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SHARP TRANSFERABILITY AND FINITE SUBLATTICES OF FREE LATTICES

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Transferable and sharply transferable lattices were defined and characterized in [**1**]. Finite sublattices of free lattices were studied in [**2**] and a characterization of them given in [**3**]. In this paper, we will show that the class of finite sharply transferable lattices coincides with the class of finite sublattices of free lattices.

We recall here the relevant definitions from [**1**]. If X and Y are non-empty subsets of a partially ordered set, then we say $X \prec Y$ holds if and only if, for every element x of X, there exists an element y of Y such that $x \leqq y$. If $\mathscr{L} = \langle L; \vee, \vee \rangle$ is a finite lattice, $x \in L$ and $U \subseteq L$, then we say $\langle x, U \rangle$ is a *minimal pair* of $\mathscr{L}$ if and only if the following three conditions are satisfied:

(i) $x \notin U$;
(ii) $x \leqq \bigvee U$;
(iii) if $U' \subseteq L$, $U' \prec U$, and $x \leqq \bigvee U'$, then $U \subseteq U'$.

Then we say the lattice $\mathscr{L}$ satisfies the condition $(T_\vee)$ if and only if there exists a linear ordering $\langle x_1, x_2, \ldots, x_n \rangle$ of all the elements of L such that

$$\text{if } \langle x_i, U \rangle \text{ is a minimal pair and } x_j \in U, \text{ then } j < i.$$

The condition $(T_\wedge)$ is the dual of $(T_\vee)$.

A lattice $\mathscr{L}$ satisfies condition (W) if and only if, whenever $a, b, c, d \in L$ and $a \wedge b \leqq c \vee d$, we have $a \leqq c \vee d$ or $b \leqq c \vee d$ or $a \wedge b \leqq c$ or $a \wedge b \leqq d$.

A lattice $\mathscr{L}$ is called *transferable* if and only if, whenever $\mathscr{L}$ is embeddable into the lattice $I(\mathscr{L}')$ of all ideals of a lattice $\mathscr{L}'$, then $\mathscr{L}$ is embeddable into $\mathscr{L}'$.

If $\varphi : \mathscr{L} \to I(\mathscr{L}')$ is an embedding, then a mapping $\psi : L \to L'$ is called *φ-normal* if and only if, for $x, y \in L$, $x \leqq y$ in $\mathscr{L}$ if and only if $\psi(x) \in \varphi(y)$. Then $\mathscr{L}$ is called *sharply transferable* if and only if, for every embedding $\varphi : \mathscr{L} \to I(\mathscr{L}')$, there is an embedding $\psi : \mathscr{L} \to \mathscr{L}'$ which is φ-normal.

In [**1**] it was shown that a finite lattice $\mathscr{L}$ is sharply transferable if and only if $\mathscr{L}$ satisfies $(T_\vee)$, $(T_\wedge)$, and (W). We will show that these conditions coincide with those used by McKenzie [**3**] to characterize finite sublattices of free lattices.

Throughout the following discussion, we let $\mathscr{L}$ and $\mathscr{L}'$ denote lattices, with underlying sets L and L', respectively, and we assume $\mathscr{L}$ is finite.

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Definition 1. (a) For any mapping $\alpha : L \to L'$ define $\alpha' : L \to L'$ so that for $x \in L$,

$$\alpha'(x) = \bigwedge \{\bigvee \alpha(U) : x \leqq \bigvee U, U \subseteq L\}.$$

(b) Define $\alpha^{(n)}$ for $n \in \omega$ by $\alpha^{(0)} = \alpha$ and $\alpha^{(i+1)} = (\alpha^{(i)})'$ for all $i \in \omega$.

We remark that if α is the inclusion map of L into the lattice freely generated by L, then $(\alpha^{(n)}|n \in \omega)$ is the lower half of the *standard limit table* for $\mathscr{L}$, as defined by McKenzie [**3**, §6].

LEMMA 1. *Let* $\alpha : L \to L'$ *be a mapping.*
(a) α' *is order-preserving.*
(b) *For all* $x \in L$, $\alpha'(x) \leqq \alpha(x)$.
(c) α *is join-preserving if and only if* $\alpha = \alpha'$.
(d) *If* α *is order-preserving, then for all* $x \in L$, $\alpha'(x) = \alpha(x) \wedge \bigwedge \{\bigvee \alpha(U) : \langle x, U\rangle$ *is a minimal pair*$\}$.

Proof. (a) is obvious. (b) follows from putting $U = \{x\}$. For (c), suppose α is join-preserving. If $U \subseteq L$ and $x \leqq \bigvee U$, then $\alpha(x) \leqq \bigvee \alpha(U)$. Thus, $\alpha(x) \leqq \alpha'(x)$.

Suppose next that $\alpha = \alpha'$. By (a), it suffices to show that $\alpha(x \vee y) \leqq \alpha(x) \vee \alpha(y)$. Let $U = \{x, y\}$. Then

$$\alpha(x \vee y) = \alpha'(x \vee y) \leqq \bigvee \alpha(U) = \alpha(x) \vee \alpha(y).$$

To prove (d), let $U_0 \subseteq L$ and $x \leqq \bigvee U_0$ hold. If $x \leqq y$ for some $y \in U_0$, then $\alpha(x) \leqq \bigvee \alpha(U_0)$. Otherwise, there clearly exists $U \subseteq L$ such that $\langle x, U\rangle$ is a minimal pair, and $U \prec U_0$. Then $\bigvee \alpha(U) \leqq \bigvee \alpha(U_0)$. In any case, we have

$$\alpha(x) \wedge \bigwedge \{\bigvee \alpha(U) : \langle x, U\rangle \text{ is a minimal pair}\} \leqq \bigvee \alpha(U_0).$$

Since this holds for every such U_0, we conclude that

$$\alpha(x) \wedge \bigwedge \{\bigvee \alpha(U) : \langle x, U\rangle \text{ is a minimal pair}\} \leqq \alpha'(x).$$

By (b), equality holds.

The next result yields a condition for a standard lower limit table to terminate.

LEMMA 2. *Let* $\alpha : L \to L'$ *be an order-preserving mapping. If* $\mathscr{L}$ *satisfies* $(T_\vee)$ *and* N *is the number of elements of* L, *then* $\alpha^{(N+1)} = \alpha^{(N)}$.

Proof. We begin by showing that for all $n \in \omega$ and $x \in L$,

(1) $\alpha^{(n+1)}(x) = \alpha(x) \wedge \bigwedge \{\bigvee \alpha^{(n)}(U) : \langle x, U\rangle \text{ is a minimal pair}\}$.

The proof is by induction. For $n = 0$, this is simply Lemma 1(d). For $n > 0$,

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by inductive hypothesis, we have

$$\begin{aligned}\alpha^{(n+1)}(x) &= \alpha^{(n)}(x) \wedge \bigwedge \{\bigvee \alpha^{(n)}(U) : \langle x, U\rangle \text{ is a minimal pair}\} \\ &= \alpha(x) \wedge [\bigwedge \{\bigvee \alpha^{(n-1)}(U) : \{x, U\} \text{ is a minimal pair}\}] \\ &\qquad \wedge [\bigwedge \{\bigvee \alpha^{(n)}(U) : \langle x, U\rangle \text{ is a minimal pair}\}].\end{aligned}$$

But, by Lemma 1(b), the first bracketed term contains the last bracketed term, so the above expression reduces to the right hand side of (1), completing the proof of (1).

Now, using $(T_\vee)$, let $\langle a_1, a_2, \ldots, a_N\rangle$ be an ordering of all the elements of L so that if $\langle a_i, U\rangle$ is a minimal pair and $a_j \in U$, then $j < i$. The lemma then follows immediately from the next statement.

Claim: Let $1 \leqq i \leqq N$ and $k \in \omega$. If $i \leqq k$, then

(2) $\alpha^{(i)}(a_i) = \alpha^{(k)}(a_i)$.

The proof is by induction on i. For $i = 1$, there exist no minimal pairs of the form $\langle a_i, U\rangle$, so (1) implies that $\alpha^{(n+1)}(a_1) = \alpha(a_1)$ for all $n \in \omega$, so (2) follows.

Next, suppose $1 < j \leqq N$ and that (2) holds whenever $i < j$. Then $j < k$ implies

$$\alpha^{(k)}(a_j) = \alpha(a_j) \wedge \bigwedge \{\bigvee \alpha^{(k-1)}(U) : \langle a_j, U\rangle \text{ is a minimal pair}\}.$$

But if $\langle a_j, U\rangle$ is a minimal pair and $a_i \in U$, then $i < j \leqq k - 1$, so by the inductive hypothesis,

$$\alpha^{(k-1)}(a_i) = \alpha^{(i)}(a_i) = \alpha^{(j-1)}(a_i).$$

Thus,

$$\begin{aligned}\alpha^{(k)}(a_j) &= \alpha(a_j) \wedge \bigwedge \{\bigvee \alpha^{(j-1)}(U) : \langle a_j, U\rangle \text{ is a minimal pair}\} \\ &= \alpha^{(j)}(a_j),\end{aligned}$$

completing the proof.

Remark 1. If $\langle x, U\rangle$ is a minimal pair, then every $u \in U$ is join-irreducible. It follows that by a slight modification of the above proof, N can be replaced by $M - 1$, where M is the number of join-irreducible elements of $\mathscr{L}$.

The following notion is taken from [**3**].

Definition 2. An epimorphism $\varphi : \mathscr{L}' \to \mathscr{L}$ is called *upper* [respectively, *lower*] *bounded* if and only if for every $x \in L$, $\varphi^{-1}\{x\}$ has a greatest [respectively, least] element.

In the next definition, we single out two properties of McKenzie's limit tables [**3**, §6].

Definition 3. Let $\varphi : L' \to L$ be a surjection.

(a) A mapping $\alpha : L \to L'$ is called a *φ-transversal* if and only if, for all $x \in L$, $\varphi(\alpha(x)) = x$.

(b) A φ-transversal α is called *cofinal* if and only if, for all $x \in L'$, there exists $n \in \omega$ such that $\alpha^{(n)}(\varphi(x)) \leqq x$.

LEMMA 3. *Let $\varphi : \mathscr{L}' \to \mathscr{L}$ be an epimorphism and let $\alpha : L \to L'$ be a cofinal φ-transversal. Then φ is lower bounded if and only if $\alpha^{(n+1)} = \alpha^{(n)}$ for some $n \in \omega$. Furthermore, in this case $\alpha^{(n)}(x)$ is the least element of $\varphi^{-1}\{x\}$, for every $x \in L$.*

Proof. This is trivial, since clearly $\alpha^{(n)}$ is a φ-transversal for every $n \in \omega$.

Definition 4. Let $\mathscr{F}$ be the lattice freely generated by the set L. The homomorphism $f : \mathscr{F} \to \mathscr{L}$ whose restriction to L is the identity is called the *standard epimorphism onto* $\mathscr{L}$.

LEMMA 4. *Let $\varphi : \mathscr{L}' \to \mathscr{L}$ be an epimorphism, and let $\alpha : L \to L'$ be a φ-transversal. Let $X \subseteq L'$ be a generating set for $\mathscr{L}'$. If, for all $x \in X$, there exists $n \in \omega$ such that $\alpha^{(n)}(\varphi(x)) \leqq x$, then α is cofinal. In particular, if $f : \mathscr{F} \to \mathscr{L}$ is the standard epimorphism onto $\mathscr{L}$, then the inclusion map $\iota : L \to F$ is a cofinal f-transversal.*

Proof. Let $Y \subseteq L'$ be the set of all $y \in L'$ such that $\alpha^{(n)}(\varphi(y)) \leqq y$ for some $n \in \omega$. Since $X \subseteq Y$, it suffices to prove Y is a sublattice of $\mathscr{L}'$. Let $x, y \in Y$. Then choose $m \in \omega$, $m > 0$, such that $\alpha^{(m)}(\varphi(x)) \leqq x$ and $\alpha^{(m)}(\varphi(y)) \leqq y$. Since $\alpha^{(m)}$ is order-preserving, we have

$$x \wedge y \geqq \alpha^{(m)}(\varphi(x)) \wedge \alpha^{(m)}(\varphi(y)) \geqq \alpha^{(m)}(\varphi(x) \wedge \varphi(y)) = \alpha^{(m)}(\varphi(x \wedge y)).$$

Therefore $x \wedge y \in Y$. Furthermore, let $U = \{\varphi(x), \varphi(y)\}$. Then $\varphi(x \vee y) \leqq \bigvee U$, so

$$\alpha^{(m+1)}(\varphi(x \vee y)) \leqq \bigvee \alpha^{(m)}(U) = \alpha^{(m)}(\varphi(x)) \vee \alpha^{(m)}(\varphi(y)) \leqq x \vee y.$$

Thus, $x \vee y \in Y$, completing the proof.

Now we have immediately:

THEOREM 1. *If a finite lattice $\mathscr{L}$ satisfies $(T_\vee)$, then the standard epimorphism onto $\mathscr{L}$ is lower bounded.*

Proof. This follows by Lemmas 1(a), 2, 3, and 4.

We consider next the converse proposition. For this we will require the following result from [**1**].

THEOREM 2. *Given a finite lattice $\mathscr{L}$, there exists a lattice $\hat{\mathscr{L}}$ and an embedding $\varphi : \mathscr{L} \to I(\hat{\mathscr{L}})$ such that if there exists a φ-normal join-preserving mapping $\psi : \mathscr{L} \to \hat{\mathscr{L}}$, then $\mathscr{L}$ satisfies $(T_\vee)$.*

THEOREM 3. *For a finite lattice $\mathscr{L}$, if the standard epimorphism $f : \mathscr{F} \to \mathscr{L}$ is lower bounded, then $\mathscr{L}$ satisfies $(T_\vee)$.*

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Proof. Let $\hat{\mathscr{L}}$ and $\varphi : \mathscr{L} \to I(\hat{\mathscr{L}})$ be as given by Theorem 2. If ψ and ψ^* are φ-normal mappings of L into $\hat{L}$, we say that $\psi \prec \psi^*$ holds if and only if, for $x \in L$, $U \subseteq L$,

(3) if $x \leqq \bigvee U$, then $\psi(x) \leqq \bigvee \psi^*(U)$.

We observe that for any φ-normal mapping ψ, there exists a φ-normal mapping ψ^* with $\psi \prec \psi^*$. Indeed, given x, U such that $x \leqq \bigvee U$, we have $\psi(x) \in \varphi(x) \subseteq \bigvee \varphi(U)$, so for each $y \in U$, we can select $y_{\langle x, u\rangle} \in \varphi(y)$ so that

$$\psi(x) \leqq \bigvee \{y_{\langle x, u\rangle} : y \in U\}.$$

Doing this for each x and U, we then define, for $y \in L$,

$$\psi^*(y) = \bigvee \{y_{\langle x, u\rangle} : x \leqq \bigvee U \text{ and } y \in U\}.$$

It is then trivial to check that ψ^* is φ-normal and that (3) holds.

Let $\iota : L \to F$ be the inclusion mapping, where $\mathscr{F} = \langle F; \wedge, \vee \rangle$. Then, by Lemmas 3 and 4 and the hypothesis, $\iota^{(N+1)} = \iota^{(N)}$ for some $N \in \omega$. Choose an arbitrary φ-normal mapping $\psi_0 : L \to \hat{L}$, and, for $0 \leqq n < N$, choose ψ_{n+1} so that $\psi_n \prec \psi_{n+1}$. Define $\beta : \mathscr{F} \to \hat{\mathscr{L}}$ to be the homomorphism whose restriction to L is ψ_N. We claim that $\beta \circ \iota^{(N)}$ is a join-preserving φ-normal mapping.

Indeed, $\iota^{(N)}$ is join-preserving by Lemma 1(b), so $\beta \circ \iota^{(N)}$ is join-preserving. If $x \in L$, then, by Lemma 3, $\iota^{(N)}(x) \leqq x$, so

$$\beta \circ \iota^{(N)}(x) \leqq \beta(x) = \psi_N(x) \in \varphi(x),$$

since ψ_N is φ-normal.

It remains to show that $\beta \circ \iota^{(N)}(x) \geqq \psi_0(x)$, which we do by proving by induction on j that for $0 \leqq j \leqq N$,

(4) for all $x \in L$, $\beta \circ \iota^{(j)}(x) \geqq \psi_{N-j}(x)$.

By definition we have $\beta \circ \iota^{(0)} = \psi_N$. If $0 \leqq j < N$, then, for $x \in L$,

$$\begin{aligned}\beta \circ \iota^{(j+1)}(x) &= \beta(\bigwedge \{\bigvee \iota^{(j)}(U) : U \subseteq L, x \leqq \bigvee U\}) \\ &= \bigwedge \{\bigvee \beta \circ \iota^{(j)}(U) : U \subseteq L, x \leqq \bigvee U\},\end{aligned}$$

the first equality by Definition 1, the second since β is a homomorphism and $\mathscr{L}$ is finite. By inductive hypothesis, for each $y \in U$, $\beta \circ \iota^{(j)}(y) \geqq \psi_{N-j}(y)$, whence

$$\begin{aligned}\beta \circ \iota^{(j+1)}(x) &\geqq \bigwedge \{\bigvee \psi_{N-j}(U) : U \subseteq L, x \leqq \bigvee U\} \\ &\geqq \psi_{N-j-1}(x) = \psi_{N-(j+1)}(x),\end{aligned}$$

the last inequality coming from (3). This proves (4), and therefore the claim. The conclusion of the theorem then follows by Theorem 2.

Combining Theorems 1 and 3, we have:

COROLLARY 1. *A finite lattice $\mathscr{L}$ satisfies* $(T_\vee)$ *if and only if the standard epimorphism onto $\mathscr{L}$ is lower bounded.*

Of course we have the dual.

COROLLARY 2. *A finite lattice $\mathscr{L}$ satisfies* $(T_\wedge)$ *if and only if the standard epimorphism onto $\mathscr{L}$ is upper bounded.*

In [3] McKenzie proved the following:

THEOREM 4. *A finite lattice $\mathscr{L}$ is embeddable into a free lattice if and only if $\mathscr{L}$ satisfies* (W) *and the standard epimorphism onto $\mathscr{L}$ is both upper and lower bounded.*

The following is the principal result of [**1**].

THEOREM 5. *A finite lattice $\mathscr{L}$ is sharply transferable if and only if $\mathscr{L}$ satisfies* (W), $(T_\vee)$, *and* $(T_\wedge)$.

As a consequence, we have our main result.

THEOREM 6. *A finite lattice is sharply transferable if and only if it is embeddable into a free lattice.*

Remark 2. We have as a corollary that the class of sharply transferable lattices is closed under sublattices. This suggests the question: does a sublattice of a lattice satisfying $(T_\vee)$ also satisfy $(T_\vee)$?

Remark 3. Corollary 1 and Lemma 2 give a considerable improvement on the upper bound for the length of a limit table given by McKenzie [**3**, § 6]. Remark 1 gives a further improvement.

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