Canad. Math. Bull. Vol. **38** (1), 1995 pp. 120–127

CONJUGACY SEPARABILITY OF GENERALIZED FREE PRODUCTS OF CERTAIN CONJUGACY SEPARABLE GROUPS

C. Y. TANG

ABSTRACT. We prove that generalized free products of finitely generated free-by-finite or nilpotent-by-finite groups amalgamating a cyclic subgroup are conjugacy separable. Applying this result we prove a generalization of a conjecture of Fine and Rosenberger [7] that groups of F-type are conjugacy separable.

1. **Introduction.** In [5] Dyer showed that generalized free products (g. f. p.) of two free groups or two finitely generated (f. g.) nilpotent groups amalgamating a cyclic subgroup are conjugacy separable (c. s.). In general it is not known whether finite extensions of c. s. groups are c. s. In fact it is not known whether finite extensions of surface groups are c. s. although surface groups are known to be conjugacy separable. However, it is known that polycyclic-by-finite groups are c. s. (Remeslennikov [10] and Formanek [8]). Also Dyer [4] proved that free-by-finite groups are c. s. Therefore we ask the natural questions whether g. f. p. of such groups with cyclic amalgamations are c. s. In this paper we prove the following:

Let A, B be f. g. free-by-finite or nilpotent-by-finite groups satisfying unique root property for elements of infinite order. Then $G = A *_H B$, where H is cyclic, is c. s. In fact using similar argument it can be shown that if A, B are f. g. free-by-finite or nilpotent-by-finite groups where H is an isolated cyclic subgroup of A and B, then $G = A *_H B$ is c. s. This gives a positive answer to a conjecture of Fine and Rosenberger [7] with slight generalization, about the c. s. of groups of F-type, which are defined by Fine and Rosenberger [7] as groups of the form:

$$G = \langle a_1, \ldots, a_n; a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}, uv \rangle \tag{1}$$

where $n \geq 2$, $\alpha_i = 0$ or $\alpha_i \geq 2$ and u, v are cyclically reduced words of infinite order on $\{a_1, \ldots, a_p\}$ and $\{a_{p+1}, \ldots, a_n\}$ respectively, where $1 \leq p \leq n-1$. In fact, in [6], Fine and Rosenberger answered positively a question of Allenby and Tang [3], whether Fuchsian groups are c. s. In the same paper they asked whether groups of the form $G = \langle a_1, \ldots, a_n; a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}, (uv)^t \rangle$ where $t \geq 2$ and n, α_i and u, v are as in (1) are c. s. Allenby [1] proved that these groups are c. s. In a private communication Rosenberger pointed out to the author that Allenby's proof actually works for groups of F-type.

The author gratefully acknowledges the partial support by the National Science and Engineering Research Council of Canada, Grant No. A-6064.

Received by the editors July 12, 1993.

AMS subject classification: Primary: 20E06, 20E26; secondary: 20E05, 20F05, 20F18.

https://doi.org/10.4153/CMB-1995-017-5 Published online by Cambridge University Press

Applying the results of this paper we prove a generalization of these results by allowing u, v to be possibly of finite orders.

Throughout this paper we made use of the following notation and terms: $N \operatorname{ch}_f G$ means N is a characteristic subgroup of finite index (f. i.) in the group G.

$a \sim_G b$ means a, b are conjugates in G. $\{a\}^G$ denotes the set of all conjugates of a in G.

If $x \in G = A *_H B$ then $\|x\|$ means the free product length of x in G.

A group G is said to have *unique root property for elements of infinite order* if $x^n = y^n$ implies $x = y$ for all $x, y \in G$ of infinite order.

A subgroup H of G is said to be *isolated* if $x^n \in H$ implies $x \in H$ for all $x \in G$.

A group G is said to be π_c if for all cyclic subgroups H of G and $x \in G \setminus H$, there exists $N \triangleleft_f G$ such that in $\bar{G} = G/N$, $\bar{x} \notin \bar{H}$.

Let $x, y \in G$ and $x \not\sim_G y$. Then x, y are said to be *conjugacy distinguishable* (c. d.) if there exists $N \triangleleft_f G$ such that in $\bar{G} = G/N$, $\bar{x} \not\sim_{\bar{G}} \bar{y}$.

We also make use of the following result of Ribes and Zalesskii [11].

THEOREM RZ. *Let G be free-by-finite and let $H_1, \ldots, H_n$ be* f. g. *subgroups of G. Then $H_1 H_2 \cdots H_n$ is closed in the profinite topology of G.*

We also adapted some of Allenby's proofs [1] for our purposes.

2. **Weak potency.** In [2] Allenby and Tang introduced the concept of potency to prove the residual finiteness (RF) of certain 1-relator groups. However, in many cases we only need a weaker version of potency.

DEFINITION 2.1. Let G be a group and $x \in G$. Then G is said to be *weakly $\langle x \rangle$-potent*, briefly, $\langle x \rangle$-wpot, if there exists a positive integer r such that for every positive integer n there exists $N_n \triangleleft_f G$ such that in $\bar{G} = G/N_n$, $\bar{x}$ is of order exactly rn. A group is said to be *weakly potent* if G is $\langle x \rangle$-wpot for all elements x of infinite order in G.

LEMMA 2.2. *Let G be* f. g. *free-by-finite or nilpotent-by-finite. Then G is weakly potent.*

PROOF. We shall only prove the case when G is free-by-finite. The case of nilpotent-by-finite is similar. Let $B \triangleleft_f G$ such that B is free. Let $x \in G$ be of infinite order. Let r be the smallest positive integer such that $x^r \in B$. Thus $|Bx| = r$ in G/B. Let $x^r \in \Gamma_{i-1}(B) \setminus \Gamma_i(B)$, $\Gamma_i(B)$ being the i-th term of the lower central series of B. Then $\Gamma_i(B) \triangleleft G$. Let $\bar{G} = G/\Gamma_i(B)$. Then $\overline{\Gamma_{i-1}(B)}$ is a f. g. free abelian subgroup of $\bar{G}$. Therefore, there exists a basis $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k\}$ such that $\bar{x}^r \in \langle \bar{y}_1 \rangle$, say, $\bar{y}_1^s = \bar{x}^r$. Let $\bar{M}_n = \Gamma_{i-1}(\bar{B})^{sn}$. Clearly $\bar{M}_n \triangleleft \bar{G}$. Let $\bar{\bar{G}} = \bar{G}/\bar{M}_n$. Then $|\bar{\bar{y}}_1| = sn$. Thus $|\bar{\bar{y}}_1^s| = n$, whence $|\bar{\bar{x}}^r| = n$. Since $|Bx| = r$ in G/B, it follows that $|\bar{\bar{x}}| = rn$. Since $\bar{\bar{B}}$ is a f. g. nilpotent group, it follows that $\bar{\bar{B}}$ is RF, whence $\bar{\bar{G}}$ is RF. Therefore, there exists $\bar{\bar{N}}_n \triangleleft_f \bar{\bar{G}}$ such that $\bar{\bar{x}}, \ldots, \bar{\bar{x}}^{rn-1} \notin \bar{\bar{N}}$. Let $\tilde{G} = \bar{\bar{G}}/\bar{\bar{N}}_n$. Then $\tilde{x}$ is of order rn in the finite group $\tilde{G}$. Let N_n be the preimage of $\bar{\bar{N}}_n$ in G. Then N_n is the required normal subgroup in G.

https://doi.org/10.4153/CMB-1995-017-5 Published online by Cambridge University Press

LEMMA 2.3. *Let $x \in G$ such that G is $\langle x\rangle$-wpot. If $h^i \sim_G h^j$ then $j = \pm i$.*

PROOF. Suppose $|i| \neq |j|$. Since G is $\langle x\rangle$-wpot, there exists an integer $r > 0$ such that for every positive integer n there exists $N_n \lhd_f G$ such that $|\bar{x}| = rn$ in $\bar{G} = G/N_n$. Let $n = |ij|$. This implies $|\bar{x}^i| = rj$ and $|\bar{x}^j| = ri$. It follows that $\bar{x}^i \not\sim_{\bar{G}} \bar{x}^j$ contradicting $x^i \sim_G x^j$. Hence $|i| = |j|$, *i.e.*, $j = \pm i$.

As an immediate consequence of Lemma 2.3, we have

LEMMA 2.4. *Let $x \in G$ such that G is $\langle x\rangle$-wpot and that $\langle x\rangle$ has unique root property in G. If $u, v \in G$ such that $u = x^i v x^j$, then either such expression is unique or $v^{-1}xv = x^{\pm 1}$.*

PROOF. If $x^i v x^j$ is not unique, let $u = x^k v x^l$ be another such expression. This implies $x^i v x^j = x^k v x^l$. Thus $v^{-1}x^{i-k}v = x^{l-j}$. Since G is $\langle x\rangle$-wpot by Lemma 2.3, $l-j = \pm(i-k)$. Thus by the unique root property of $\langle x\rangle$ in G, we have $v^{-1}xv = x^{\pm 1}$.

3. **Main results.** In order to prove our main results we need to prove some finite separability property of conjugacy classes from certain cyclic subgroups of a given group. Throughout the following consideration, unless otherwise stated, we assume that G is a finite extension of a group B, where B is either free or nilpotent. We also assume G has unique root property for elements of infinite order.

LEMMA 3.1. *Let $x, h \in G$ such that $\{x\}^G \cap \langle h\rangle = \emptyset$. Then there exists $N \lhd_f G$ such that, in $\bar{G} = G/N$, $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{h}\rangle = \emptyset$.*

PROOF. $\{x\}^G \cap \langle h\rangle = \emptyset$ implies $x \not\sim_G h^i$ for all integers i. Thus if $|h|$ is finite, since G is c. s., it follows that there exists $N \lhd_f G$ such that $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{h}\rangle = \emptyset$. Hence we can assume $\langle h\rangle$ to be infinite cyclic.

CASE 1: x IS OF INFINITE ORDER. Since $B \lhd_f G$, there exists a positive integer m such that x^m, $h^m \in B$. We show that $\{x^m\}^G \cap \langle h^m\rangle = \emptyset$. Suppose this is not so. Then there exists $g \in G$ such that $g^{-1}x^m g = h^{km}$. Since G has unique root property, $g^{-1}xg = h^k$ contradicting $\{x\}^G \cap \langle h\rangle = \emptyset$. Hence $\{x^m\}^G \cap \langle h^m\rangle = \emptyset$. Now $x^m \in B$ and $\langle h^m\rangle \subset B$. Therefore, by Dyer (Lemmas 6 and 8 [5]) there exists $N \lhd_f B$ such that in $\bar{B} = B/N$, $\{\bar{x}^m\}^{\bar{G}} \cap \langle \bar{h}^m\rangle = \emptyset$. Since $B \lhd_f G$, we can assume $N \operatorname{ch}_f B$, which implies $N \lhd_f G$. Clearly N is the required normal subgroup. For if, in $\bar{G} = G/N$, $\bar{x} \sim_{\bar{G}} \bar{h}^i$ then $\bar{x}^m \sim_{\bar{G}} \bar{h}^{im}$ contradicting $\{\bar{x}^m\}^{\bar{G}} \cap \langle \bar{h}^m\rangle = \emptyset$. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{h}\rangle = \emptyset$.

CASE 2: x IS OF FINITE ORDER, SAY, α. Let β be the smallest positive integer such that $h^\beta \in B$. Suppose $\alpha \not| \beta$. Let $(\alpha, \beta) = d$. Thus $\alpha = \alpha_1 d$ and $\beta = \beta_1 d$. Let $N \lhd_f G$ such that $N \subset B$ and $N \cap \langle x\rangle = \emptyset$. Let $\bar{G} = G/N$. We show that $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{h}\rangle = \emptyset$. Suppose $\bar{x} \sim_{\bar{G}} \bar{h}^i$. Since $|\bar{x}| = \alpha$, it follows that $|\bar{h}^i| = \alpha$. Thus, if $|\bar{h}| = n$, then $n = k\alpha$. Also $n = l\beta$, since $h^\beta \in B$. Therefore, $k\alpha = k\alpha_1 d = l\beta_1 d = l\beta$. This implies $k\alpha_1 = l\beta_1$. Since $(\alpha_1, \beta_1) = 1$, we must have $\beta_1 \mid k$. Let $k = r\beta_1$. Now $\langle \bar{h}^k\rangle$ is a subgroup of $\langle \bar{h}\rangle$ of order α. This implies elements of order α in $\langle \bar{h}\rangle$ are of the form $\bar{h}^{jk}$ where $(j, \alpha) = 1$. Therefore, $\bar{x} \sim_{\bar{G}} \bar{h}^i$

https://doi.org/10.4153/CMB-1995-017-5 Published online by Cambridge University Press

implies $\bar{h}^i = \bar{h}^{jk}$ where $(j,\alpha) = 1$. But $\bar{x} \sim_{\bar{G}} \bar{h}^{jk}$ implies $\bar{x}^d \sim_{\bar{G}} \bar{h}^{jkd} = \bar{h}^{jr\beta_1 d} = \bar{h}^{jr\beta} \in \bar{B}$. Since $\alpha \not| \beta$, $d < \alpha$. This implies $\bar{x}^d \neq 1$, whence $\bar{x}^d \notin \bar{B}$. Since $\bar{B} \lhd \bar{G}$, it follows that $\bar{x}^d \not\sim_{\bar{G}} \bar{h}^{jr\beta}$. This is a contradiction, whence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{h} \rangle = \emptyset$ if $\alpha \not| \beta$.

Next suppose $\alpha \mid \beta$, say, $\beta = l\alpha$. Again let $N \lhd_f G$ such that $N \subset B$. Let $\bar{G} = G/N$. If $\bar{x} \sim_{\bar{G}} \bar{h}^i$ then $|\bar{h}^i| = \alpha$. Now $|\bar{h}| = k\beta = kl\alpha$. This implies $i = jkl$, where $(j,\alpha) = 1$. By Lemma 2.2, G is $\langle h \rangle$-wpot. Therefore, we can choose N such that $|\bar{h}| = k\beta$ and $(k,\alpha) = d \neq 1$. Let $k = k_1 d$ and $\alpha = \alpha_1 d$. Then $\bar{x} \sim_{\bar{G}} \bar{h}^{jkl}$ implies $\bar{x}^{\alpha_1} \sim_{\bar{G}} \bar{h}^{jkl\alpha_1} = \bar{h}^{jk_1 dl\alpha_1} = \bar{h}^{jk_1\beta} \in \bar{B}$. Since $d \neq 1$, $\alpha_1 < \alpha$, whence $\bar{x}^{\alpha_1} \notin \bar{B}$. Thus as in the case $\alpha \not| \beta$ we reach a contradiction. Hence $\{\bar{x}\}^{\bar{G}} \cap \langle \bar{h} \rangle = \emptyset$. This completes the proof.

In [11] Ribes and Zalesskii proved that if $H_1, \ldots, H_n$ are f. g. subgroups of a free-by-finite group then the product set $H_1 H_2 \cdots H_n$ is closed under the profinite topology of the group. Also, in [12] Stebe proved that if H, K are subgroups of a f. g. nilpotent group then HK is closed under the profinite topology of the group. Using their results we prove the following lemma in the form we need:

LEMMA 3.2. *Let H, K be* f. g. *subgroups of G. If $x \in G \setminus HK$, then there exists $N \lhd_f G$ such that in $\bar{G} = G/N$, $\bar{x} \notin \overline{HK}$.*

PROOF. Let $C = H \cap B$ and $D = K \cap B$. Clearly C and D are of f. i. in H and K respectively. Let $\{i = h_1, h_2, \ldots, h_\alpha\}$ and $\{1 = k_1, k_2, \ldots, k_\beta\}$ be the coset representatives of C and D in H and K respectively. Then,

$$\begin{aligned} HK &= \{ch_i dk_j \; ; \; c \in C, d \in D, 1 \leq i \leq \alpha, 1 \leq j \leq \beta\} \\ &= \{h_i c' dk_j \; ; \; c' \in C, d \in D, 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}. \end{aligned}$$

Thus $x \notin HK$ implies $h_i^{-1} x k_j^{-1} \notin CD$ for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$. If $h_i^{-1} x k_j^{-1} \in B$ then by Theorem RZ or by Stebe [12], there exists $N_{ij} \lhd_f B$ such that in $\bar{B} = B/N_{ij}$, $\bar{h}_i^{-1} \bar{x} \bar{k}_j^{-1} \notin \bar{C}\bar{D}$. Since $B \lhd_f G$, we can assume $N_{ij} \operatorname{ch}_f B$, whence $N_{ij} \lhd_f G$. By abuse of notation, let $\bar{G} = G/N_{ij}$ then $\bar{h}_i^{-1} \bar{x} \bar{k}_j^{-1} \notin \bar{C}\bar{D}$, which implies $\bar{x} \notin \bar{h}_i \overline{CD} \bar{k}_j$. If $h_i^{-1} x k_j^{-1} \notin B$, we can let $N_{ij} = B$. Then $\bar{h}_i^{-1} \bar{x} \bar{k}_j^{-1} \neq 1$, which implies $\bar{x} \neq \bar{h}_i \bar{k}_j$. Let

$$N = \bigcap_{\substack{1 \leq i \leq \gamma \\ 1 \leq j \leq \beta}} N_{ij}.$$

Then $N \lhd_f G$ is the needed normal subgroup.

COROLLARY 3.3. *Let $x, y \in G$ and let H, K be* f. g. *subgroups of G. If $x \notin HyK$ then there exists $N \lhd_f G$ such that in $\bar{G} = G/N$, $\bar{x} \notin \bar{H}\bar{y}\bar{K}$.*

PROOF. Since $x \notin HyK$ if and only if $xy^{-1} \notin H(yKy^{-1})$, by Lemma 3.2, there exists $N \lhd_f G$ such that in $\bar{G} = G/N$, $\overline{xy}^{-1} \notin \bar{H}(\bar{y}\bar{K}\bar{y}^{-1})$. Hence $\bar{x} \notin \bar{H}\bar{y}\bar{K}$.

We need the following lemma to prove our main result.

LEMMA 3.3. *Let $h \in G$ be of infinite order such that $h \not\sim_G h^{-1}$. Then there exists $N \lhd_f G$ such that in $\bar{G} = G/N$, $\bar{h}^m \not\sim_{\bar{G}} \bar{h}^i$ for $\bar{h}^i \neq \bar{h}^m$, where $h^m \in B$.*

PROOF. Let C be the centralizer of h^m in G and $D = CB$. Let $\{d_1 = 1, d_2, \ldots, d_r\}$ be the coset representatives of D in G. Thus if $g \in G$ then $g = cbd_i$ where $c \in C$,

$b \in B$. This implies $\{h^m\}^G = \bigcup_{i=1}^r \{d_i^{-1}b^{-1}h^m b d_i; b \in B\}$. Since G is $\langle h^m\rangle$-wpot, by Lemma 2.3, $\{h^m\}^G \cap \langle h^m\rangle \subseteq \{h^{\pm m}\}$. If $h^m \sim_G h^{-m}$ then $g^{-1}h^m g = h^{-m}$ for some $g \in G$. Since G has unique root property for elements of infinite order, this implies $g^{-1}hg = h^{-1}$ contradicting $h \not\sim_G h^{-1}$. Therefore $\{h^m\}^G \cap \langle h\rangle = \{h^m\}$. In particular, $\{d_i^{-1}b^{-1}h^m bdi\} \cap \langle h^m\rangle \subseteq \{h^m\}$. If $i \neq 1$ and $d_i^{-1}b^{-1}h^m bd_i = h^m$, then $bd_i \in C$. This implies $d_i \in CB$ contradicting $i \neq 1$. Hence $\{d_i^{-1}b^{-1}h^m bd_i\} \cap \langle h^m\rangle = \emptyset$ for $i \neq 1$. Therefore, by Dyer ([5] Lemmas 6 and 8), there exists $X \triangleleft_f B$ such that in $\tilde{B} = B/X$, $\{\tilde{d}_i^{-1}\tilde{b}^{-1}\tilde{h}^m\tilde{b}\tilde{d}_i\} \cap \langle\tilde{h}^m\rangle = \emptyset$ for $i \neq 1$. Since $B \triangleleft_f G$, as before, we can assume $X \triangleleft_f G$. To complete the proof we need to find $N \triangleleft_f G$ such that in $\bar{G} = G/N$, $\{\bar{h}^m\}^{\bar{G}} \cap \langle\bar{h}^m\rangle = \{\bar{h}^m\}$.

Suppose $|\tilde{h}^m| = n$ and suppose $h^m \in \Gamma_{k-1}(B) \setminus \Gamma_k(B)$. Let $L = \Gamma_{k-1}^n$. Since $L \operatorname{ch} B$, $L \triangleleft G$. Let $\bar{\bar{G}} = G/L$. Then $|\bar{\bar{h}}^m| = n$. Now $\bar{\bar{B}}$ is f. g. nilpotent, whence RF. It follows that $\bar{\bar{G}}$ is RF. Thus there exists $Y \triangleleft_f G$ such that in $\hat{G} = G/Y$, $|\hat{h}| = mn$. Let $N = X \cap Y$. We shall show that N is the required normal subgroup of f. i. in G. Let $\bar{G} = G/N$. If $\bar{b}^{-1}\bar{h}^m\bar{b} = \bar{h}^{jm}$, $1 < j < n$, then $\hat{b}^{-1}\hat{h}^m\hat{b} = \hat{h}^{jm}$. Since $L \subset X$ and $\bar{b}^{-1}\bar{h}^m\bar{b} = \bar{h}^m$ we have $\hat{b}^{-1}\hat{h}^m\hat{b} = \hat{h}^m$. This implies $\bar{h}^m = \bar{h}^{jm}$. Thus $\hat{h}^{(j-1)m} = 1$ contradicting $|\hat{h}^m| = n$. Hence $\{\bar{h}^m\}^{\bar{B}} \cap \langle\bar{h}^m\rangle = \{\bar{h}^m\}$. Moreover, by the choice of X, $\{\bar{h}^m\}^{\bar{G}} \cap \langle\bar{h}^m\rangle = \{\bar{h}^m\}$. This proves the lemma.

We are now ready to prove our main results.

THEOREM 3.4. *Let A, B be* f. g. *free-by-finite or nilpotent-by-finite groups with unique root property for elements of infinite order. Let $G = A *_H B$ where $H = \langle h\rangle$. Then G is* c. s.

PROOF. Let $x, y \in G$ such that $x \not\sim_G y$. We can assume x, y to be of minimal lengths in their respective conjugacy classes. Also by Dyer's result which states that the g. f. p. of two c. s. groups amalgamating a finite subgroup is c. s. (Theorem 4 [5]), we need only consider the case when $\langle h\rangle$ is infinite cyclic.

CASE 1. $\|x\| = \|y\| = 0$. This implies $x = h^i$, $y = h^j$ and $h^i \not\sim_G h^j$. In particular $h^i \not\sim_A h^j$ and $h^i \not\sim_B h^j$. Since A, B are c. s., there exist $N_A \triangleleft_f A$ and $N_B \triangleleft_f B$ such that $\bar{h}^i \not\sim_{\bar{A}} \bar{h}^j$ and $\bar{h}^i \not\sim_{\bar{B}} \bar{h}^j$ where $\bar{A} = A/N_A$ and $\bar{B} = B/N_B$. Now A, B are both $\langle h\rangle$-wpot. Let r_1, r_2 be positive integers such that for each positive integer n there exist $N_n \triangleleft_f A$ and $M_n \triangleleft_f B$ such that $|N_n h| = r_1 n$ and $|M_n h| = r_2 n$ in A/N_n and B/M_n respectively. Clearly $h^i \not\sim_G h^j$ implies $i \neq j$. Suppose $|i| \neq |j|$. Let $k = |i|\,|j|$. Let $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$, where $\bar{A} = A/N_{r_2 k}$ and $\bar{B} = B/M_{r_1 k}$. Thus $|\bar{h}| = r_1 r_2 k$. This implies $|\bar{h}^i| = r_1 r_2 |j|$ and $|\bar{h}^j| = r_1 r_2 |i|$. Since $|i| \neq |j|$, it follows that $|\bar{h}^i| \not\sim_{\bar{G}} |\bar{h}^j|$. Therefore we can assume $j = -i$, *i.e.* $h^i \not\sim_G h^{-i}$. This implies $h \not\sim_G h^{-1}$. Let C and D be the free or nilpotent groups of f. i. in A and B respectively. Let $C \cap D = \langle h^m\rangle$. Then, by Lemma 3.3, there exist $N \triangleleft_f A$ and $M \triangleleft_f B$ such that in $\bar{A} = A/N$ and $\bar{B} = B/M$, $\bar{h}^{im} \not\sim_A \bar{h}^j$ and $\bar{h}^{im} \not\sim_B \bar{h}^j$ for $\bar{h}^j \neq \bar{h}^{im}$ in $\bar{A}$ or $\bar{B}$. Moreover, since A, B are both $\langle h\rangle$-wpot, we can assume $N \cap \langle h\rangle = M \cap \langle h\rangle = \langle h^{kim}\rangle$. Let $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$. We need to show $\bar{h}^i \not\sim_{\bar{G}} \bar{h}^{-i}$. Suppose $\bar{h}^i \sim_{\bar{G}} \bar{h}^{-i}$. Then $\bar{h}^{im} \sim_{\bar{G}} h^{-im}$. But by the choice of N and M, $\bar{h}^{-im}$ is the only conjugate in $\bar{G}$ of $\bar{h}^{im}$ in $\langle\bar{h}\rangle$. Moreover, A, B being $\langle h\rangle$-wpot, we can choose M and N such that $\bar{h}^{im} \neq \bar{h}^{-im}$ in both $\bar{A}$ and $\bar{B}$. It follows that $\bar{h}^{im} \not\sim_{\bar{G}} \bar{h}^{-im}$. Hence $\bar{h}^i \not\sim_{\bar{G}} \bar{h}^{-i}$. Thus $\bar{x} \not\sim_{\bar{G}} \bar{y}$. Hence by Dyer's theorem [5], x, y are c. d. in G.

CASE 2. $\|x\| \neq \|y\|$ and $\|x\|, \|y\| \leq 1$. We shall only treat the case of $\|x\| = 0$ and $\|y\| = 1$. The other case is similar. Since y is of the minimal length in $\{y\}^G$, $y \notin \langle h \rangle$. Thus by, Lemma 3.1, there exists $M_B \triangleleft_f B$ such that in $\bar{B} = B/M_B$, $\{\bar{y}\}^{\bar{B}} \cap \langle \bar{h} \rangle = \emptyset$. Let $M_B \cap \langle h \rangle = \langle h^\beta \rangle$. Since B is $\langle h \rangle$-wpot, this implies there exists a positive integer r_2, say, such that for each positive integer n there exists $M_n \triangleleft_f B$ such that $M_n \cap \langle h \rangle = \langle h^{r_2 n} \rangle$. Similarly there exists $N_n \triangleleft_f A$ such that $N_n \cap \langle h \rangle = \langle h^{r_1 n} \rangle$ for some positive integer r_1. Thus,

$$M_{r_1\beta} \cap \langle h \rangle = \langle h^{r_1 r_2 \beta} \rangle = N_{r_2\beta} \cap \langle h \rangle \subset \langle h^\beta \rangle.$$

Let $\bar{\bar{G}} = \bar{\bar{A}} \ast_{\bar{\bar{H}}} \bar{\bar{B}}$, where $\bar{\bar{A}} = A/N_{r_2\beta}$ and $\bar{\bar{B}} = B/M_{r_1\beta}$. If $\bar{\bar{y}} \sim_{\bar{\bar{B}}} \bar{\bar{h}}^k$, then $\bar{y} \sim_{\bar{B}} \bar{h}^k$ contradicting $\{\bar{y}\}^{\bar{B}} \cap \langle h \rangle = \emptyset$. Therefore $\bar{\bar{y}} \not\sim_{\bar{\bar{B}}} \bar{\bar{h}}^k$. This implies $\bar{\bar{y}} \not\sim_{\bar{\bar{G}}} \bar{\bar{h}}^k$. As in Case 1, x, y are c. d. in G.

CASE 3. $\|x\|, \|y\| \geq 2$ and $\|x\| \neq \|y\|$. Since A, B are both π_c and $\langle h \rangle$-wpot, there exist $N \triangleleft_f A$ and $M \triangleleft_f B$ such that in $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$, $\|\bar{x}\| = \|x\| \neq \|y\| = \|\bar{y}\|$. Hence $\bar{x} \not\sim_{\bar{G}} \bar{y}$. It follows that x, y are c. d. in G.

CASE 4. $\|x\| = \|y\| \geq 2$. Let $x = u_1 u_2 \cdots u_r$ and $y = v_1 v_2 \cdots v_r$ be reduced words in $G = A \ast_H B$. Then, by Dyer [5], $x \sim_G y$ if and only if for some $1 \leq i \leq r$ the system of equations:

$$\left.\begin{aligned} u_{i+1} &= x_0^{-1} v_1 x_1 \\ u_{i+2} &= x_1^{-1} v_2 x_2 \\ &\vdots \\ u_{i+r} &= x_{r-1}^{-1} v_r x_0 \end{aligned}\right\} \quad I(i)$$

has a solution of H. Since $x \not\sim_G y$, $I(i)$ has no solution in H for all $1 \leq i \leq r$. We shall show that for each i, there exist finite images $\bar{A}$, $\bar{B}$ of A, B respectively such that the corresponding system of equations $\overline{I(i)}$ has no solution in $\bar{H}$.

If for each i, $I(i)$ having no solution in H implies that there exists k (depending on i) such that $u_{i+k} \notin H v_k H$, then by Corollary 3.3, there exists $N \triangleleft_f A$ such that in $\bar{A} = A/N$, $\bar{u}_{i+k} \notin \bar{H} \bar{v}_k \bar{H}$. Similarly for $\bar{B} = B/M$ if $u_{i+k}, v_k \in B$. Since A, B are $\langle h \rangle$-wpot, it is easy to see that we can assume $N \cap H = M \cap H$. This implies $\bar{x} \not\sim_{\bar{G}} \bar{y}$, where $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$. It follows that x, y are c. d. in G. Hence we can assume that, for each i, $I(i)$ has no solution in H, but for some i, each

$$u_{i+j} = x_{j-1}^{-1} v_j x_j \tag{3.1}$$

where $1 \leq j \leq r$, has a solution in H. Therefore, we need to show for this case there also exist finite images of $\bar{A}$, $\bar{B}$ of A, B respectively such that in $\bar{G} = \bar{A} \ast_{\bar{H}} \bar{B}$, $\bar{x} \not\sim_{\bar{G}} \bar{y}$. So, suppose (3.1) has a solution, say, $u_{i+j} = h^\alpha v_j h^\beta$. Then, by Lemma 2.4, either $v_j^{-1} h v_j = h^{\pm 1}$ or the solution is unique.

(i) Suppose none of the solutions for (3.1) is unique. This implies $v_j^{-1}hv_j = h^{\pm 1}$ for each $1 \leq j \leq r$. This means each solution $u_{i+j} = h^{\alpha}v_jh^{\beta}$ can be substituted by $h^{\alpha+d}v_jh^{\alpha-d}$ if $v_j^{-1}hv_j = h^{-1}$ or by $h^{\alpha-d}v_jh^{\alpha+d}$ if $v_j^{-1}hv_j = h$, where d is an arbitrary integer. It follows that, by suitable choices of d, we can find integers $\alpha_0, \alpha_1, \ldots, \alpha_r$ such that $u_{i+j} = h^{-\alpha_{j-1}}v_jh^{\alpha_j}$, $1 \leq j \leq r$. Since $x \not\sim_G y$, $\alpha_0 \neq \alpha_r$. We note that if $v_j^{-1}hv_j = h^{-1}$ occurs (a) an even number of times for $1 \leq j \leq r$ then if we replace $h^{-\alpha_0}$ by $h^{-\beta_0}$ where $\alpha_0 - \beta_0 = d$, then $\alpha_r - \beta_r = d$, *i.e.*, $\beta_r = \alpha_r - d$. On the other hand if $v_j^{-1}hv_j = h^{-1}$ occurs (b) an odd number of times for $1 \leq j \leq r$ then if we replace $h^{-\alpha_0}$ by $h^{-\beta_0}$ where $\alpha_0 - \beta_0 = d$ then $\beta_r = \alpha_r + d$. In case (a) $\alpha_r - \alpha_0 = \beta_r - \beta_0 \neq 0$. In case (b) $\alpha_r - \alpha_0 = \beta_r - \beta_0 - 2d$. Thus, in case (a), let m be the maximum of all $|\alpha_r - \alpha_0|$ for $0 \leq i < r$. Since A, B are $\langle h\rangle$-wpot, we can choose $N_{r_2n} \lhd_f A$ and $M_{r_1n} \lhd_f B$ so that in $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$, where $\bar{A} = A/N_{r_2n}$ and $\bar{B} = B/M_{r_1n}$, $\bar{h}^{\beta_r-\beta_0} \neq 1$. Thus $\bar{x} \not\sim_{\bar{G}} \bar{y}$ whence x, y are c. d. in G. In case (b), since $\alpha_r - \alpha_0 \neq 0$ and d can be chosen arbitrarily, we must have $\beta_r - \beta_0$ to be odd. Thus, as in case (a), if we choose n even for N_{r_2n} and M_{r_1n} then $\bar{h}^{\beta_r-\beta_0} \neq 1$. Therefore, again, we get x, y c. d. in G.

(ii) Assume, for each $0 \leq i < r$, there exists j such that $u_{i+j} = x_{j-1}^{-1}v_jx_j$ has a unique solution in H. This implies that, for each i, there are only finitely many possible solutions for the x_j's. Moreover, since $x \not\sim_G y$, no combinations of these solutions is a solution of $I(i)$ for each i. Since A, B are $\langle h\rangle$-wpot, there exist $N \lhd_f A$ and $M \lhd_f B$ such that $N \cap \langle h\rangle = M \cap \langle h\rangle = \langle h^t\rangle$ such that $t > 2m$ where m is the maximum of all $|\alpha_k|$ for which h^{α_k} is a solution to an equation $u_{i+j} = x_{j-1}^{-1}v_jx_j$ where $0 \leq i, j < r$. Thus the corresponding system of equations $\overline{I(i)}$, has no solution in $\bar{G} = \bar{A} *_{\bar{H}} \bar{B}$ where $\bar{A} = A/N$ and $\bar{B} = B/M$. Therefore, $\bar{x} \not\sim_{\bar{G}} \bar{y}$, whence x, y are c. d. in G. This completes the proof.

In proving Lemmas 2.4, 3.1 and 3.3 we made use of the unique root property for elements of infinite order. It is not difficult to show that by assuming $\langle h\rangle$ to be an isolated subgroup of G we can also prove Lemmas 2.4, 3.1 and 3.3. Thus we have:

THEOREM 3.5. *Let $G = A *_H B$, where A, B are* f. g. *free-by-finite or nilpotent-by-finite groups and $H = \langle h\rangle$ is an isolated subgroup of A and B. Then G is* c. s.

We now apply Theorem 3.4 to prove that groups of the form:

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n; a_1^{\alpha_1}, \ldots, \alpha_m^{\alpha_m}, b_1^{\beta_1}, \ldots, b_n^{\beta_n}, (uv)^t\rangle \tag{3.2}$$

where $t \geq 1$ and u, v are cyclically reduced words on $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ respectively, are c. s. In the case when $t = 1$ and u, v are of infinite order, G is a group of F-type. These groups are conjectured by Fine and Rosenberger [7] to be c. s.

THEOREM 3.6. *Let G be given by (3.2). Then G is* c. s.

PROOF.

CASE 1. If one of u, v, say, v is of infinite order, then let,

$$A = \langle a_1, \ldots, a_m, x; a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, (ux)^t\rangle$$
$$B = \langle b_1, \ldots, b_n; b_1^{\beta_1}, \ldots, b_n^{\beta_n}\rangle.$$

https://doi.org/10.4153/CMB-1995-017-5 Published online by Cambridge University Press

Since A, B are free products of cyclic groups, they are free-by-finite. Moreover, by Magnus, Karrass and Solitar ([9] p. 194), A, B have unique root property for elements of infinite order. Since $G = A *_{x=y} B$, by Theorem 3.4, G is c. s.

CASE 2. Both u and v are of finite orders. Since elements of finite orders in G are conjugates of a_i^k and b_i^k, WLOG we can assume $G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n; a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, b_1^{\beta_1}, \ldots, b_n^{\beta_n}, (a_1^k b_1^l)^t \rangle$. Let $c = (\alpha_1, k)$ and $d = (\beta_1, l)$. Let $W = \langle x, y; x^\gamma, y^\delta, (x^r y^s)^r \rangle$, where $\gamma = \frac{\alpha_1}{c}$, $\delta = \frac{\beta_1}{d}$, $r = \frac{k}{c}$ and $s = \frac{l}{d}$. Since $(\gamma, r) = (\delta, s) = 1$,

$$W \approx W_1 = \langle x, y; x^\gamma, y^\delta, (xy)^t \rangle.$$

By Fine and Rosenberger [6], W_1 is c. s., whence W is c. s. Since $|a_1^c| = |x|$ is finite, $L = \langle a_1, a_1^{\alpha_1} \rangle *_{a_1^c = x} W$ is c. s. by Dyer [5]. Let $M = L *_{y=b_1^d} \langle b_1; b_1^{\beta_1} \rangle$. Again M is c. s. Now, let

$$P = \langle a_2, \ldots, a_m, b_2, \ldots, b_n; a_2^{\alpha_2}, \ldots, a_m^{\alpha_m}, b_2^{\beta_2}, \ldots, b_n^{\beta_n} \rangle.$$

Then $G = P * M$, whence G is c. s.

COROLLARY 3.7. *Groups of F-type are* c. s.

REFERENCES

1. R. B. J. T. Allenby, *Conjugacy separability of a class of* 1*-relator products*, Proc. Amer. Math. Soc. **116**(1993), 621–628.

2. R. B. J. T. Allenby and C. Y. Tang, *The residual finiteness of some* 1*-relator groups with torsion*, J. Algebra **71**(1981), 132–140.

3. ———, *Conjugacy separability of certain* 1*-relator groups with torsion*, J. Algebra **103**(1986), 619–637.

4. J. L. Dyer, *Separating conjugates in free-by-finite groups*, J. London Math. Soc. (2) **20**(1979), 215–221.

5. ———, *Separating conjugates in amalgamated free products and HNN extensions*, J. Austral. Math. Soc. Ser. A **29**(1980), 35–51.

6. B. Fine and G. Rosenberger, *Conjugacy separability of Fuchsian groups and related questions*, Contemp. Math. **109**(1990), 11–18.

7. ———, *Generalized algebraic properties of Fuchsian groups*, Groups, St. Andrews **1**, London Math. Soc. Lecture Note Series **160**, 1989.

8. E. Formanek, *Conjugacy separability in polycyclic groups*, J. Algebra **42**(1976), 1–10.

9. W. Magnus, A. Karrass and D. Solitar, *Combinatorial groups theory*, Pure Appl. Math. **XIII**, Wiley-Interscience, New York, London, Sydney, 1966.

10. V. M. Remeslennikov, *Conjugacy in polycyclic groups*, Algebra i Logika **8**(1969), 712–725, Russian; Translation: Algebra and Logic **8**(1969), 404–11.

11. L. Ribes and P. A. Zalesskii, *On the profinite topology on a free group*, Bull. London Math. Soc. **25**(1993), 37–43.

12. P. F. Stebe, *Residual solvability of an equation in nilpotent groups*, Proc. Amer. Math. Soc. **54**(1976), 57–58.

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
N2L 3G1

https://doi.org/10.4153/CMB-1995-017-5 Published online by Cambridge University Press