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ON APPROXIMATIONS TO SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS OF THE URYSOHN TYPE

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1. **Introduction.** This note will derive a priori estimates of the errors due to replacing the given integral operator A by a similar operator A^* of the same type when successive approximations are applied to the integral equation $\varphi = A\varphi$.

The existence and uniqueness of solutions to this equation follow easily by applying a well known fixed point theorem in a Banach space to the above mapping [1, 2]. Moreover, sufficient conditions for the existence and uniqueness of a solution to Urysohn's equation are stated explicitly in a note by the author [3].

2. **Formulation of the problem.** We recall that Urysohn's integral equation is defined as

(2.1) $$\varphi = A\varphi \qquad \text{where } A \text{ is given by:}$$

(2.2) $$A\varphi := \lambda \int_G F(x, y; \varphi(y))\, dy + f(x).$$

Here G is a closed bounded set in E^n. The function F is assumed to be measurable for each value of $\varphi(y)$ and for each $x, y \in G$. We also assume that

$$\int_G F(x, y\colon \varphi(y))\, dy \in L^2(G)$$

for each $\varphi \in S$, where S denotes a closed sphere of radius $\rho > 0$ about θ in $L^2(G)$.(1) Moreover we assume that F satisfies a generalized Lipschitz condition in G, namely:

(2.3) $$|F(x, y; u_1) - F(x, y; u_2)| \le a(x, y)\, |u_1 - u_2|$$

for any u_1, u_2, with $a(x, y)$ satisfying:

(2.4) $$0 < \|a\|^2 := \int_G \int_G a^2(x, y)\, dx\, dy \le A^2$$

(where A is a positive constant). We also assume that λ satisfies:

(2.5) $$|\lambda| \le \|a\|^{-1}$$

(1) i.e. the space of real-valued, square integrable functions on T with norm:

$$\|x\|^2 := \textstyle\int_G x^2(t)\, dt.$$

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and that $f \in L^2(G)$. We may then conclude [3] that the iteration

$$\varphi_{n+1} := A\varphi_n \quad \text{with } n = 0, 1, \ldots; \quad \varphi_0 \in S \tag{2.6}$$

converges to the unique solution of (2.1), provided that ρ satisfies:

$$\left\| f+\lambda \int_G F(x, y; \theta)\, dy \right\| \leq \rho(1-K) \quad \text{with } K := |\lambda|\, \|a\|. \tag{2.7}$$

Unfortunately, due to computational difficulties, such as evaluation of the integrals, the method of successive approximations is not always suitable for calculating approximations to a solution of (2.1) in applications. Therefore a somewhat different procedure for generating approximations to the solution is desired.

The approach chosen in this note is the consideration of another integral equation of the same type, with a different integrand which is "similar" to that in (2.1). In short, we introduce a perturbed operator A^* of similar structure to the operator A where A^* is assumed to be much simpler to compute in order to generate approximations to the solution of (2.1).

3. **Results.** The following theorem gives us the desired a priori estimates of the errors induced by replacing A by A^* in (2.6).

THEOREM. *Let A be the operator of* §2 *under the hypotheses there assumed. Let $F^*(x, y, u)$ be another function similar to $F(x, y, u)$. Suppose that for any $\varphi \in S$ we have:*

$$|F^*(x, y; \varphi(y)) - F(x, y; \varphi(y))| \leq \omega(x, y) \tag{3.1}$$

and for any u

$$|F^*(x, y; u) - F^*(x, y; \theta)| \leq \alpha(x, y)\, |u| \tag{3.2}$$

where α and ω are nontrivial L^2 functions on $G \times G$. Furthermore suppose that we have

$$\|\alpha\| \leq \|a\|. \tag{3.3}$$

Then the iteration

$$\varphi^*_{n+1} := A^*\varphi^*_n, \qquad n = 0, 1, \ldots, \quad \textit{where} \quad \varphi^*_0 := \varphi_0 \in S \tag{3.4}$$

with

$$A^*\varphi := \lambda \int_G F^*(x, y; \varphi(y))\, dy + f(x), \tag{3.5}$$

with $\varphi \in S$, can be carried out indefinitely, and all φ^*_n, $n=0, 1$ remain within the sphere S of radius ρ and centre θ, provided that:

$$\|A^*\theta\| \leq \rho(1-K). \tag{3.6}$$

Then we have also:

$$\|\varphi^*_n - \varphi^*_1\| \leq \frac{K}{1-K}\, \|\varphi_0 - \varphi^*_1\| + \frac{2\, \|\omega\|\, |\lambda| \sqrt{\text{meas}(G)}}{1-K} \tag{3.7}$$

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and

$$\|\varphi_n-\varphi_n^*\| \leq \frac{\|\omega\|\ |\lambda|\sqrt{\text{meas}(G)}}{1-K} \tag{3.8}$$

and

$$\|\varphi-\varphi_1^*\| \leq \frac{K}{1-K}\|\varphi_0-\varphi_1^*\|+\frac{\|\omega\|\ |\lambda|\sqrt{\text{meas}(G)}}{1-K} \tag{3.9}$$

The changes $\|\varphi_n^*-\varphi_{n+1}^*\|$ *are strictly decreasing, at least as long as*

$$\|\varphi_{n+1}^*-\varphi_n^*\| > \frac{2\ \|\omega\|\ |\lambda|\sqrt{\text{meas}(G)}}{1-K} \tag{3.10}$$

Proof. (i) First let us estimate $\|A\varphi-A^*\varphi\|$, $\forall\varphi\in S$. From (2.2) and (3.5) it follows that:

$$|A\varphi-A^*\varphi|^2 \leq |\lambda|^2\left|\int_G |F(x,y;\varphi(y))-F^*(x,y;\varphi(y))|\,dy\right|^2$$

using (3.1) and the Schwarz inequality we find

$$\|A\varphi-A^*\varphi\| \leq |\lambda|\ \|\omega\|\sqrt{\text{meas}(G)} =: \varepsilon \tag{3.11}$$

We may assume that ε is a small positive number.

(ii) Let us find an estimate for $\|A^*\varphi-A^*\theta\|$. Again using (3.5), (3.2), the Schwarz inequality, and (3.3), we find that for any $\varphi\in S$:

$$\|A^*\varphi-A^*\theta\| \leq |\lambda|\ \|\alpha\|\ \|\varphi\| \leq K\rho,$$

hence $\|A^*\varphi\|\leq\|A^*\theta\|+K\rho$, and because of (3.6) we have

$$\|A^*\varphi\| \leq \rho, \qquad \forall\varphi\in S. \tag{3.12}$$

Therefore, A^* maps the sphere S with radius ρ and centre θ into itself and the iterations (3.4) are defined for $n=0, 1, \ldots$ and all the φ_n^* remain within the sphere S.

(iii) Next, let us estimate $\|\varphi-\varphi_1^*\|$. We have $\|\varphi-\varphi_1^*\|\leq\|\varphi-\varphi_1\|+\|\varphi_1-\varphi_1^*\|$. Using the inequality [3]:

$$\|\varphi-\varphi_n\| \leq \frac{K^n}{1-K}\|\varphi_1-\varphi_0\|,$$

and the relations (2.6) and (3.4) we find that:

$$\|\varphi-\varphi_1^*\| \leq \frac{K}{1-K}\|\varphi_1-\varphi_0\|+\varepsilon$$

From

$$\|\varphi_1-\varphi_0\| \leq \|\varphi_1-\varphi_1^*\|+\|\varphi_1^*-\varphi_0\| = \|\varphi_1^*-\varphi_0\|+\|A\varphi_0-A^*\varphi_0\|$$

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and (3.11) we get: $\|\varphi_1-\varphi_0\| \le \|\varphi_1^*-\varphi_0\|+\varepsilon$, and hence

$$\|\varphi-\varphi_1^*\| \le \frac{K}{1-K}\,\|\varphi_1^*-\varphi_0\|+\frac{\varepsilon}{1-K} \tag{3.13}$$

which together with (3.11) proves (3.9).

(iv) Now, we consider $\|\varphi_n-\varphi_n^*\|$. Since

$$\|\varphi_n-\varphi_n^*\| = \|A\varphi_{n-1}-A^*\varphi_{n-1}^*\| \le \|A\varphi_{n-1}-A\varphi_{n-1}^*\|+\|A\varphi_{n-1}^*-A^*\varphi_{n-1}^*\|$$

we have

$$\|\varphi_n-\varphi_n^*\| \le K\,\|\varphi_{n-1}-\varphi_{n-1}^*\|+\varepsilon, \qquad n = 1, 2, \ldots$$

By induction we find that:

$$\|\varphi_n-\varphi_n^*\| \le \frac{\varepsilon}{1-K} = \frac{\|\omega\|\,|\lambda|\,\sqrt{\text{meas}(G)}}{1-K} \tag{3.14}$$

and this proves (3.8).

(v) Let us now find an estimate for $\|\varphi_n^*-\varphi_1^*\|$. From $\|\varphi_n^*-\varphi_1^*\| \le \|\varphi_n^*-\varphi_n\|+\|\varphi_n-\varphi_1^*\|$ we find with the help of (3.14) that:

$$\|\varphi_n^*-\varphi_1^*\| \le \frac{\varepsilon}{1-K}+\|\varphi_n-\varphi_1^*\|. \tag{3.15}$$

Using the fact that A is a contraction mapping [3], it can easily be shown that:

$$\|\varphi_n-\varphi_1\| \le \frac{K}{1-K}\,\|\varphi_1-\varphi_0\|.$$

Hence (3.13) remains valid if one replaces φ by φ_n, yielding:

$$\|\varphi_n-\varphi_1^*\| \le \frac{K}{1-K}\,\|\varphi_1^*-\varphi_0\|+\frac{\varepsilon}{1-K}.$$

From this and (3.15) we obtain (3.7).

(vi) Finally, we estimate $\|\varphi_{n+1}^*-\varphi_n^*\|$. We have:

$$\|\varphi_n^*-\varphi_{n+1}^*\| \le \|A^*\varphi_{n-1}^*-A\varphi_{n-1}^*\|+\|A\varphi_{n-1}^*-A\varphi_n^*\|+\|A\varphi_n^*-A^*\varphi_n^*\|$$

Hence $\|\varphi_n^*-\varphi_{n+1}^*\| \le 2\varepsilon+K\|\varphi_{n-1}^*-\varphi_n^*\|$ and therefore:

$$\|\varphi_{n-1}^*-\varphi_n^*\|-\|\varphi_n^*-\varphi_{n+1}^*\| \ge (1-K)\left(\|\varphi_{n-1}^*-\varphi_n^*\|-\frac{2\varepsilon}{1-K}\right).$$

Because of (3.10) we conclude that: $\|\varphi_{n-1}^*-\varphi_n^*\| > \|\varphi_n^*-\varphi_{n+1}^*\|$ which completes the proof of the theorem.

REMARK. The main part of this proof is similar to the proof of Urabe's theorem [4], [5].

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