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ON THE EXISTENCE OF POSITIVE DECAYING ENTIRE SOLUTIONS FOR A CLASS OF SUBLINEAR ELLIPTIC EQUATIONS

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1. Introduction. In recent years there has been a growing interest in the existence and asymptotic behavior of entire solutions for second order nonlinear elliptic equations. By an entire solution we mean a solution of the elliptic equation under consideration which is guaranteed to exist in the whole Euclidean N-space $\mathbf{R}^N$, $N \geqq 2$. For standard results on the subject the reader is referred to the papers [**2-7**, **9-21**].

The study of entire solutions, which at an early stage was restricted to simple equations of the form $\Delta u + f(x, u) = 0$, $x \in \mathbf{R}^N$, Δ being the N-dimensional Laplacian, has now been extended and generalized to elliptic equations of the type

$$\text{(A)} \qquad Lu + f(x, u, Du) = 0, \quad x \in \mathbf{R}^N,$$

where

$$L = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij} + \sum_{i=1}^{N} b_i(x)D_i,$$

$$D_i = \partial/\partial x_i,\ D_{ij} = \partial^2/\partial x_i \partial x_j,\ 1 \leqq i, j \leqq N, \quad \text{and}$$

$$D = (D_1, \dots, D_N).$$

Thus various existence theorems have been obtained which are applicable to (A) in which f may depend genuinely on Du; see e.g. [**3**, **6**, **7**, **12**, **13**, **14**, **17**, **20**]. Needless to say, however, not all such equations can be covered by the existing theories of entire solutions. For example, it is not known if the equation

$$\text{(B)} \qquad \Delta u + c(x)|Du|^\delta = 0, \quad x \in \mathbf{R}^N,$$

$\delta > 0$ being a constant, possesses an entire solution other than constant functions which are obviously solutions of (B).

The objective of this paper is to develop existence theorems of nonconstant positive entire solutions for equation (A) subject to the condition

$$f(x, u, 0) \equiv 0 \quad \text{for } (x, u) \in \mathbf{R}^N \times \mathbf{R}_+.$$

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We start with equation (B) (Section 2) and show that (B) has a positive *decaying* entire solution, that is, a positive entire solution tending uniformly to zero as $|x| \to \infty$, if $0 < \delta < 1$ and if $c(x)$ is a positive locally Hölder continuous function in $\mathbf{R}^N$ which is "small" in some sense. Then, in Section 3 we consider more general equations of the form

$$\text{(C)} \qquad Lu + f(x, Du) = 0, \quad x \in \mathbf{R}^N,$$

with the same L as in (A) and $f(x, 0) \equiv 0$ for $x \in \mathbf{R}^N$, and derive criteria for (C) to have a positive decaying entire solution under the hypothesis that $f(x, Du)$ is "sublinear" with respect to Du. Finally, in Section 4 we attempt to generalize the results of Section 3 to equation (A) in which f depends on u as well and is sublinear with respect to u and Du. The main theorems are proved by means of the supersolution-subsolution method, or the method of barriers. The sublinearity and the smallness of the functions in the structure hypotheses for (A) (or (B) or (C)) are needed in constructing suitable supersolutions and subsolutions which guarantee the existence of the desired entire solution of the respective equation.

2. The equation (B). We begin by considering the simplest equation

$$\text{(B)} \qquad \Delta u + c(x)|Du|^{\delta} = 0, \quad x \in \mathbf{R}^N,$$

where $N \geqq 3$, $0 < \delta < 1$, and $c(x)$ is positive and locally Hölder continuous in $\mathbf{R}^N$ (with exponent $\theta \in (0, 1)$). Put

$$c^*(r) = \max_{|x|=r} c(x), \quad c_*(r) = \min_{|x|=r} c(x), \quad r \geqq 0.$$

Suppose that

$$\text{(2.1)} \qquad \int_R^{\infty} r^{-(N-1)} \left(\int_R^r s^{(N-1)(1-\delta)} c^*(s) ds \right)^{1/(1-\delta)} dr < \infty$$

for some (and hence any) $R > 0$, and define the functions $y, z:[0, \infty) \to (0, \infty)$ by

$$\text{(2.2)} \qquad \begin{cases} y(r) = (1 - \delta)^{1/(1-\delta)} \displaystyle\int_r^{\infty} t^{-(N-1)} \\ \qquad \times \left(\displaystyle\int_R^t s^{(N-1)(1-\delta)} c_*(s) ds \right)^{1/(1-\delta)} dt & \text{for } r \geqq R, \\ y(r) = y(R) & \text{for } 0 \leqq r < R, \end{cases}$$

$$\text{(2.3)} \qquad \begin{cases} z(r) = (1 - \delta)^{1/(1-\delta)} \displaystyle\int_r^{\infty} t^{-(N-1)} \\ \qquad \times \left(\displaystyle\int_R^t s^{(N-1)(1-\delta)} c^*(s) ds \right)^{1/(1-\delta)} dt & \text{for } r \geqq R, \\ z(r) = z(R) & \text{for } 0 \leqq r < R. \end{cases}$$

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Then, it is easy to verify that

$$y \in C^{2+\theta'}_{\text{loc}}[0, \infty) \quad \text{and} \quad z \in C^{2+\theta'}_{\text{loc}}[0, \infty)$$

for some $\theta' \in (0, 1)$ and that $y(r)$ and $z(r)$ satisfy the following differential equations for $r > 0$:

$$(r^{N-1}y'(r))' + r^{N-1}c_*(r)|y'(r)|^\delta = 0,$$

$$(r^{N-1}z'(r))' + r^{N-1}c^*(r)|z'(r)|^\delta = 0.$$

Therefore, the functions $v(x) = y(|x|)$ and $w(x) = z(|x|)$ are of class $C^{2+\theta'}_{\text{loc}}(\mathbf{R}^N)$ and satisfy the differential inequalities

$$0 = \Delta v(x) + c_*(|x|)|Dv(x)|^\delta \leqq \Delta v(x) + c(x)|Dv(x)|^\delta,$$

$$0 = \Delta w(x) + c^*(|x|)|Dw(x)|^\delta \geqq \Delta w(x) + c(x)|Dw(x)|^\delta$$

in $\mathbf{R}^N$. Since $v(x) \leqq w(x)$ in $\mathbf{R}^N$, from a theorem of Akô and Kusano [**1**] it follows that (B) has a positive entire solution $u(x)$ such that $v(x) \leqq u(x) \leqq w(x)$ in $\mathbf{R}^N$. Since

$$\lim_{|x|\to\infty} w(x) = \lim_{r\to\infty} y(r) = 0$$

by (2.1), $u(x)$ is a decaying entire solution of (B). Thus, (2.1) ensures the existence of a positive decaying entire solution of (B).

If (2.1) is replaced by a stronger condition

$$\text{(2.4)} \qquad \int_R^\infty r^{(N-1)(1-\delta)}c^*(r)dr < \infty,$$

then (2.2) and (2.3) imply that

$$\lim_{r\to\infty} r^{N-2}y(r) = \frac{(1-\delta)^{1/(1-\delta)}}{N-2}\left(\int_R^\infty s^{(N-1)(1-\delta)}c_*(s)ds\right)^{1/(1-\delta)},$$

$$\lim_{r\to\infty} r^{N-2}z(r) = \frac{(1-\delta)^{1/(1-\delta)}}{N-2}\left(\int_R^\infty s^{(N-1)(1-\delta)}c^*(s)ds\right)^{1/(1-\delta)},$$

so that the solution $u(x)$ of (B) obtained above satisfies

$$\text{(2.5)} \qquad k_1|x|^{2-N} \leqq u(x) \leqq k_2|x|^{2-N}, \quad |x| \geqq 1,$$

for some positive constants k_1 and k_2.

An entire solution of (B) satisfying (2.5) is called a *minimal* positive entire solution, because any positive function satisfying $\Delta u \leqq 0$ in some exterior domain in $\mathbf{R}^N$, $N \geqq 3$, cannot decay faster than a constant multiple of $|x|^{2-N}$ as $|x| \to \infty$.

Suppose in particular that $c(x)$ in (B) satisfies

$$c_1|x|^\alpha \leqq c(x) \leqq c_2|x|^\alpha, \quad |x| \geqq 1,$$

for some constants α, $c_1 > 0$ and $c_2 > 0$. Then, (2.1) holds if and only if

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$\alpha < \delta - 2$, while (2.4) holds if and only if $\alpha < (N - 1)\delta - N$. Noting that $(N - 1)\delta - N < \delta - 2$, we conclude that: (i) (B) has a positive decaying entire solution if $\alpha < \delta - 2$; (ii) (B) has a minimal positive entire solution if $\alpha < (N - 1)\delta - N$; and (iii) if $(N - 1)\delta - N \leqq \alpha < \delta - 2$, then the decaying entire solution obtained is not minimal, that is, the order of its decay at infinity is lower than any constant multiple of $|x|^{2-N}$.

3. The equation (C). Let us now turn to the consideration of more general elliptic equations of the form

$$\text{(C)} \qquad Lu + f(x, Du) = 0, \quad x \in \mathbf{R}^N, N \geqq 2,$$

where L is given by

$$\text{(3.1)} \qquad L = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij} + \sum_{i=1}^{N} b_i(x)D_i.$$

We use the notation:

$$\text{(3.2)} \qquad A(x) = \sum_{i,j=1}^{N} a_{ij}(x)x_i x_j/|x|^2,$$

$$\text{(3.3)} \qquad B(x) = \left[\sum_{i=1}^{N} (b_i(x)x_i + a_{ii}(x)) - A(x)\right]/|x|, \quad x \neq 0.$$

The conditions we assume for L and f are as follows:

(L_1) $a_{ij}(x) = a_{ji}(x)$ for all $x \in \mathbf{R}^N$, $1 \leqq i, j \leqq N$, and there is a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqq a_0|\xi|^2 \quad \text{for all } (x, \xi) \in \mathbf{R}^N \times \mathbf{R}^N.$$

(L_2) $a_{ij} \in C^{1+\theta}_{\text{loc}}(\mathbf{R}^N)$, $b_i \in C^{\theta}_{\text{loc}}(\mathbf{R}^N)$, $1 \leqq i, j \leqq N$, for some $\theta \in (0, 1)$, and there is a constant $K > 0$ such that

$$||a_{ij}||_{\theta,\Omega(x)} \leqq K, \ ||b_i||_{\theta,\Omega(x)} \leqq K \quad \text{for all } x \in \mathbf{R}^N, \ 1 \leqq i, j \leqq N,$$

where $||\cdot||_{\theta,\Omega(x)}$ denotes the norm in the space $C^{\theta}(\Omega(x))$,

$$\Omega(x) = \{y \in \mathbf{R}^N : |y - x| \leqq 1\}.$$

(L_3) there exists a function $B_* \in C^{\theta}_{\text{loc}}(0, \infty)$ such that

$$B_*(r) \leqq \min_{|x|=r} B(x)/A(x), \ r > 0,$$

$\exp(-\int_R^r B_*(s)ds)$ is bounded on $[R, \infty)$ and

$$\int_R^\infty \exp\left(-\int_R^r B_*(s)ds\right)dr < \infty \quad \text{for any } R > 0.$$

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(F_1) $f(x, p)$ ($p = (p_1, \ldots, p_N)$) is locally Hölder continuous (with exponent θ) in $\mathbf{R}^N \times \mathbf{R}^N$.

(F_2) (Nagumo's condition) For any bounded domain $\Omega \subset \mathbf{R}^N$ there is a constant $\rho(\Omega) > 0$ such that

$$|f(x, p)| \leqq \rho(\Omega)(1 + |p|^2), \quad (x, p) \in \Omega \times \mathbf{R}^N.$$

(F_3) There exist a positive function $c \in C^{\theta}_{\mathrm{loc}}(\mathbf{R}^N)$ and a nonnegative function $\varphi \in C^{\theta}_{\mathrm{loc}}[0, \infty)$ such that

$$(3.4) \qquad 0 \leqq f(x, p) \leqq c(x)\varphi(|p|), \quad (x, p) \in \mathbf{R}^N \times \mathbf{R}^N.$$

Moreover, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$,

$$(3.5) \qquad \Phi(\xi) = \int_0^{\xi} dt/\varphi(t) \text{ exists for any } \xi > 0,$$

and

$$(3.6) \qquad \varphi(\lambda t) \leqq \Lambda_{\varphi}(\lambda)\varphi(t) \quad \text{for } \lambda > 0 \text{ and } t \geqq 0,$$

for some $\Lambda_{\varphi} \in C^{\theta}_{\mathrm{loc}}(0, \infty)$.

(F_4) There exist a constant $\delta \in (0, 1)$ and an open set $\Omega_0 \subset \mathbf{R}^N$ containing the origin such that

$$\inf_{x \in \Omega_0} \left[\liminf_{p \to 0} f(x, p)/|p|^{\delta}\right] > 0.$$

To state our main results we need the following functions defined for $r > 0$:

$$B_*(r) \text{ as in } (L_3), \qquad B^*(r) = \max_{|x|=r} B(x)/A(x),$$

$$(3.7) \qquad p_*(r) = \exp\left(\int_1^r B_*(s)ds\right), \quad p^*(r) = \exp\left(\int_1^r B^*(s)ds\right),$$

$$\pi_*(r) = \int_r^{\infty} ds/p_*(s), \qquad \pi^*(r) = \int_r^{\infty} ds/p^*(s).$$

THEOREM 3.1. *Suppose that* (L_1)-(L_3) *and* (F_1)-(F_4) *are satisfied.*
(i) *Suppose that*

$$\Phi(\infty) = \lim_{\xi \to \infty} \Phi(\xi) = \infty,$$

where $\Phi(\xi)$ *is defined by* (3.5), *and*

$$(3.8) \qquad \int_R^{\infty} \frac{1}{p_*(r)} \Phi^{-1}\left(\int_R^r p_*(s)\Lambda_{\varphi}(1/p_*(s))c^*(s)ds\right)dr < \infty$$

for some $R > 0$, *where*

$$c^*(r) = \max_{|x|=r} c(x)/A(x)$$

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and Φ^{-1} *is the inverse function of* Φ. *Then, equation* (C) *has a positive decaying entire solution.*

(ii) *Suppose that*

$$\int_R^\infty p_*(r)\Lambda_\varphi(1/p_*(r))c^*(r)dr < \infty \tag{3.9}$$

for some $R > 0$, *then, regardless of the value of* $\Phi(\infty)$, *equation* (C) *has a positive decaying entire solution* $u(x)$ *such that*

$$k_1\pi^*(|x|) \leqq u(x) \leqq k_2\pi_*(|x|), \quad |x| \geqq 1, \tag{3.10}$$

for some positive constants k_1 *and* k_2.

Proof. We adopt the supersolution-subsolution method due to Akô and Kusano [**1**] (see also [**16**]): If there exist bounded functions v, $w \in C^{2+\theta}_{\mathrm{loc}}(\mathbf{R}^N)$, $\theta \in (0, 1)$, such that $v(x) \leqq w(x)$, $x \in \mathbf{R}^N$,

$$Lv(x) + f(x, Dv(x)) \geqq 0, \quad x \in \mathbf{R}^N, \tag{3.11}$$

and

$$Lw(x) + f(x, Dw(x)) \leqq 0, \quad x \in \mathbf{R}^N, \tag{3.12}$$

then (C) has an entire solution $u(x)$ satisfying $v(x) \leqq u(x) \leqq w(x)$, $x \in \mathbf{R}^N$. (Such functions $v(x)$ and $w(x)$ are called a *subsolution* and a *supersolution* of (C), respectively.)

We begin with the construction of a subsolution $v(x)$ of (C). In view of (F_4) there exist positive constants P_0, R_0 and a nonnegative function $c_0 \in C_0^\infty(\mathbf{R}^N)$ such that

$$\operatorname{supp} c_0 = \{x:|x| \leqq 2R_0\} \subset \Omega_0 \quad \text{and}$$

$$f(x, p) \geqq c_0(x)|p|^\delta \quad \text{for } x \in \mathbf{R}^N, |p| \leqq P_0. \tag{3.12}$$

Define

$$\begin{cases} y(r) = (1 - \delta)^{1/(1-\delta)} \displaystyle\int_r^\infty \frac{1}{p^*(t)} \\ \qquad \times \left(\displaystyle\int_{R_0}^t [p^*(s)]^{1-\delta} c_{0*}(s)ds\right)^{1/(1-\delta)} dt & \text{for } r \geqq R_0, \\ y(r) = y(R_0) & \text{for } 0 \leqq r < R_0, \end{cases} \tag{3.13}$$

where

$$c_{0*}(r) = \min_{|x|=r} c_0(x)/A(x).$$

Then, we see that $y \in C^{2+\theta'}_{\mathrm{loc}}[0, \infty)$ for some $\theta' \in (0, 1)$, $y'(r) < 0$ for $r \in (R_0, \infty)$, $y(r) \to 0$ as $r \to \infty$, and

$$(p^*(r)y'(r))' + p^*(r)c_{0*}(r)|y'(r)|^\delta = 0, \quad r > 0. \tag{3.14}$$

Note that (3.14) is equivalent to

$$y''(r) + B^*(r)y'(r) + c_{0*}(r)|y'(r)|^{\delta} = 0. \tag{3.15}$$

Furthermore, since $1/p^*(r)$ is bounded on $[R_0, \infty)$ by (L_3) and $c_{0*}(r) = 0$ for $r \geqq 2R_0$, (3.13) implies that $y'(r)$ is bounded on $[0, \infty)$, and so there is a constant μ such that

$$0 < \mu < 1 \quad \text{and} \quad \mu|y'(r)| \leqq P_0 \quad \text{for } r \in [R_0, \infty). \tag{3.16}$$

If we define $v(x) = \mu y(|x|)$, $x \in \mathbf{R}^N$, then using (3.12), (3.15), (3.16) and noting that $y'(r) < 0$ for $r > R_0$, we obtain

$$\begin{aligned} &Lv(x) + f(x, Dv(x)) \\ &\geqq Lv(x) + c_0(x)|Dv(x)|^{\delta} \\ &= \mu[A(x)y''(r) + B(x)y'(r) + \mu^{\delta-1}c_0(x)|y'(r)|^{\delta}] \\ &\geqq \mu[A(x)(y''(r) + B^*(r)y'(r)) \\ &\qquad\qquad + (B(x) - B^*(r))y'(r) + c_0(x)|y'(r)|^{\delta}] \\ &\geqq \mu(-A(x)c_{0*}(r) + c_0(x))|y'(r)|^{\delta} \geqq 0 \quad \text{for } r = |x| \geqq R_0, \end{aligned}$$

and $Lv(x) + f(x, Dv(x)) = 0$ for $|x| < R_0$. This shows that $v(x) = \mu y(|x|)$ is a subsolution of (C) if (3.16) is satisfied. (Condition (3.8) or (3.9) is not needed here.)

To construct a supersolution $w(x)$ of (C) assume that $\Phi(\infty) = \infty$ and (3.8) holds. We put

$$z_0(r) = \Phi^{-1}\left(\int_{R_1}^{r} p_*(s)\Lambda_{\varphi}(1/p_*(s))c^*(s)ds\right), \quad r \geqq R_1, \tag{3.17}$$

where $R_1 > 0$ is a fixed constant, and define $z(r)$ by

$$\begin{cases} z(r) = \int_r^{\infty} z_0(t)/p_*(t)dt & \text{for } r \geqq R_1, \\ z(r) = z(R_1) & \text{for } 0 \leqq r < R_1. \end{cases} \tag{3.18}$$

It is easy to see that $z \in C^{2+\theta'}_{\text{loc}}[0, \infty)$ for some $\theta' \in (0, 1)$. Using (3.17), (3.18), (3.5) and (3.6), we obtain

$$\begin{aligned} (p_*(r)z'(r))' &= -p_*(r)c^*(r)\Lambda_{\varphi}(1/p_*(r))\varphi(z_0(r)) \\ &\leqq -p_*(r)c^*(r)\varphi(z_0(r)/p_*(r)) \\ &= -p_*(r)c^*(r)\varphi(-z'(r)), \quad r \geqq R_1, \end{aligned}$$

which, in view of (3.7) and $z'(r) < 0$, $r \geqq R_1$, reduces to

$$z''(r) + B_*(r)z'(r) + c^*(r)\varphi(|z'(r)|) \leqq 0, \quad r \geqq R_1. \tag{3.19}$$

From (3.4) and (3.19) it follows that the function $w(x) = z(|x|)$, $x \in \mathbf{R}^N$, satisfies

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$$\begin{aligned}
&Lw(x) + f(x, Dw(x))\\
&\leqq Lw(x) + c(x)\varphi(|Dw(x)|)\\
&= A(x)z''(r) + B(x)z'(r) + c(x)\varphi(|z'(r)|)\\
&= A(x)(z''(r) + B_*(r)z'(r))\\
&+ (B(x) - A(x)B_*(r))z'(r) + c(x)\varphi(|z'(r)|)\\
&\leqq (-A(x)c^*(r) + c(x))\varphi(|z'(r)|) \leqq 0 \text{ for } r = |x| \geqq R_1.
\end{aligned}$$

Since $Lw(x) + f(x, Dw(x)) = 0$ for $|x| < R_1$, $w(x) = z(|x|)$ is a supersolution of (C).

From (3.13), (3.18) and (3.7) we see that

$$\text{(3.20)}\quad \lim_{r\to\infty} y(r)/\pi^*(r) = (1-\delta)^{1/(1-\delta)}\left(\int_{R_0}^{\infty} [p^*(s)]^{1-\delta} c_{0*}(s)ds\right)^{1/(1-\delta)},$$

$$\text{(3.21)}\quad \lim_{r\to\infty} z(r)/\pi_*(r) = \Phi^{-1}\left(\int_{R_1}^{\infty} p_*(s)\Lambda_\varphi(1/p_*(s))c^*(s)ds\right).$$

The limit (3.20) is finite, while the limit (3.21) is finite or infinite, and so noting that $\pi_*(r) \geqq \pi^*(r)$ for $r \geqq 1$, we conclude that $\mu y(r) \leqq z(r)$ for $r \geqq 0$ provided $\mu > 0$ is chosen small enough. With this choice of μ, we have $v(x) \leqq w(x)$ for all $x \in \mathbf{R}^N$. Therefore, there exists a positive entire solution $u(x)$ of equation (C) satisfying $v(x) \leqq u(x) \leqq w(x)$ in $\mathbf{R}^N$. That $u(x)$ is a decaying solution follows from the fact that

$$\lim_{|x|\to\infty} w(x) = \lim_{r\to\infty} z(r) = 0 \quad \text{by (3.18).}$$

If (3.9) holds, then the limit (3.21) is finite, and hence the solution $u(x)$ satisfies (3.10). If (3.9) holds but $\Phi(\infty) < \infty$, then it suffices to choose $R_1 > 0$ so that

$$\int_{R_1}^{\infty} p_*(s)\Lambda_\varphi(1/p_*(s))c^*(s)ds < \Phi(\infty)$$

and repeat the same argument as above. This completes the proof of Theorem 3.1.

Example. 3.1. Consider the equation

$$\text{(3.22)}\quad \Delta u + \frac{c(x)|Du|^\beta}{1 + |Du|^\alpha}[\log(1 + |Du|)]^\gamma = 0,\ x \in \mathbf{R}^N,\ N \geqq 3,$$

where α, β, γ are positive constants and $c(x)$ is a nonnegative function of class $C^\theta_{\text{loc}}(\mathbf{R}^N)$, $\theta \in (0, 1)$, such that $c(0) > 0$.

The operator $L = \Delta$ satisfies (L_1)-(L_3), and the functions in (3.7) for this operator become

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$$B^*(r) = B_*(r) = (N-1)/r, \quad p^*(r) = p_*(r) = r^{N-1} \quad \text{and}$$
$$\pi^*(r) = \pi_*(r) = r^{2-N}/(N-2).$$

Suppose that $\alpha < \beta + \gamma < 1$. Then, the function

$$f(x, p) = c(x)|p|^{\beta}[\log(1 + |p|)]^{\gamma}/(1 + |p|^{\alpha})$$

satisfies (F_1)-(F_4); in particular, (F_3) holds with the choice $\varphi(t) = t^{\beta+\gamma-\alpha}$ for which

$$\Lambda_{\varphi}(\lambda) = \lambda^{\beta+\gamma-\alpha} \quad \text{and} \quad \Phi(\xi) = \xi^{1+\alpha-\beta-\gamma}/(1 + \alpha - \beta - \gamma),$$

and (F_4) holds with $\delta = \beta + \gamma$ and $\Omega_0 = \{x : c(x) > c(0)/2\}$. From Theorem 3.1 it follows that if

$$\int_R^{\infty} r^{-(N-1)}\left(\int_R^r s^{(N-1)(1+\alpha-\beta-\gamma)}c^*(s)ds\right)^{1/(1+\alpha-\beta-\gamma)} dr < \infty$$

for some $R > 0$, then (3.22) has a positive decaying entire solution, and that a stronger condition

$$\int_R^{\infty} r^{(N-1)(1+\alpha-\beta-\gamma)}c^*(r)dr < \infty$$

guarantees the existence of a decaying entire solution $u(x)$ such that

$$k_1|x|^{2-N} \leqq u(x) \leqq k_2|x|^{2-N}, \ |x| \geqq 1,$$

for some positive constants k_1 and k_2.

Example 3.2. There is a class of elliptic equations having positive entire solutions which decay exponentially as $|x| \to \infty$. Consider the equation

$$(3.23) \quad \Delta u + \sum_{i=1}^{N} b_i(x)D_i u + c(x)|Du|^{\delta} = 0, \quad x \in \mathbf{R}^N, \ N \geqq 2,$$

where $c(x)$ is as in Example 3.1 and $b_i(x)$, $1 \leqq i \leqq N$, satisfy

$$\|b_i\|_{\theta,\Omega(x)} \leqq K, \quad x \in \mathbf{R}^N,$$

for some constant $K > 0$ and

$$\beta = \liminf_{|x|\to\infty}\left[\sum_{i=1}^{N} b_i(x)x_i/|x|\right] > 0.$$

If $0 < \delta < 1$ and

$$\int_R^{\infty} e^{(1-\delta)\gamma r}c^*(r)dr < \infty$$

for some $\gamma < \beta$ and $R > 0$, then (3.23) possesses an entire solution $u(x)$ such that

$$0 < u(x) \leqq ke^{-\gamma|x|}, \quad x \in \mathbf{R}^N,$$

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for some constant $k > 0$. This follows from (ii) of Theorem 3.1 combined with the observation that, since in this case

$$B(x)/A(x) = \sum_{i=1}^{N} b_i(x)x_i/|x| + (N - 1)/|x| > \gamma \quad \text{for } |x| > R_0$$

provided $R_0 > 0$ is large enough, a continuous function on $(0, \infty)$ which equals γ on $[2R_0, \infty)$ can be chosen as $B_*(r)$, so that (L_3) holds for

$$L = \Delta + \sum_{i=1}^{N} b_i(x)D_i$$

and $p_*(r)$ and $\pi_*(r)$ can be taken to be

$$p_*(r) = m_1 e^{\gamma r}, \quad \pi_*(r) = m_2 e^{-\gamma r}, \quad r \geqq 2R_0,$$

for some positive constants m_1 and m_2.

4. The equation (A). We are now in a position to deal with general elliptic equations of the form

(A) $\qquad Lu + f(x, u, Du) = 0, \quad x \in \mathbf{R}^N, N \geqq 2,$

where L is as in (C) and f depends on both u and Du. With regard to (A) we assume in addition to (L_1)-(L_3) that:

(F_1^*) $f(x, u, p)$ is locally Hölder continuous (with exponent θ) in $\mathbf{R}^N \times \mathbf{R}_+ \times \mathbf{R}^N$;

(F_2^*) (Nagumo's condition) For any bounded domain $\Omega \subset \mathbf{R}^N$ and any constant $J > 0$ there is a constant $\rho(\Omega, J) > 0$ such that

$$|f(x, u, p)| \leqq \rho(\Omega, J)(1 + |p|^2)$$

for $x \in \Omega$, $0 < u \leqq J$ and $p \in \mathbf{R}^N$;

(F_3^*) There exist nonnegative functions $c \in C^{\theta}_{\text{loc}}(\mathbf{R}^N \times \mathbf{R}_+)$ and $\varphi \in C^{\theta}_{\text{loc}}[0, \infty)$ such that

(4.1) $\qquad 0 \leqq f(x, u, p) \leqq c(x, u)\varphi(|p|), \quad (x, u, p) \in \mathbf{R}^N \times \mathbf{R}_+ \times \mathbf{R}^N,$

where φ is exactly as in (F_3) and c satisfies

(4.2) $\qquad c(x, \lambda u) \leqq \psi(\lambda)c(x, u) \quad \text{for } \lambda > 0, (x, u) \in \mathbf{R}^N \times \mathbf{R}_+,$

for some positive function $\psi \in C^{\theta}_{\text{loc}}(0, \infty)$;

(F_4^*) There exist an open set $\Omega_0 \subset \mathbf{R}^N$ containing the origin and constants γ, δ such that $0 < \delta < 1$, $\gamma + \delta < 1$ and

$$\inf_{x \in \Omega_0} \left[\liminf_{(u,p) \to (0,0)} f(x, u, p)/u^{\gamma}|p|^{\delta} \right] > 0.$$

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The main results of this section are as follows. The functions defined by (3.5)-(3.7) are also used therein.

THEOREM 4.1. *In addition to* (L_1)-(L_3) *and* (F_1^*)-(F_4^*) *assume that* $c(x, u)$ *is nondecreasing in* u *for each fixed* x, $\gamma \geqq 0$ *in* (F_4^*), *and*

$$(4.3) \qquad \lim_{\lambda \to \infty} \lambda^{-1}\psi(\lambda)\Lambda_\varphi(\lambda) = 0.$$

(i) *Suppose that* $\Phi(\infty) = \infty$ *and*

$$(4.4) \qquad \int_R^\infty \frac{1}{p_*(r)} \Phi^{-1}\left(\int_R^r p_*(s)\Lambda_\varphi(1/p_*(s))c^*(s, 1)ds\right)dr < \infty$$

for some $R > 0$, *where*

$$c^*(r, 1) = \max_{|x|=r} c(x, 1)/A(x).$$

Then, equation (A) *has a positive decaying entire solution.*

(ii) *If*

$$(4.5) \qquad \int_R^\infty p_*(r)\Lambda_\varphi(1/p_*(r))c^*(r, \pi_*(r))dr < \infty$$

for some $R > 0$, *where*

$$c^*(r, \pi_*(r)) = \max_{|x|=r} c(x, \pi_*(|x|))/A(x),$$

then, regardless of the value of $\Phi(\infty)$, *equation* (A) *has a positive decaying entire solution* $u(x)$ *such that*

$$(4.6) \qquad k_1\pi^*(|x|) \leqq u(x) \leqq k_2\pi_*(|x|), \quad |x| \geqq 1,$$

for some positive constants k_1 *and* k_2.

THEOREM 4.2. *In addition to* (L_1)-(L_3) *and* (F_1^*)-(F_4^*) *assume that* $c(x, u)$ *is nonincreasing in* u *for each fixed* x, $\gamma \leqq 0$ *in* (F_4^*) *and* (4.3) *holds. If*

$$(4.7) \qquad \int_R^\infty p_*(r)\Lambda_\varphi(1/p_*(r))c^*(r, \pi^*(r))dr < \infty$$

for some $R > 0$, *where*

$$c^*(r, \pi^*(r)) = \max_{|x|=r} c(x, \pi^*(|x|))/A(x),$$

then equation (A) *has a positive decaying entire solution* $u(x)$ *which satisfies* (4.6) *for some contants* $k_1 > 0$ *and* $k_2 > 0$.

In the proofs of these theorems given below extensive use is made of a function $h_0 \in C^{2+\theta}_{\mathrm{loc}}(\mathbf{R}^N)$ with the properties:

(i) $Lh_0(x) \leqq 0$ and $h_0(x) > 0$ in $\mathbf{R}^N$;

(ii) For any positive function $h \in C^2(\mathbf{R}^N)$ satisfying $Lh(x) \leqq 0$ in $\mathbf{R}^N$,

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(4.8) $\quad h_0(x) = O(h(x))$ as $|x| \to \infty$.

It can be shown that under hypotheses (L_1)-(L_3) such a function $h_0(x)$ exists and enjoys the following properties:

(I) $\quad \lim_{|x|\to\infty} h_0(x) = 0;$

(II) $\quad \pi^*(|x|) = O(h_0(x))$ as $|x| \to \infty;$

(III) If $g \in C^{\theta}_{\mathrm{loc}}(\mathbf{R}^N)$ has compact support and $g(x) \geqq 0$, $\not\equiv 0$ in $\mathbf{R}^N$, then the equation

(4.9) $\quad Lu = -g(x), \quad x \in \mathbf{R}^N,$

has a unique solution $u \in C^{2+\theta}_{\mathrm{loc}}(\mathbf{R}^N)$ which tends uniformly to 0 as $|x| \to \infty$. Furthermore, $u(x)$ satisfies

(4.10) $\quad k_1 h_0(x) \leqq u(x) \leqq k_2 h_0(x), \quad x \in \mathbf{R}^N,$

for some positive constants k_1 and k_2.

The existence of $h_0(x)$ is proved in [**8**, Theorem 2.1]. For the proof of (I) and (III), see [**6**, Theorem 3.3] and [**8**, Theorem 2.2]. Property (II) follows from the maximum principle applied to $Mh_0(x) - \pi^*(|x|)$ for sufficiently large $M > 0$.

Proof of Theorem 4.1. As in the proof of Theorem 3.1 it suffices to construct a function $V(x)$ (a *subsolution* of (A)) satisfying

$$LV(x) + f(x, V(x), DV(x)) \geqq 0, \quad x \in \mathbf{R}^N,$$

and a function $W(x)$ (a *supersolution* of (A)) satisfying

$$LW(x) + f(x, W(x), DW(x)) \leqq 0, \quad x \in \mathbf{R}^N,$$

so that the inequality $V(x) \leqq W(x)$ holds throughout $\mathbf{R}^N$.

Part (i). Let $w(x)$ be a positive decaying entire solution of the equation

(4.11) $\quad Lw + c(x, 1)\varphi(|Dw|) = 0, \quad x \in \mathbf{R}^N,$

where c and φ are as in (F_3^*). The existence of $w(x)$ follows from (4.4) and (i) of Theorem 3.1 applied to (4.11). Put

$$M_1 = \sup_{x\in\mathbf{R}^N} w(x),$$

and choose $\lambda > 0$ so that

(4.12) $\quad \lambda^{-1}\psi(\lambda)\Lambda_{\varphi}(\lambda)\psi(M_1) \leqq 1,$

which is possible because of (4.3). Define $W(x) = \lambda w(x)$, $x \in \mathbf{R}^N$. Then, using (F_3^*), (4.11) and (4.12), we see that

$$LW(x) + f(x, W(x), DW(x))$$

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$$\begin{aligned}
&\leqq LW(x) + c(x, W(x))\varphi(|DW(x)|)\\
&= \lambda[Lw(x) + \lambda^{-1}c(x, \lambda w(x))\varphi(\lambda|Dw(x)|)]\\
&\leqq \lambda[Lw(x) + \lambda^{-1}\psi(\lambda)\psi(M_1)\Lambda_\varphi(\lambda)c(x, 1)\varphi(|Dw(x)|)]\\
&\leqq \lambda[Lw(x) + c(x, 1)\varphi(|Dw(x)|)] = 0, \quad x \in \mathbf{R}^N,
\end{aligned}$$

implying that $W(x)$ is a supersolution of (A). Note that since

$$LW(x) \leqq -f(x, W(x), DW(x)) \leqq 0, \quad x \in \mathbf{R}^N,$$

from (4.8) with $h(x) = W(x)$ there is a constant $M_2 > 0$ such that

(4.13) $M_2 h_0(x) \leqq W(x), \quad x \in \mathbf{R}^N.$

To obtain a subsolution of (A), we first observe that hypothesis (F_4^*) implies the existence of positive constants P_0, R_0, U_0 and a nonnegative function $c_0 \in C_0^\infty(\mathbf{R}^N)$ such that

$$\operatorname{supp} c_0 = \{x:|x| \leqq R_0\} \subset \Omega_0 \quad \text{and}$$

(4.14) $f(x, u, p) \geqq c_0(x)u^\gamma|p|^\delta$

for $x \in \mathbf{R}^N$, $0 < u \leqq U_0$ and $|p| \leqq P_0$. Consider the equation

(4.15) $Lv + c_0(x)[h_0(x)]^\gamma|Dv|^\delta = 0, \quad x \in \mathbf{R}^N.$

By Theorem 3.1 there exists a positive decaying entire solution $v(x)$ of (4.15). From the property (III) of $h_0(x)$ (with

$$g(x) = c_0(x)[h_0(x)]^\gamma|Dv(x)|^\delta$$

in (4.9)) it follows that

(4.16) $M_3 h_0(x) \leqq v(x) \leqq M_4 h_0(x), \quad x \in \mathbf{R}^N,$

for some constants $M_3 > 0$ and $M_4 > 0$. Using (4.16) and the fact that $Lv(x) = 0$ for $|x| > R_0$ and applying a standard argument based on the $W^{2,q}$ estimates of solutions and the Sobolev imbedding theorem (see e.g. [**16**, Theorem 2]), we conclude that $|Dv(x)|$ is bounded in $\mathbf{R}^N$. We now define $V(x) = \mu v(x)$, $x \in \mathbf{R}^N$, where $\mu > 0$ is chosen small enough so that $0 < \mu < 1$, $\mu^{\gamma+\delta-1}M_3^\gamma \geqq 1$, and

$$\mu v(x) \leqq W(x), \quad \mu v(x) \leqq U_0, \quad \mu|Dv(x)| \leqq P_0, \quad x \in \mathbf{R}^N;$$

such a choice of μ is possible because of (4.13), (4.16) and the boundedness of $|Dv(x)|$. We then see that $V(x)$ is a subsolution of (A), since in view of (4.14)-(4.16),

$$\begin{aligned}
&LV(x) + f(x, V(x), DV(x))\\
&= \mu Lv(x) + f(x, \mu v(x), \mu|Dv(x)|)\\
&\geqq \mu Lv(x) + c_0(x)[\mu v(x)]^\gamma[\mu|Dv(x)|]^\delta
\end{aligned}$$

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$$\geqq \mu\{Lv(x) + \mu^{\gamma+\delta-1}M_3^\gamma c_0(x)[h_0(x)]^\gamma |Dv(x)|^\delta\}$$
$$\geqq \mu\{Lv(x) + c_0(x)[h_0(x)]^\gamma |Dv(x)|^\delta\} = 0, \quad x \in \mathbf{R}^N.$$

Since $V(x) \leqq W(x)$, $x \in \mathbf{R}^N$, there exists a positive entire solution $u(x)$ of (A) such that $V(x) \leqq u(x) \leqq W(x)$ in $\mathbf{R}^N$. It is obvious that

$$\lim_{|x|\to\infty} u(x) = 0.$$

Part (ii). Define $\hat{\pi}_*(r)$ by

$$\hat{\pi}_*(r) = \pi_*(r) \text{ for } r \geqq 1, \ \hat{\pi}_*(r) = \pi_*(1) \text{ for } 0 \leqq r < 1,$$

and consider the equation

$$(4.17) \quad Lw + c(x, \hat{\pi}_*(|x|))\varphi(|Dw(x)|) = 0, \quad x \in \mathbf{R}^N.$$

Applying (ii) of Theorem 3.1 to (4.17) and arguing as in part (i), we obtain a positive decaying entire solution $w(x)$ of (4.17) satisfying

$$(4.18) \quad M_5 h_0(x) \leqq w(x) \leqq M_6 \hat{\pi}_*(|x|), \quad x \in \mathbf{R}^N,$$

where M_5 and M_6 are positive constants, and we can show that the function $W(x) = \lambda w(x)$, $x \in \mathbf{R}^N$, is a supersolution of (A) provided $\lambda > 0$ is sufficiently large. Exactly as in part (i) we can find a subsolution $V(x)$ of (A) satisfying $V(x) \leqq W(x)$, $x \in \mathbf{R}^N$. Therefore, equation (A) has an entire solution $u(x)$ such that $V(x) \leqq u(x) \leqq W(x)$ in $\mathbf{R}^N$. Combining (4.18) with inequalities of type (4.16) satisfied by $V(x)$, we have

$$(4.19) \quad M_7 h_0(x) \leqq u(x) \leqq M_8 \hat{\pi}_*(|x|), \quad x \in \mathbf{R}^N,$$

for some constants $M_7 > 0$ and $M_8 > 0$.

On the other hand, from the property (II) of $h_0(x)$ there is a constant $M_9 > 0$ such that $M_9 \pi^*(|x|) \leqq h_0(x)$ for $|x| \geqq 1$, which together with (4.19) implies the desired asymptotic behavior (4.6) of the solution $u(x)$. This completes the proof of Theorem 4.1.

Proof of Theorem 4.2. Consider the equation

$$(4.20) \quad Lw + c(x, h_0(x))\varphi(|Dw|) = 0, \quad x \in \mathbf{R}^N.$$

The nonincreasing nature of $c(x, u)$ with respect to u implies that

$$c(x, h_0(x)) \leqq c(x, M_9\pi^*(|x|)) \text{ for } |x| \geqq 1,$$

and so (ii) of Theorem 3.1 shows that (4.20) has a positive decaying entire solution $w(x)$ such that

$$M_{10} h_0(x) \leqq w(x) \leqq M_{11} \hat{\pi}_*(|x|), \quad x \in \mathbf{R}^N,$$

for some constants $M_{10} > 0$ and $M_{11} > 0$, where $\hat{\pi}_*(r)$ is as above. Define the function $W(x) = \lambda w(x)$, $x \in \mathbf{R}^N$, where $\lambda > 0$ is chosen so large that

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$$\lambda^{-1}\psi(\lambda)\Lambda_\varphi(\lambda)\psi(M_{10}) \leqq 1.$$

Then, noting that

$$c(x, w(x)) \leqq c(x, M_{10}h_0(x)),$$

we see that $W(x)$ is a supersolution of (A). A subsolution $V(x)$ of (A) such that $V(x) \leqq W(x)$, $x \in \mathbf{R}^N$, can be constructed in essentially the same manner as in the proof of the preceding theorem, and hence there exists an entire solution $u(x)$ of (A) lying between $V(x)$ and $W(x)$ for every $x \in \mathbf{R}^N$. The details are left to the reader.

COROLLARY 4.1. *In addition to* (L_1), (L_2) *and* (F_1^*)-(F_4^*) *assume that* $c(x, u)$ *is nondecreasing in* u *for each fixed* x *and* (4.3) *holds. Suppose moreover that there is a constant* $\upsilon > 1$ *such that* $B(x)/A(x) \geqq \upsilon/|x|$ *for all sufficiently large* $|x|$.

(i) *Suppose that* $\Phi(\infty) = \infty$ *and*

$$(4.21) \quad \int_R^\infty t^{-\upsilon}\Phi^{-1}\left(\int_R^r s^\upsilon \Lambda_\varphi(ks^{-\upsilon})c^*(s, 1)ds\right)dr < \infty$$

for any $k > 0$ *and some* $R > 0$, *where*

$$c^*(r, 1) = \max_{|x|=r} c(x, 1)/A(x).$$

Then, there exists a decaying positive entire solution of (A).

(ii) *If*

$$(4.22) \quad \int_R^\infty r^\upsilon \Lambda_\varphi(kr^{-\upsilon})c^*(r, r^{1-\upsilon})dr < \infty$$

for any $k > 0$ *and some* $R > 0$, *then there exists a decaying positive entire solution* $u(x)$ *of* (A) *such that*

$$(4.23) \quad k_1 h_0(x) \leqq u(x) \leqq k_2(1 + |x|)^{1-\upsilon}, \quad x \in \mathbf{R}^N,$$

for some positive constants k_1 *and* k_2.

Proof. In view of the assumption $B(x)/A(x) \geqq \upsilon/|x|$ with $\upsilon > 1$ we can take $B_*(r) = \upsilon/r$ for large r, so that (L_3) holds for L and the corresponding functions $p_*(r)$ and $\pi_*(r)$ can be taken to be $p_*(r) = m_1 r^\upsilon$ and $\pi_*(r) = m_2 r^{1-\upsilon}$ for some constants m_1 and m_2. The conclusions of Corollary 4.1 now follow from Theorem 4.1.

Example 4.1. Consider the equation

$$(4.24) \quad \Delta u + c(x)u^\gamma |Du|^\delta = 0, \quad x \in \mathbf{R}^N, N \geqq 3,$$

where γ and δ are constants such that $0 < \delta < 1$, $\gamma + \delta < 1$, and $c(x)$ is a nonnegative locally Hölder continuous function in $\mathbf{R}^N$ with $c(0) > 0$. Clearly, (L_1)-(L_3) and (F_1^*)-(F_4^*) hold for (4.24); in particular (F_3^*) holds with $c(x, u) = c(x)u^\gamma$ and $\varphi(t) = t^\delta$, so that

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$$\Phi(\xi) = \xi^{1-\delta}/(1 - \delta) \quad \text{and}$$

$$\Lambda_\varphi(\lambda) = \lambda^\delta, \ \psi(\lambda) = \lambda^\gamma.$$

Noting that in this case $p^*(r) = p_*(r) = r^{N-1}$, $\pi^*(r) = \pi_*(r) = r^{2-N}/(N - 2)$, we conclude from (i) of Theorem 4.1 that if $\gamma \geqq 0$ and

$$\int_R^\infty r^{-(N-1)}\left(\int_R^r s^{(N-1)(1-\delta)}c^*(s)ds\right)^{1/(1-\delta)} dr < \infty$$

for some $R > 0$, then (4.24) has a decaying positive entire solution, and from (ii) of Theorem 4.1 and Theorem 4.2 that if either $\gamma \geqq 0$ or $\gamma < 0$ and

$$\int_R^\infty r^{(N-1)(1-\delta)-(N-2)\gamma}c^*(r)dr < \infty$$

for some $R > 0$, then (4.24) has a positive entire solution which decays like a constant multiple of $|x|^{2-N}$ as $|x| \to \infty$.

Example 4.2. Our final example concerns the equation

$$(4.25) \quad \Delta u + \sum_{i=1}^N x_i D_i u + c(x)u^\gamma |Du|^\delta = 0, \quad x \in \mathbf{R}^N, \ N \geqq 2,$$

where $c(x)$ is as in Example 4.1 and γ, δ are nonnegative constants with $0 < \gamma + \delta < 1$. Hypotheses (L_1)-(L_3) and (F_1^*)-(F_4^*) are satisfied by (4.25). In particular, since for

$$L = \Delta + \sum_{i=1}^N x_i D_i,$$

$$B(x)/A(x) = |x| + (N - 1)/|x| \to \infty \quad \text{as } |x| \to \infty,$$

(L_3) holds with the choice

$$B_*(r) = B^*(r) = r + (N - 1)/r,$$

to which there correspond

$$p_*(r) = p^*(r) = e^{-1/2}r^{N-1}e^{r^2/2}$$

and

$$\pi_*(r) = \pi^*(r) = e^{1/2}\int_r^\infty s^{1-N}e^{-s^2/2}ds.$$

Noting that

$$\lim_{r\to\infty} r^N e^{r^2/2}\pi_*(r) = e^{1/2}$$

and applying (ii) of Theorem 4.1, we conclude that the condition

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$$(4.26) \quad \int_R^\infty r^{(N-1)(1-\delta)-\gamma N} e^{(1-\delta-\gamma)r^2/2} c^*(r)dr < \infty \text{ for some } R > 0$$

guarantees the existence of a positive entire solution $u(x)$ satisfying

$$k_1|x|^{-N}e^{-|x|^2/2} \leqq u(x) \leqq k_2|x|^{-N}e^{-|x|^2/2}, \quad |x| \geqq 1,$$

for some positive constants k_1 and k_2. Condition (4.26) is satisfied if, for example,

$$0 < c(x) \leqq c_1 e^{-|x|^2}, \ x \in \mathbf{R}^N, \text{ for some constant } c_1 > 0.$$

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