyperelliptic curve of genus 3, or the sum of two smooth elliptic curves meeting transversely at 2 points, according to whether C is irreducible or not. In both cases, D is a curve of arithmetic genus 3. Tensoring with $K_X + \tilde{F}$ the decomposition sequence $0 \to \mathcal{O}_D(-D) \to \mathcal{O}_{\tilde{F}} \to \mathcal{O}_D \to 0$ and taking global sections, one obtains the following exact sequence:

$$0 \to \mathrm{H}^0(D, K_D) \to \mathrm{H}^0(\tilde{F}, K_{\tilde{F}}) \to \mathrm{H}^0(D, \mathcal{O}_D(K_X + \tilde{F})).$$

Since $h^0(D, K_D) = 3$, and $h^0(\tilde{F}, K_{\tilde{F}}) = 5$ the image V of $\mathrm{H}^0(\tilde{F}, K_{\tilde{F}}) \to \mathrm{H}^0(D, \mathcal{O}_D(K_X + \tilde{F}))$ has dimension 2. On the other hand, V contains the restriction to D of $\mathrm{H}^0(X, K_X)$, that has dimension 3 since ϕ maps D to a conic. So we have a contradiction, and we must conclude that if X is of type I, then $|F|$ (and thus also $\phi^*|F|$) does not contain multiple fibres. $\square$

§3. The examples

We describe here the known examples of surfaces X such that the canonical map of X is 2-1 on a canonically embedded surface Σ, and we also present some new ones. We collect at the end of the section some lemmas that are needed in the description of the examples.

1. Examples with X regular.

The first example of a surface X mapped non-birationally onto a canonical surface by the canonical system was found independently by several authors ([Be1], [Ca1], [VZ]). One of the possible descriptions of the canonical image Σ is the following: Σ is a quintic surface in $\mathbf{P}^3$, defined by the vanishing of the determinant of a generic symmetric 5×5 matrix M of linear forms. The singularities of Σ are 20 nodes, occurring precisely where the rank of M drops by 2, and they form an even set. The double cover of Σ branched over the nodes is a regular surface X with $p_g = 4$. In [Ci] p.126 Ciliberto has remarked that the same method can be used to produce similar examples, with Σ a canonical complete intersection in a projective space. Notice that, if Σ is of type $(3,3)$, $(2,2,3)$ and $(2,2,2,2)$, then the examples thus obtained are not on the Castelnuovo line.

2. **Examples with $p_g(X) = 5$ and $q(X) = 2$.**

Surfaces Σ with $p_g = 5$ and $K^2 = 8$ are on the Castelnuovo line and have been described in detail in [Ho4]. If the canonical map is birational, then the canonical image is isomorphic to the canonical model, and it is the intersection of a quadric and a quartic in $\mathbf{P}^4$. If the quadric is singular, then Σ is a Castelnuovo surface of type $(0,1,1)$ or $(0,0,2)$, according to whether the quadric is singular at one point or along a line. In the former case Σ carries two different pencils of genus 3, while in the latter case Σ is necessarily singular at the points of intersection with the singular line of the quadric. (For a generic choice of the quartic, these singularities will be 4 nodes).

Let now A be a principally polarized abelian surface, let K be the Kummer surface of A and let $q : A \to K$ be the projection onto the quotient. If D is a symmetric theta divisor, then the linear system $|2D|$ is the pull-back from K of a linear system $|H|$. If D is irreducible, then $|H|$ embeds K as a quartic surface in $\mathbf{P}^3$; if D is reducible, then $|H|$ maps K 2-1 onto the smooth quadric in $\mathbf{P}^3$. Let $f : \Sigma \to K$ be the double cover branched on a curve B of $|2H|$ not meeting the singular set of K. If B is smooth, then the singularities of Σ are 32 nodes, which are the inverse image of the 16 singular points of K. If B has simple double points, then Σ has extra rational double points above the singularities of B. The sheaf $f_*\mathcal{O}_\Sigma$ splits as $\mathcal{O}_K \oplus \mathcal{O}_K(-H)$ and, using 1.1, one computes: $K^2_\Sigma = 2H^2 = 8$, $p_g(\Sigma) = p_g(K) + h^0(\mathcal{O}_K(H)) = 5$, $q(\Sigma) = h^1(\mathcal{O}_K) + h^1(\mathcal{O}_K(-H)) = 0$. Denote by $\phi : X \to \Sigma$ the map obtained from $q : A \to K$ by base-changing with f, and by $\alpha : X \to A$ the map that completes the square as in diagram 2.1. The map ϕ is branched precisely over the inverse image of the singularities of K, while α is branched on $q^*B \in |4D|$, and X is singular only above the singularities of q^*B. One has: $\alpha_*\mathcal{O}_X = \mathcal{O}_A \oplus \mathcal{O}_A(-2D)$, and thus one may compute the invariants of X as above, and obtain: $p_g(X) = 5 = p_g(\Sigma)$, $K^2_X = 16$, $q(X) = 2$. (In fact, α is the Albanese map of X). So the canonical map of X is the composition of ϕ with the canonical map of Σ. If D is irreducible, then by our construction Σ is isomorphic to the intersection in $\mathbf{P}^4$ of the cone over K with a quadric not passing through the vertex of the cone, and so it is a canonical surface. As we have already explained at the beginning, when the quadric is singular, namely when B is cut out on K by a singular quadric of $\mathbf{P}^3$, Σ is a Castelnuovo surface. In this case, it is easy to check that the genus 3 fibres are mapped to plane sections of K by ϕ, and that their inverse images in X are connected genus 5 curves; thus

X is a surface of type I.

Assume now that D is reducible: then A is isomorphic to the product $E_1 \times E_2$ of two elliptic curves, with origins O_i and, if $\pi_i : A \to E_i$, $i = 1, 2$, are the projections, then $D = \pi_1^{-1}(O_1) + \pi_2^{-1}(O_2)$. The map $\pi_1 \circ \alpha : X \to E_1$, is an elliptic pencil of genus 3 curves. We wish to show that, for a generic choice of $B \in |2H|$, the generic fibres of this pencil are not hyperelliptic. The subspace $V = q^*\mathrm{H}^0(K, 2H) \subset \mathrm{H}^0(A, 4D)$ is the subspace of even sections. It is possible to find a basis $\sigma_1^i, \dots \sigma_3^i, \tau^i$ of $\mathrm{H}^0(E_i, 4O_i)$, $i = 1, 2$, such that the σ_j^i's are even and the τ^i's are odd. So V is spanned by the products $\sigma_j^1 \sigma_k^2$, $i, j = 1, 2, 3$ and by $\tau^1 \tau^2$. It follows that the restriction of V to a generic fibre of π_1 contains sections that are not even. So, for a generic choice of $B \in |2H|$, the inverse image in X of a generic fibre of π_1 is not hyperelliptic by Lemma 3.4.

The maps π_1 and $\pi_1 \circ \alpha$ are compatible with the involutions on A, on X and on E_1, and so they induce linear pencils $p_1 : K \to \mathbf{P}^1$ and $p_1 \circ f : \Sigma \to \mathbf{P}^1$. The generic fibre of $p_1 \circ f$ is the same as the generic fibre of $q_1 \circ \alpha$, and so it is a non-hyperelliptic curve of genus 3. By Lemma 1.1 of [AK], the canonical map of Σ is not composed with a pencil and it has degree ≤ 2; on the other hand, the restriction of $|K_\Sigma|$ to a smooth fibre F of $p_1 \circ f$ is a subsystem of $|K_F|$. So we must conclude that the restriction of the canonical map of Σ to F is an embedding. Moreover, the system $f^*|2H|$, and so it separates the fibres of $p_1 \circ f$. So we conclude that the canonical map of Σ is birational and Σ is a Castelnuovo surface (of type $(0, 1, 1)$). The pull-back of the genus 3 pencil $p_1 \circ f$ factors through the elliptic pencil $\pi_1 \circ \alpha$, thus it is not connected and X is a surface of type II.

3. Surfaces of type II with $p_g = 6$, $q = 1$.

From Propositions 5.1 and 5.3, it follows that these examples arise from divisors Σ of bidegree $(4, 3)$ in $\mathbf{P}^2 \times \mathbf{P}^1$ with only nodes as singularities and having the following properties: 1) the pencil on Σ induced by the projection $p : \mathbf{P}^2 \times \mathbf{P}^1 \to \mathbf{P}^1$ has 4 double fibres, 2) Σ is smooth away from the double fibres. Such a surface Σ has 32 nodes, 8 on each of the double fibres, and these form an even set. The double cover $\phi : X \to \Sigma$ branched over the nodes is a surface of type II. In Section 6 we produce explicitly such an example. This enables us to describe a 16-dimensional family of non isomorphic surfaces X of type II with the above invariants. Let U_1 be the open subset of the 4-fold product of $\mathbf{P}^1$ with itself consisting of the

4-tuples (z_1, z_2, z_3, z_4) such that $z_i \neq z_j$ for $i \neq j$; let U_2 be the open subset of the 4-fold product of the space $\mathbf{P}^5$ of conics with itself consisting of the 4-tuples (Q_1, Q_2, Q_3, Q_4) such that Q_i is reduced, $i = 1, \dots 4$, and Q_1^2, Q_2^2, Q_3^2, Q_4^2 represent independent points in the space $\mathbf{P}^{14}$ of quartics; finally, denote by $U_3 \subset \mathbf{P}^{55}$ the open subset of irreducible divisors of bidegree $(4, 3)$ in $\mathbf{P}^2 \times \mathbf{P}^1$, and let $Z \subset U_1 \times U_2 \times U_3$ be the closed subset consisting of the points $(z_1, z_2, z_3, z_4; Q_1, Q_2, Q_3, Q_4; \Sigma)$ such that the fibre of Σ over z_i is Q_i^2, for $i = 1, \dots 4$. It is easy to check that the projection of Z onto $U_1 \times U_2$ is surjective, and that the fibre of this projection over a point of $U_1 \times U_2$ is naturally isomorphic to $\mathbf{P}^3$ minus the coordinate planes. So Z is a smooth quasiprojective variety of dimension 27. Let now U_0 be the open subset of Z consisting of the points such that the singularities of Σ are only nodes: the example of Section 6 shows that U_0 is nonempty. Moreover, the number $\nu(\Sigma)$ of nodes is a lower-semicontinuous function of $\Sigma \in U_0$ and, by Lemma 2.5, one always has $\nu(\Sigma) \geq 32$. This minimum is attained in the example, and so there is a nonempty open subset $U \subset U_0$ such that Σ has precisely 32 nodes, occurring on the double fibres. Notice that the restriction to U of the projection onto U_3 is a Galois cover of its image with Galois group S_4, the group action consisting simply in changing the ordering of the double fibres of Σ. We abuse notation and also denote by U the image of U in U_3. The double covers of surfaces $\Sigma \in U$, branched over the nodes, form a 27-dimensional family W of surfaces of type II with $q = 1$ and $p_g = 6$. The group $\mathbf{PGL}(2) \times \mathbf{PGL}(1)$ acts naturally on U, and thus on W. On the other hand, it is easy to show that two surfaces X, $X' \in W$ are isomorphic iff they belong to the same orbit of the action of $\mathbf{PGL}(2) \times \mathbf{PGL}(1)$. Thus the geometric invariant theory quotient of W by $\mathbf{PGL}(2) \times \mathbf{PGL}(1)$ is an irreducible variety of dimension 16, parametrizing non-isomorphic surfaces.

4. An infinite family of surfaces of type I with q=2.

These examples are due to Beauville (see [Ca2], 2.9). Let A, K, $q : A \to K$, D and H be as in Example 2. For $n \geq 3$, consider a smooth hypersurface G of bidegree $(1, n)$ in $\mathbf{P}^3 \times \mathbf{P}^1$: the projection onto $\mathbf{P}^1$ exhibits G as a $\mathbf{P}^2$-bundle and the hypersurfaces of bidegree $(1, k)$, $k \geq 0$ induce effective tautological hyperplane sections of G. Assume that G intersects the singular locus of $Y = K \times \mathbf{P}^1$ transversely, and the intersection at smooth points of Y is transversal. (This certainly happens for a generic choice of G). Then the surface $\Sigma = G \cap Y$ has $16n$ nodes at the intersections with the singular locus of Σ and is smooth elsewhere. Using adjunction on $\mathbf{P}^3 \times \mathbf{P}^1$, one sees that the hypersurfaces of bidegree $(1, n-2)$ induce canonical curves of Σ, and therefore the canonical system $|K_\Sigma|$ is very ample. A straightforward computation yields: $p_g(\Sigma) = 4n-3$, $K_\Sigma^2 = 12n-16$. So Σ is a Castelnuovo surface, and G, with the natural projection onto $\mathbf{P}^1$, is isomorphic to the $\mathbf{P}^2$-bundle $\mathbf{P}_{a,b,c}$ containing it. Since the canonical divisor of Σ is induced by the hypersurfaces of bidegree $(1, n-2)$, one has $a+b+c = p_g(\Sigma)-3 = 4n-6$, and $a \geq n-2$.

Now denote by X the pull-back of Σ to $A \times \mathbf{P}^1$ via the map $q \times 1$: the surface X is smooth, the projection $X \to A$ is the Albanese map, and $q \times 1$ restricts to a double cover $\phi : X \to \Sigma$ branched over the nodes Σ. Using adjunction on $A \times \mathbf{P}^1$ and $K \times \mathbf{P}^1$, one checks immediately that $p_g(X) = p_g(\Sigma)$. So ϕ is the canonical map of X. Moreover, it is easy to see that the inverse image of a fibre F of the projection $\Sigma \to \mathbf{P}^1$ is connected, and thus X is of type I.

The results that follow show that the surfaces Σ of Example 4 exist for all the admissible values of a, b and c.

LEMMA 3.1. *Let $n \geq 3$ and $n-2 \leq a \leq b \leq c$ be integers such that $a+b+c = 4n-6$; then there exists a smooth divisor $G \in \mathbf{P}^3 \times \mathbf{P}^1$ of bidegree $(1,n)$, and an isomorphism $\mathbf{P}_{a,b,c} \to G$ such that hypersurfaces of bidegree $(1, n-2)$ pull back to tautological hyperplane sections of $\mathbf{P}_{a,b,c}$.*

Write $4n-6 = 3\epsilon + \rho$, with ρ and ϵ integers, $0 \leq \rho < 3$; a generic hypersurface G of bidegree $(1,n)$, with the polarization given by hypersurfaces of bidegree $(1, n-2)$, is isomorphic to $\mathbf{P}_{a,b,c}$ with the tautological hyperplane section, where a, b, c are as follows:

$\rho = 0$, $a = b = c = \epsilon$;
$\rho = 1$, $a = b = \epsilon$, $c = \epsilon + 1$,
$\rho = 1$, $a = \epsilon$, $b = c = \epsilon + 1$.

Proof. Let T be the tautological hyperplane section and L be the fibre of the projection $p : \mathbf{P}_{a,b,c} \to \mathbf{P}^1$; the divisor $T' = T - (n-2)L$ is base point free, and the corresponding morphism $g : \mathbf{P}_{a,b,c} \to \mathbf{P}^{n+2}$ is birational. More precisely, if $a > n-2$ then g is an embedding and if $a = n-2$ then the image of g is one over $\mathbf{P}_{b,c}$. Let $h : \mathbf{P}_{a,b,c} \to \mathbf{P}^3$ be the morphism associated to a generic 3-dimensional subsystem of $|T'|$: h has degree n and maps the fibres of $\mathbf{P}_{a,b,c}$ linearly to planes in $\mathbf{P}^3$. The morphism $h \times p : \mathbf{P}_{a,b,c} \to \mathbf{P}^3 \times \mathbf{P}^1$

embeds $\mathbf{P}_{a,b,c}$ as a divisor of type $(1, n)$, and hypersurfaces of bidegree $(1, n-2)$ pull-back to elements of $|T|$ via $h \times p$.

To prove the second part of the statement, consider the space $\mathbf{P}^{4n+3}$ of divisors of bidegree $(1, n)$, and the dense open subset $U \subset \mathbf{P}^{4n+3}$ consisting of the smooth divisors. If k is an integer, then $h^0(G, \mathcal{O}_G(1, k))$ is ower semi-continuous function of $G \in U$. If, say, $\rho = 0$, then we have shown that there exists $G_0 \in U$ such that G_0 with the polarization given by hypersurfaces of bidegree $(1, n-2)$ is isomorphic to $\mathbf{P}_{\epsilon,\epsilon,\epsilon}$ with the tautological hyperplane sections. This is equivalent to the condition $h^0(G_0, \mathcal{O}_{G_0}(1, n-\epsilon-3) = 0$. Then, by semi-continuity, one has $h^0(G, \mathcal{O}(1, n-\epsilon-3) = 0$ on a dense open set $U_1 \subset U$, and so G is isomorphic to $\mathbf{P}_{\epsilon,\epsilon,\epsilon}$ for every $G \in U_1$. The same argument shows the statement for $\rho = 1, 2$. ☐

LEMMA 3.2. *Let A be an abelian surface with an irreducible principal polarization, let $K \subset \mathbf{P}^3$ be the corresponding Kummer quartic, and let G be a smooth hypersurface of bidegree $(1, n)$ in $\mathbf{P}^3 \times \mathbf{P}^1$: there exists $\gamma \in \mathbf{PGL}(3)$ such that $\Sigma_\gamma = G \cap (\gamma K \times \mathbf{P}^1)$ has $16n$ nodes, occurring at the intersections of G with the singular locus of $K \times \mathbf{P}^1$, and is smooth elsewhere.*

Proof. The proof consists simply in counting dimensions. Let $(\mathbf{P}^3)^*$ be the space of planes in $\mathbf{P}^3$, let $K^* \subset (\mathbf{P}^3)^*$ be the dual surface of K, and let $\psi : \mathbf{P}^1 \to (\mathbf{P}^3)^*$ be the map that associates to $z \in \mathbf{P}^1$ the plane $G \cap (\mathbf{P}^3 \times \{z\})$. We say that $\gamma \in \mathbf{PGL}(3)$ is "good" if γK^* and $\psi(\mathbf{P}^1)$ intersect transversely at smooth points, and moreover the intersection points are regular values of ψ and do not lie on the exceptional curves corresponding to the nodes of K. We are going to show that if γ is "good", then it satisfies the claim. Remark first of all that the points of the curve $\psi(\mathbf{P}^1)$ correspond to planes that are tangent to γK at most at one smooth point. So the surface Σ_γ has nodes at the points of intersections with the singular set of $\gamma K \times \mathbf{P}^1$. To show that Σ is smooth elsewhere, notice that $\Sigma_\gamma \cap (\mathbf{P}^3 \times \{z\})$ is just the intersection of γK with the plane $\psi(z)$, and so Σ_γ can be singular only at points $(x, z) \in G$ such that the plane $\psi(z)$ is tangent to γK at x. A computation in local coordinates shows that these points are also smooth if ψ is regular at z and the curve $\psi(\mathbf{P}^1)$ meets γK^* transversely at $\psi(z)$. In order to conclude the proof it is enough to remark that the γ's that are not "good" form a subset of dimension at most 14. This is a consequence of the following facts: 1) the subset of $\mathbf{PGL}(3)$ consisting of the elements that map a point $x_1 \in \mathbf{P}^3$ to a point x_2 has dimension 12, 2) the subset of

$\mathbf{PGL}(3)$ consisting of the elements that map a chosen point $x_1 \in \mathbf{P}^3$ and a line L_1 through x_1 to a point x_2 and a line L_2 through x_2 has dimension 10 . $\square$

The two previous lemmas together yield the following:

PROPOSITION 3.3. *Let A be an abelian surface with an irreducible principal polarization, and let $a \leq b \leq c$ be integers such that $a + b + c \equiv 2 \,(\mathrm{mod}\ 4)$ and $a \geq n := (a + b + c + 6)/4$; then there exist X and Σ as in Example 4 such that Σ is a Castelnuovo surface of type (a, b, c) and A is the Albanese variety of X.*

We close the section by proving the lemma needed in Example 2.

LEMMA 3.4. *Let E be an elliptic curve with origin O; let $B \in |4O|$ be a reduced divisor and let $f : C \to E$ be the double cover branched on B, with $\mathcal{L} = 2O$. Then C is hyperelliptic if and only if B is symmetric with respect to the elliptic involution.*

Proof. As it is explained in Section 1, C is isomorphic to a divisor $D \subset \mathcal{L}$; the line bundle $\mathcal{L}$ has a natural linearization and, if B is symmetric, then D is easily seen to be also symmetric. Thus the involution on $\mathcal{L}$ induces an involution of C, whose fixed points are the inverse images of points of order 2 of E. Since B is symmetric and reduced, it does not contain any point of order 2, and so the involution has 8 fixed points on C. By the Hurwitz formula, the quotient of C by the involution is rational, and thus C is hyperelliptic.

Conversely, assume that C is hyperelliptic and denote by $\phi : C \to \phi(C)$ the canonical map, with $\phi(C)$ a plane conic. If $g : E \to \mathbf{P}^1$ is the quotient map of the elliptic involution, then by 1.1 the canonical system $\mathrm{H}^0(C, K_C)$ contains $f^*\mathrm{H}^0(E, 2O) = f^*g^*\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ as a subsystem. So one has a map $\bar{f} : \phi(C) \to \mathbf{P}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{f} & E \\ \phi \downarrow & & \downarrow g \\ \phi(C) & \xrightarrow{\bar{f}} & \mathbf{P}^1 \end{array} \tag{3.1}$$

If we denote by i_1 the hyperelliptic involution on C and by i_2 the elliptic involution on E, then it follows immediately: $f \circ i_1 = i_2 \circ f$. In particular, if R is the ramification divisor of f, then $i_1(R) = R$. Applying the Hurwitz

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formula to f, one sees that R is a canonical divisor of C; since R is reduced, it contains no Weierstrass point. Thus, R may be written as $x + i_1(x) + y + i_1(y)$, for some $x, y \in C$ and $B = f(x) + i_2(f(x)) + f(y) + i_2(f(y))$ is a symmetric divisor. ∎

§4. Surfaces of type I with $q(X) \geq 2$

In this section we study surfaces of type I with $q \geq 2$ and we show that they are all obtained as in Example 4 of Section 3. More precisely we prove the following:

THEOREM 4.1. *Let $\phi : X \to \Sigma$ be as in Assumption 2.1; if X is of type I and $q \geq 2$, then $p_g(X) \equiv 1 \pmod 4$, $q(X) = 2$, the Albanese surface A of X has an irreducible principal polarization, and X can be constructed as in Example 4 of Section 3, with $n = (p_g(X)+3)/4$.*

The proof of Theorem 4.1 requires some preliminary steps: first we show that the Albanese variety A of X is a surface and the Albanese map is surjective, and then we prove that A is isomorphic to the Prym variety of the unramified double cover $\tilde{F} \to F$, with F a generic fibre. Then it follows that A is principally polarized and that there is a map $f : \Sigma \to K$, where K is the Kummer quartic K of A. Finally we show that f can be extended to a morphism $g : \mathbf{P}_{a,b,c} \to \mathbf{P}^3$. For the rest of the section, we will assume that $\phi : X \to \Sigma$ is as in Assumption 2.1 and that X is of type I and that $q(X) \geq 2$. In particular, the pull-back to X of the genus 3 pencil $|F|$ is a linear pencil $|\tilde{F}|$ of genus 5.

LEMMA 4.2. *The irregularity $q(X)$ is equal to 2.*

Proof. It suffices to show that $q(X) \leq 2$. Notice that for a generic fibre F, the restriction map $\mathrm{H}^0(\Sigma, K_\Sigma + F) \to \mathrm{H}^0(F, K_F)$ is surjective, by the regularity of Σ. So $\phi^*\mathrm{H}^0(\Sigma, K_\Sigma + F) \subset \mathrm{H}^0(X, K_X + \tilde{F}) \to \mathrm{H}^0(\tilde{F}, K_{\tilde{F}})$ is a subspace whose image via the restriction map $\mathrm{H}^0(X, K_X + \tilde{F}) \to \mathrm{H}^0(\tilde{F}, K_{\tilde{F}})$ has dimension 3. Since the pencil $|\tilde{F}|$ is linear, by Ramanujan vanishing one has $\mathrm{H}^1(X, K_X + \tilde{F}) = 0$ and therefore the cokernel of the above restriction map is isomorphic to $\mathrm{H}^1(X, K_X)$. Since $\tilde{F}$ has genus 5, it follows that $q(X) \leq 2$. ∎

LEMMA 4.3. *The Albanese map $\alpha : X \to A$ is surjective.*

Proof. Assume by contradiction that $\alpha(X) = B$ is a curve. We start by showing that this forces the pencil $|F|$ to be isotrivial.

Let $\alpha : X \to B$ be the Albanese pencil. In this case, diagram 2.1 reduces to:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & B \\ \phi \downarrow & & \downarrow q \\ \Sigma & \xrightarrow{f} & \mathbf{P}^1 \end{array} \tag{4.1}$$

Because the pencil $|\tilde{F}|$ is linear, each fibre maps surjectively onto B. Restricting the above diagram to a generic $\tilde{F}$ yields:

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\alpha'} & B \\ \phi' \downarrow & & \downarrow q \\ F & \xrightarrow{f'} & \mathbf{P}^1 \end{array} \tag{4.2}$$

Let n be the degree of f' and α': by the Hurwitz formula for α' one has $2n \leq 8$. The map $\phi' : \tilde{F} \to F$ is obtained from $q : B \to \mathbf{P}^1$ by base change and normalization. Let D be the branch locus of q: since ϕ' is unramified, $f'^*D = 2D'$ for a suitable divisor D' on F. Since D consists of distinct points, it follows that n is even. If n were equal to 2, then F would be hyperelliptic, contradicting the assumption that the canonical map of Σ is birational. So we are left with $n = 4$. Applying the Hurwitz formula to α' again, one sees that α' is unramified. Therefore f' is branched precisely on the points of D, and so, by the Riemann's construction (cf. [GH], p.255), there are only a finite number of possibilities for F up to isomorphism. We have thus proven that the pencil $|F|$ is isotrivial. We will finish the proof by showing that this cannot happen.

Consider the pull-back $|F'|$ of $|F|$ to the minimal desingularization S of Σ and consider a fibre F'_0 containing an exceptional curve of S. It is well known (cf. [Se]) that there are only the following two possibilities for a singular fibre F'_0 in an isotrivial fibration: either 1) F'_0 consists of a smooth multiple curve D, possibly together with some strings of smooth rational curves meeting D at distinct points, such that D intersects each string transversely at one of its ends, or 2) F'_0 is a collection of strings of rational curves whose ends all meet at one point, not necessarily transversely.

On the other hand F'_0 is the total transform of a fibre containing a node of Σ, namely of a plane quartic C with precisely a double point at the node and, possibly, other singularities. It is easy to convince oneself that in case 1) C can only be a double conic, and in case 2) C can only be a pair of irreducible conics intersecting only at one point. Case 1) is ruled out by Proposition 2.5. In case 2) the pull-back of C to X would be a union of rational curves, and therefore it would be contracted by α, but this is impossible. □

PROPOSITION 4.4. *The Albanese variety A of X is a principally polarized abelian surface, and the polarization D of A is irreducible.*

The above proposition is a consequence of the following lemmas, that describe the Prym variety Z of the cover $\tilde{F} \to F$, for a generic F and show that Z is naturally isomorphic to A.

We start by reviewing quickly the properties of Prym varieties that we need; for more details and proofs the reader may consult Chapter 12 of [LB]. Let J be the Jacobian of $\tilde{F}$, and let $\gamma : \tilde{F} \to J$ be the period map with base point $x_0 \in \tilde{F}$. The Abel-Prym map with base point x_0, $\beta : \tilde{F} \to Z$, is defined as the composition $\hat{\imath} \circ \gamma$, where $\hat{\imath} : J \to Z$ is a surjective morphism of abelian varieties with connected kernel. Z is an abelian surface having a natural principal polarization D, the restriction of β to $\tilde{F}$ is an embedding and the image of F is a divisor algebraically equivalent to $2D$, by Welters criterion ([LB], p.373).

LEMMA 4.5. *The polarization D on Z is irreducible.*

Proof. Assume by contradiction that D is reducible: then Z is a product $E_1 \times E_2$ of elliptic curves and D is algebraically equivalent to $\pi_1^{-1}(O_1)+\pi_2^{-1}(O_2)$, where π_i is the projection onto E_i, $i = 1, 2$, and $O_i \in E_i$. For a suitable choice of O_1 and O_2, the curve $\beta(\tilde{F})$ in $E_1 \times E_2$ is linearly equivalent to $\pi_1^{-1}(2O_1) + \pi_2^{-1}(2O_2)$ Denote by j_i the involution on E_i that fixes O_i, $i = 1, 2$: then $\beta(\tilde{F})$is invariant under $j_1 \times 1$ and $1 \times j_2$. The quotient of $\beta(\tilde{F})$ by the diagonal automorphism $j_1 \times j_2$ is isomorphic to F, and, via this isomorphism, $j_1 \times 1$ induces an involution of F whose quotient is a plane section of the smooth quadric in $\mathbf{P}^3$. So F is hyperelliptic, but this contradicts the assumption that the canonical map of Σ is birational. □

Given any map $h : \tilde{F} \to Y$, with Y a complex torus, there exist a unique morphism of tori $\bar{\psi} : J \to Y$ and a unique translation $\tau : Y \to Y$ such that $\tau \circ h = \bar{\psi} \circ \gamma$. If, moreover, the map h is equivariant with respect to the $\mathbf{Z}/2$-actions given by the involution on $\tilde{F}$ and by multiplication by -1 on

Y, then the kernel of $\hat{i}$ is mapped to 0 by $\bar{\psi}$; thus $\bar{\psi}$ induces a morphism $\psi : Z \to Y$ such that $\tau \circ h = \bar{\psi} \circ \beta$. We are interested in the case in which h is the restriction to $\tilde{F}$ of the Albanese map $\alpha : X \to A$. Consistently with the above notation, we denote by $\bar{\psi} : J \to A$ and $\psi : Z \to A$ the morphisms induced by the restriction of α.

LEMMA 4.6. *The morphism $\psi : Z \to A$ is an isomorphism.*

By the above discussion, ψ is an isomorphism iff $\bar{\psi}$ is surjective and has connected kernel. In turn, if we consider the dual map of $\bar{\psi}$, $\bar{\psi}^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(\tilde{F})$, then the above conditions are equivalent to $\bar{\psi}^*$ being injective. So assume that $\bar{\psi}^*$ is not injective and consider a torsion element $\xi \in \ker \bar{\psi}^*$ of order $m > 1$. Let $r : X' \to X$ be the unramified $\mathbf{Z}/m$-cover given by ξ: the restriction of r to a generic fibre $\tilde{F}$ is a disjoint union of m components isomorphic to $\tilde{F}$. Using the Stein factorization of the pull-back to X' of the pencil $\tilde{F}$, one gets the following commutative diagram, where the vertical arrows are pencils with fibre $\tilde{F}$ and the horizontal arrows are connected $\mathbf{Z}/m$-covers:

$$\begin{array}{ccc} X' & \xrightarrow{r} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{\bar{r}} & \mathbf{P}^1 \end{array} \tag{4.3}$$

The map $\bar{r}$ is ramified at at least 2 points, while r, which is obtained from $\bar{r}$ by base change and normalization, is unramified. This implies that the fibres of the pencil $|\tilde{F}|$ over the branch points of $\bar{r}$ are m-tuple fibres. But this contradicts Proposition 2.5. □

Finally, we put all the previous results together and get:

Proof of Theorem 4.1. Consider the basic diagram 2.1: we wish to show that the map $f : \Sigma \to K$ can be extended to a map $\bar{f} : \mathbf{P}_{a,b,c} \to \mathbf{P}^3$ that maps the fibres of $\mathbf{P}_{a,b,c}$ linearly to planes of $\mathbf{P}^3$. By Lemma 4.6, the Albanese variety A can be identified with the Prym variety Z of a generic $\tilde{F}$. So the fibres $\tilde{F}$ are mapped to divisors of $|2D|$, where D is the principal polarization on A (see Proposition 4.4); as a consequence, the fibres F on Σ are mapped isomorphically to plane sections of K with respect to the embedding as a quartic in $\mathbf{P}^3$. If F is a smooth fibre of Σ, then there is a natural linear isomorphism between the fibre of $\mathbf{P}_{a,b,c}$ containing F and the plane in $\mathbf{P}^3$ containing $f(F)$. So we can define a rational map

$\bar{f} : \mathbf{P}_{a,b,c} \to \mathbf{P}^3$, such that its restriction to Σ extends to the morphism f. Let now F_0 be a fibre of $\mathbf{P}_{a,b,c}$ containing an indeterminacy point of $\bar{f}$: the restriction of $\bar{f}$ to F_0 is a degenerate projectivity, whose singular locus can either be a point or a line. In the former case, the scheme theoretic image via f of the curve $\Sigma \cap F_0$ would be a 4–tuple line. This is impossible, because a Kummer surface has no such plane section. If the indeterminacy locus of $\bar{f}$ on F_0 were a line not contained in Σ, then the curve $\Sigma \cap F_0$ would be contracted to a point, but this is of course impossible. So the only possibility left is that the indeterminacy locus of $\bar{f}$ on F_0 is a line R contained in Σ. Remark that every other component of $\Sigma \cap F_0$ is contracted by f, and so the pull-back to X of every component of $\Sigma \cap F_0$ different from R is contracted by α. Arguing as in the proof of Proposition 2.5, one shows that R can contain at most 4 nodes of Σ, so the pull-back $\tilde{R}$ of R to X is a curve of genus 0 or 1. On the other hand, the scheme-theoretic image Δ of the fibre of $|\tilde{F}|$ containing $\tilde{R}$ is supported on $\alpha(\tilde{R})$, but this is impossible because Δ is an ample divisor on an abelian surface. So we conclude that $\bar{f}$ is indeed a regular map. If we denote by $p : \mathbf{P}_{a,b,c} \to \mathbf{P}^1$ the projection map, then the map $\bar{f} \times p : \mathbf{P}_{a,b,c} \to \mathbf{P}^3 \times \mathbf{P}^1$ embeds $\mathbf{P}_{a,b,c}$ as a divisor G of bidegree $(1, n)$, for a suitable value of n, and Σ is mapped isomorphically to $G \cap (K \times \mathbf{P}^1)$. To determine n, we use adjunction on $\mathbf{P}^3 \times \mathbf{P}^1$ and remark that divisors of bidegree $(1, n-2)$ cut out canonical curves on Σ, and so $p_g(X) = p_g(\Sigma) = a + b + c + 3 = 4n - 3 \equiv 1 (\mathrm{mod}\ 4)$, $n = (p_g(X) + 3)/4$. By Proposition 2.3, i), Σ has $16n$ nodes, occurring at the intersections of G with the singular locus of $K \times \mathbf{P}^1$. So the intersection of G with the singular locus of $K \times \mathbf{P}^1$ is transversal. □

§5. Surfaces of type *II*

In this section we describe surfaces of type *II* in detail and we show that the invariants of these surfaces are bounded.

So here X and Σ are as in Assumption 2.1, and moreover the pull-back $\tilde{F}$ of a generic F is disconnected. The Stein factorization of the pencil $|\tilde{F}|$ gives rise to the following commutative diagram, where p denotes the pencil $|F|$ and $\tilde{p}$ denotes the connected fibration on X through which $|\tilde{F}|$ factors:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \Sigma \\ \tilde{p} \downarrow & & \downarrow p \\ B & \xrightarrow{\bar{\phi}} & \mathbf{P}^1 \end{array} \tag{5.1}$$

The curve B is smooth and the map $\bar{\phi}$ is a double cover. We introduce a new invariant of X, the genus g of B. Notice that $g \leq q(X)$

THEOREM 5.1. *The surface Σ has precisely $2g+2$ double fibres, occurring at the branch points of $\bar{\phi}$, and $c \leq g$. Conversely, if Σ a Castelnuovo surface of type (a,b,c) with only nodes as singularities, with $c \leq g$, having $2g+2$ double fibres, and smooth outside the double fibres, then Σ has $16g+16$ nodes which form an even set, and the double cover $\phi : X \to \Sigma$ branched over the nodes is a surface of type II.*

Proof. For $i = 1, \dots 2g+2$, denote by x_i the ramification points of $\bar{\phi}$, by $y_i \in \mathbf{P}^1$ the image of x_i, and by F_i and $\tilde{F}_i$ the fibres of p and $\tilde{p}$ over y_i and x_i respectively. In diagram 5.1, the map ϕ is obtained from $\bar{\phi}$ by base change and normalization: since ϕ is unramified in codimension 1, the F_i's are double fibres, $i = 1, \dots 2g+2$, and they contain all the nodes of Σ. So, by Proposition 2.5, Σ has precisely $16g+16$ nodes. In order to show that $c \leq g$, we construct explicitly X as the normalization of a divisor in a $\mathbf{P}^2$-bundle. Denote by $\tilde{\mathbf{P}}_{a,b,c}$ the pull-back of $\mathbf{P}_{a,b,c}$ to B; so $\tilde{\mathbf{P}}_{a,b,c} = \mathrm{Proj}(\bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(a) \oplus \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(b) \oplus \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(c))$. If T and L are the tautological hyperplane section and a fibre of $\mathbf{P}_{a,b,c}$, and $\tilde{T}$, $\tilde{L}$, are the pull-backs of T and L to $\tilde{\mathbf{P}}_{a,b,c}$, then the fibre product W of p and $\bar{\phi}$ is a divisor in $\tilde{\mathbf{P}}_{a,b,c}$ linearly equivalent to $4\tilde{T}-(a+b+c-2)\tilde{L}$. The singular locus of W consists of $2g+2$ double curves, that are the intersections of W with the fibres of $\tilde{\mathbf{P}}_{a,b,c}$ over $x_1, \dots x_{2g+2}$. One has: $K_{\tilde{\mathbf{P}}_{a,b,c}} = -3\tilde{T} + \tilde{p}^*(K_B) + (a+b+c)\tilde{L}$ and $K_{\tilde{\mathbf{P}}_{a,b,c}} + W = \tilde{T} + \tilde{p}^*(K_B)$. So the canonical curves of X correspond to sections of $\tilde{T} + \tilde{p}^*(K_B + \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(2))$ vanishing on the double curves of W, namely to sections of $\tilde{T} + \tilde{p}^*(K_B - (x_1 + \ldots + x_{2g+2} + \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(2))$. The Hurwitz formula shows that K_B is linearly equivalent to $x_1 + \ldots + x_{2g+2} - \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(2)$, and so the canonical system of X is the pull-back of $\mathrm{H}^0(\tilde{\mathbf{P}}_{a,b,c}, \tilde{T})$. By assumption, we have $p_g(X) = p_g(\Sigma) = a+b+c+3$; on the other hand, $h^0(\tilde{\mathbf{P}}_{a,b,c}, \tilde{T}) = h^0(B, \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(a)) + h^0(B, \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(b)) + h^0(B, \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(c))$. Applying 1.1 to the double cover $\bar{\phi}$ (with $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}(g+1)$) yields for any integer $k \geq 0$: $h^0(B, \bar{\phi}^*\mathcal{O}_{\mathbf{P}^1}(k)) = h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k)) + h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k-g-1)) = k+1+h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k-g-1))$. So it follows that $c \leq g$.

Conversely, assume that Σ is a divisor in $\mathbf{P}_{a,b,c}$ linearly equivalent to $4T-(a+b+c-2)L$, with only nodes as singularities, with exactly $2g+2$ double fibres, $c \leq g$, and assume that Σ is smooth away from the double fibres. Then by Proposition 2.5 Σ has $16g+16$ nodes. Let $D \subset \mathbf{P}_{a,b,c}$ be the sum of the fibres of $p : \mathbf{P}_{a,b,c} \to \mathbf{P}^1$ that are double for $\Sigma \to \mathbf{P}^1$: D is smooth, it is linearly equivalent to $2D'$, where $D' = (g+1)F$, it contains all the nodes of Σ and, finally, the restriction of D to Σ is a union of double curves. Therefore, by Proposition 1.1, the nodes of Σ form an even set. If $\phi : X \to \Sigma$ is the double cover branched over the nodes, then the above computations show that X s a smooth surface such that $p_g(X) = p_g(\Sigma)$. Let $\epsilon : S \to \Sigma$ be the minimal resolution of the singularities of Σ, let E be the exceptional divisor of ϵ, let Z be the sum of the strict transforms of supports of the double fibres of Σ and let $\tilde{\phi} : \tilde{X} \to S$ be obtained from ϕ by base change with ϵ: $\tilde{\phi}$ is a smooth double cover branched on E, and the line bundle $\mathcal{L}$ associated with the cover is equal to $(g+1)\epsilon^*F - Z$. The restriction of $\mathcal{L}$ to a generic fibre F is trivial, so the inverse image in $\tilde{X}$ (and in X) of a generic F is disconnected, and X is of type *II*. ∎

We will now give bounds for the invariants of X and relations between them.

LEMMA 5.2. *The following relations between the invariants of X hold:*

$$0 \leq q - g = 3g + 3 - p_g.$$

Proof. The inequality $q - g \geq 0$ is a consequence of the fact that there exists a dominant map $\tilde{p} : X \to B$, with B a curve of genus g. The equality $q - g = 3g + 3 - p_g$ is equivalent to the formula 2.5, since by Proposition 5.1 Σ has $16g + 16$ nodes. ∎

PROPOSITION 5.3. *The numerical possibilities for the invariants of X are the following:*

a) $p_g = 3g+3, \quad q = g, \quad a = b = c = g, \quad 0 <= g \leq 26;$
b) $p_g = 3g+2, \quad q = g+1, \quad a = g-1, b = c = g, \quad 0 < g \leq 17;$
c) $p_g = 3g+1, \quad q = g+2, \quad a = b = g-1, c = g =$ *or* $a = g-2, b = c = g,$ $0 < g \leq 8.$

Proof. The topological Euler characteristic of the minimal desingularization S of Σ can be computed from Noether's formula as follows:

$$c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 9p_g(S) + 19.$$

On the other hand the following formula (see [BPV], p.97), in which $e(D)$ represents the topological characteristic of the support of a divisor D, expresses $c_2(S)$ in terms of the base and of the singular fibres of the pull-back

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to S of the fibration F on Σ, that we also denote by F:

$$c_2(S) = e(\mathbf{P}^1)e(F) + \sum_{F' \text{ singular}} e(F') - e(F).$$

(The term $e(F') - e(F)$ is always non-negative, see [BPV], p.97). From Proposition 2.5, it follows that if F' is the pull-back of a double conic on Σ, then $e(F) = 10$ or 11, according to whether the conic is smooth or not. So, recalling that by Proposition 5.1 there are $2g+2$ double fibres on Σ, and comparing the two expressions for $c_2(S)$ one obtains: $c_2(S) \geq 2(-4) + (2g+2)14$, namely $9p_g \geq 28g + 1$. The statement now follows in view of Lemma 5.2. $\square$

Remark 5.4. Notice that at least for $g = 1$ possibilities a) and b) actually occur, as it is shown by Examples 3 and 2 of Section 3. We do not know examples for possibility c).

§6. Appendix: a computation with Macaulay

We describe here how we have used Macaulay ([BS]) to show the existence of a divisor Σ of bidegree $(4,3)$ in $\mathbf{P}^2 \times \mathbf{P}^1$ with the properties required in Example 3 of Section 3. We use the notation introduced there.

We consider homogeneous coordinates (s,t) in $\mathbf{P}^1$ and (x_0, x_1, x_2) in $\mathbf{P}^2$ and set: $z_1 = (1,0)$, $z_2 = (0,1)$, $z_3 = (1,-1)$, $z_4 = (1,1)$; $Q_1 = x_0^2 + x_1^2 + x_2^2$, $Q_2 = x_0^2 + x_1^2 - x_2^2$, $Q_3 = x_0x_1 + x_0x_2 + x_1x_2$, $Q_4 = x_0^2 + 5x_0x_1 + 7x_0x_2 + 2x_1^2 + 11x_1x_2 + 3x_2^2$. We start by writing down the equation h of Σ:

```
Macaulay version 3.0, created 12 September 1994

1% ring R ! characteristic (if not 31991) ?
! number of variables ? 5
! 5 variables, please ? stx[0]-x[2]
! variable weights (if not all 1) ?
! monomial order (if not rev. lex.) ?
largest degree of a monomial : 217
1% ideal f1
! number of generators ? 1
! (1,1) ? x[0]2+x[1]2+x[2]2
1% ideal f2
! number of generators ? 1
```

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```
! (1,1) ? x[0]2+x[1]2-x[2]2
1% ideal f3 ! number of generators ? 1
! (1,1) ? x[0]x[1]+x[0]x[2]+x[1]x[2]
1% ideal f4 ! number of generators ? 1
! (1,1) ? x[0]2+5x[0]x[1]+2x[1]2+7x[0]x[2]+11x[1]x[2]+3x[2]2
1% poly s1 (s-t)*(s+t)*s
1% poly s2 (s-t)*(s+t)*t
1% poly s3 (s-t)*st
1% poly s4 (s+t)*st
1% mult f1 f1 g1
1% mult f2 f2 g2
1% mult f3 f3 g3
1% mult f4 f4 g4
1% mult s1 g1 h1
1% mult s2 g2 h2
1% mult s3 g3 h3
1% mult s4 g4 h4
1% add h1 h2 h
1% add h h3 h
1% add h h4 h
%
```

Next we show that the singularities of Σ are at most nodes. This is a local computation, that has to be repeated for each of the 6 standard open affine subsets. Consider for instance $U = \{sx_0 \neq 0\} \subset \mathbf{P}^1 \times \mathbf{P}^2$: we identify U with $\mathbf{A}^3 \subset \mathbf{P}^3$ and then consider the closure in $\mathbf{P}^3$ of $\Sigma \cap U$, defined by the equation h_{s0}. The plane at infinity is $w = 0$. The ideal I of the locus in $\mathbf{P}^3$ of the singular points of $h_{s0} = 0$ that are not nodes is generated by the derivatives of h_{s0} and by the 3×3 minors of the Hessian matrix hh_{s0} of h_{s0}. Computing the standard basis of I and using it to reduce w^{30} one gets 0, namely $w^{30} \in I$ and therefore the singularities of $\Sigma \cap U$ are at most nodes. Here is the transcript of the Macaulay session (slightly edited):

```
1% ring Ss0
! characteristic (if not 31991) ?
! number of variables ? 4
! 4 variables, please ? tx[1]x[2]w
! variable weights (if not all 1) ?
! monomial order (if not rev. lex.) ?
```

```
largest degree of a monomial : 512
1% rmap fs0 R Ss0
! s ---> ? w
! t ---> ? t
! x[0] ---> ? w
! x[1] ---> ?x[1]
! x[2] ---> ? x[2]
1% ev fs0 h hs0
1% setring Ss0
1% jacob hs0 jhs0
1% jacob jhs0 hhs0
1% flatten jhs0 jhs0
1% wedge hhs0 3 whs0
[189k][252k]
1% flatten whs0 whs0 [315k]
1% concat whs0 jhs0
1% lift-std whs0 whs0std
6.7.8.[378k]9.10.11.[441k]12.[504k][567k]
13.14.15.[630k][692k]16.[755k][818k][881k]
17.[944k][1007k]18.[1070k]19.20.[1133k]21.[1196k]
computation complete after degree 21
% ideal w
! number of generators ? 1
! (1,1) ? w30
% reduce whs0std w red
% type red 0
```

The computation in the other 5 affine open sets goes exactly in the same way. Now, to finish the computation it is enough to show that the singular locus of Σ, that we already know to be reduced and of dimension 0, has length 32. In fact, by Proposition 2.5, each of the 4 double fibres contains 8 nodes and therefore Σ is smooth outside the double fibres. First one embeds Σ in $\mathbf{P}^5$ via the Segre embedding:

```
1% ring S ! characteristic (if not 31991) ? !
number of variables ? 11 [126k]
! 11 variables, please ? stx[0]-x[2]y[0]-y[5]
! variable weights (if not all 1) ? 1:5 2:6
! monomial order (if not rev. lex.) ? 5 1 1 1 1 1 1
```

```
largest degree of a monomial : 217 512 512 512 512 512 512
1% fetch h h
1% ideal j
! number of generators ? 6
! (1,1) ? sx[0]-y[0]
! (1,2) ? sx[1]-y[1]
! (1,3) ? sx[2]-y[2]
! (1,4) ? tx[0]-y[3]
! (1,5) ? tx[1]-y[4]
! (1,6) ? tx[2]-y[5]
1% concat h j
1% std h hst
23.4.5.6.7.8.9.10.11.12.13.14.[189k]15.16.17.
computation complete after degree 17
1% elim hst helim
1% ring R
! characteristic (if not 31991) ?
! number of variables ? 6
! 6 variables, please ? y[0]-y[5]
! variable weights (if not all 1) ?
! monomial order (if not rev. lex.) ?
largest degree of a monomial : 117
1% fetch helim h
```

Now h is the ideal of Σ in $\mathbf{P}^5$; the singular locus of Σ is defined by the equations of Σ and by the 3×3 minors of the Jacobian matrix of the equations of Σ:

```
1% std h hst 23.4.5.6.7.8. computation complete after degree 8
1% jacob hst jh
1% wedge jh 3 sing
[252k][315k][378k][441k][504k][567k][630k]
1% flatten sing sing
[692k][755k][818k][881k][944k]
1% concat sing hst
1% std sing singst 0123.4.5.6.7.8. computation complete after
degree 8
1% degree singst
codimension : 5
```

```
degree : 32
```

The last line shows that the singularities of Σ are 32, as required.

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