$$= \frac{1}{12n^2} - \frac{1}{4n^3} + 3\frac{1}{12n^4} \mp \ldots$$

$$> \frac{1}{12n^2} - \frac{1}{4n^3}$$

since the series is alternating and the terms converge monotonically to zero. Therefore, if we define

$$\varepsilon_n = \frac{1}{12n^2} - \sum_{n=1}^{\infty} \sigma_n$$

we conclude that

$$0 < \varepsilon_n < \frac{1}{4n^3}$$

as stated in the theorem. This completes the proof.

Concluding remarks

Our method does not lead to an error term $O\left(\frac{1}{n^4}\right)$ since the terms of order $\frac{1}{n^3}$ for σ_n do not cancel. It would be desirable to modify this geometric reasoning to achieve such a cancellation (perhaps using telescopic cancellation, if necessary).

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107.20 Euler's constant and the speed of convergence

Introduction

Inspired in part by [1, 2], we present an elementary and unified approach to defining Euler's constant γ, and to obtaining bounds on the associated speed of convergence. These bounds give a modest refinement (with entirely different proof) of those obtained in the much-cited paper [3]. See the Proposition below.

Definition of γ

Since $f(x) = \frac{1}{x}$ is decreasing on $(0, \infty)$ and $\int_n^{n+1} \frac{1}{x}\,dx = \ln\left(\frac{n+1}{n}\right)$, we have

$$\frac{1}{n+1} \leqslant \ln\left(\frac{n+1}{n}\right) \leqslant \frac{1}{n} \qquad (n = 1, 2, 3, \ldots). \tag{1}$$

Consider now the sequence

$$(a_n) = \left(1, \ln\frac{2}{1}, \frac{1}{2}, \ln\frac{3}{2}, \frac{1}{3}, \ln\frac{4}{3}, \frac{1}{4}, \ln\frac{5}{4}, \ldots\right).$$

By (1) and the Alternating Series test, a.k.a. Leibniz's test, the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges to some number γ, which is Euler's constant. (The coarsest of estimates here gives $1 - \ln 2 \approx 0.3068 < \gamma < 1$. In fact, $\gamma \approx 0.5772$.)

Denote by γ_{2n-1} a partial sum having an odd number of terms and by γ_{2n} the next partial sum, having an even number of terms:

$$\gamma_{2n-1} = 1 - \ln\frac{2}{1} + \frac{1}{2} - \ln\frac{3}{2} + \ldots + \frac{1}{n-1} - \ln\frac{n}{n-1} + \frac{1}{n},$$

$$\gamma_{2n} = 1 - \ln\frac{2}{1} + \frac{1}{2} - \ln\frac{3}{2} + \ldots + \frac{1}{n-1} - \ln\frac{n}{n-1} + \frac{1}{n} - \ln\frac{n+1}{n}.$$

Here then,

$$\lim_{n\to\infty} \gamma_{2n-1} = \lim_{n\to\infty} \gamma_{2n} = \gamma,$$

with γ_{2n-1} decreasing to γ and γ_{2n} increasing to γ. In each of these there is a lot of cancellation, and we get

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n+1) = \gamma_{2n} \leqslant \gamma \leqslant \gamma_{2n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n.$$

Speed of Convergence

In [3] are the estimates $\frac{1}{2(n+1)} \leqslant \gamma_{2n-1} - \gamma \leqslant \frac{1}{2n}$. We improve these by continuing the current line of investigation.

Proposition:

$$\frac{1}{2(n+1)} \leqslant \gamma - \gamma_{2n} \leqslant \frac{1}{2}\ln\left(1 + \frac{1}{n}\right) \leqslant \gamma_{2n-1} - \gamma \leqslant \frac{1}{2n}.$$

Proof: Since $f(x) = \frac{1}{x}$ is convex on $(0, \infty)$, the midpoint rule applied on $[n, n+2]$ is an underestimate of the definite integral there, and so

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$$\frac{2}{n+1} \leqslant \ln\left(\frac{n+2}{n}\right) = \ln\left(\frac{n+2}{n+1}\right) + \ln\left(\frac{n+1}{n}\right).$$

That is

$$\frac{1}{n+1} - \ln\left(\frac{n+2}{n+1}\right) \leqslant \ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1}. \tag{2}$$

Likewise, the trapezium rule applied on $[n, n+1]$ is an overestimate, and so

$$\ln\left(\frac{n+1}{n}\right) \leqslant \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right).$$

That is

$$\ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1} \leqslant \frac{1}{n} - \ln\left(\frac{n+1}{n}\right). \tag{3}$$

Inequalities (2) and (3) together read

$$\frac{1}{n+1} - \ln\left(\frac{n+2}{n+1}\right) \leqslant \ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1} \leqslant \frac{1}{n} - \ln\left(\frac{n+1}{n}\right).$$

Therefore, setting

$$b_n = a_n - a_{n+1},$$

we have

$$b_{n+2} \leqslant b_{n+1} \leqslant b_n. \tag{4}$$

Now, by the right-hand inequality in (4),

$$\begin{aligned}
\gamma_{2n-1} - \gamma &= \left(\ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1}\right) + \left(\ln\left(\frac{n+2}{n+1}\right) - \frac{1}{n+2}\right) + \left(\ln\left(\frac{n+3}{n+2}\right) - \frac{1}{n+3}\right) + \dots \\
&= b_{n+1} + b_{n+3} + b_{n+5} + b_{n+7} + \dots \\
&= \frac{1}{2}\left(b_{n+1} + b_{n+1} + b_{n+3} + b_{n+3} + b_{n+5} + b_{n+5} + \dots\right) \\
&\leqslant \frac{1}{2}\left(b_n + b_{n+1} + b_{n+2} + b_{n+3} + b_{n+4} + b_{n+5} + \dots\right) \\
&= \frac{1}{2}a_n = \frac{1}{2n}.
\end{aligned}$$

By the left-hand inequality in (4), we obtain very similarly

$$\begin{aligned}
\gamma_{2n-1} - \gamma &\geqslant \frac{1}{2}\left(b_{n+1} + b_{n+2} + b_{n+3} + b_{n+4} + b_{n+5} + \dots\right) \\
&= \frac{1}{2}a_{n+1} = \frac{1}{2}\ln\left(\frac{n+1}{n}\right).
\end{aligned}$$

Therefore

$$\frac{1}{2}\ln\left(\frac{n+1}{n}\right) \leqslant \gamma_{2n-1} - \gamma \leqslant \frac{1}{2n}.$$

In an almost identical way – we leave the details to the reader – one obtains

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$$\frac{1}{2(n+1)} \leqslant \gamma - \gamma_{2n} \leqslant \frac{1}{2}\ln\left(\frac{n+1}{n}\right).$$

Remark

The estimates in the Proposition offer $2(\gamma - \gamma_{2n})$ and $2(\gamma_{2n-1} - \gamma)$ as refinements of inequalities (1). Consequently,

$$\left(1 + \frac{1}{n}\right)^{n} \leqslant e^{2n(\gamma_{2n-1}-\gamma)} \leqslant e \leqslant e^{2(n+1)(\gamma-\gamma_{2n})} \leqslant \left(1 + \frac{1}{n}\right)^{n+1}.$$

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107.21 Proof Without Words: An inverse tangent inequality

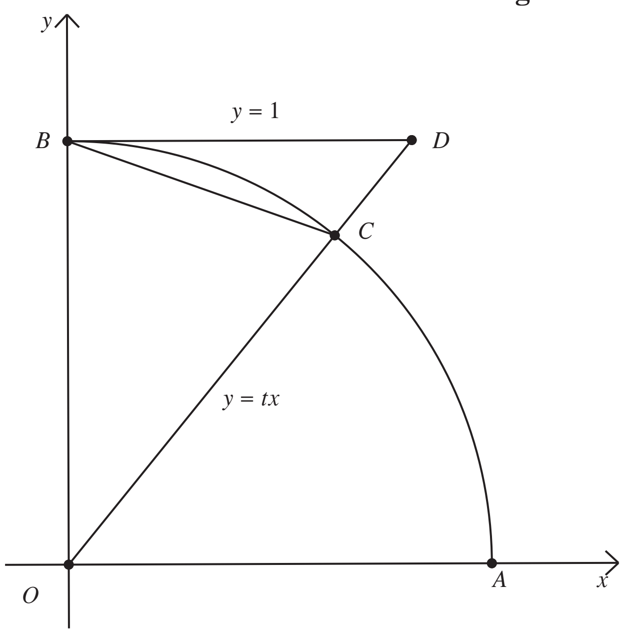

FIGURE 1