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POLYNOMIAL MODULES OVER MACAULAY MODULES

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In [2] we introduced a concept of a Macaulay module over a right noetherian ring by saying that all associated primes of the module have the same codimension. That is to say that a module M over a right noetherian ring R is Macaulay if $K \dim R/P = K \dim R/Q$ for all $P, Q \in \operatorname{Ass} M$. Our main aim here is to extend Nagata's useful result [6], that Macaulay rings are stable under polynomial adjunction, to a noncommutative setting. Specifically, we prove where x is a commuting indeterminate, that the polynomial module $M[x] = M \otimes_R R[x]$ is a Macaulay $R[x]$-module if and only if M is a Macaulay R-module. But actually, we prove a more general result. We show that when M is any module over a right noetherian ring, the associated primes of $M[x]$ are precisely the extensions of the associated primes of M.

Before continuing, we wish to contrast our terminology with the established one in the theory of commutative noetherian rings. The basic distinction is that our definition is weaker: a Macaulay ring in the sense of Nagata (see [7, p. 82]) is Macaulay in our sense. (The now fairly standard commutative notion of a Cohen-Macaulay ring is also weaker—this coinciding with Nagata's notion of "locally Macaulay".) The distinction is perhaps best understood in terms of FBN rings, for example, noetherian PI-rings. For such rings are known to be Macaulay [1] if and only if every prime ideal of codimension less than the Krull dimension of the ring contains a regular element. In particular these rings coincide, in the commutative 1-dimensional case, with the objects of study in Matlis' treatise [5] on 1-dimensional Cohen-Macaulay rings; for his definition is that every maximal ideal contains a regular element. But we should point out that although any 2-dimensional local commutative noetherian domain is Macaulay in our sense, it is possible for the maximal ideal to have depth 1. Such rings are not Macaulay by any of the usual commutative definitions.

We will assume familiarity with the notions of associated prime, primary module and primary decomposition as set forth in [2]. For any submodule W of a polynomial module over $R[x]$ we denote by $L_i(W)$ the module of leading coefficients of polynomials in W of degree i together with 0.

LEMMA 1. *Let P be a prime ideal of a right noetherian ring R and M a P-primary R-module with annihilator P. Then $M[x]$ is a $P[x]$-primary $R[x]$-module.*

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Proof. Let $Q \in \mathrm{Ass}_{R[x]} M[x]$ and let $N \neq 0$ be an $R[x]$-submodule of $M[x]$ killed by Q. We note that $N^* = \bigcup L_i(N)$ is a nonzero R-submodule of M.

Now an element of N has the form $a_0 + a_1 x + \cdots + a_i x^i$, where $a_k \in M$. Thus if $q \in R$ is the leading coefficient of a polynomial in Q of degree j then $a_i q = 0$. Thus $N^* L_j(Q) = 0$ and so, since $\mathrm{Ass}\, M = \{P\}$ by hypothesis, $L_j(Q) \subseteq P$. But $P = L_j(P[x])$ and $P[x] \subseteq Q$ because $M[x]P[x] = 0$.

This argument shows, by one of the standard proofs of the Hilbert Basis Theorem, that $Q = P[x]$. □

In our present context the set comp M of composition series primes of M (assuming finite generation, of course—see [2]) is easier to deal with than Ass M. We mention that this set, the study of which originated with Jategoankar, plays a crucial role in the proof [4] of his principal ideal theorem for noetherian PI-rings.

COROLLARY 2. *If M is a finitely generated R-module, R right noetherian, then*

$$\mathrm{comp}_{R[x]} M[x] = \{P[x] \mid P \in \mathrm{comp}_R M\}.$$

Moreover, $P[x]$ occurs with the same multiplicity in a critical composition series for $M[x]$ as P occurs in a critical composition series for M.

Proof. Let $0 = M_0 \subset \cdots \subset M_n = M$ be a critical composition series of M where, say, M_i/M_{i-1} is α_i-critical. Then $0 = M_0[x] \subset M_1[x] \subset \cdots \subset M_n[x] = M[x]$ is a critical composition series of $M[x]$ since $M_i[x]/M_{i-1}[x] \simeq (M_i/M_{i-1})[x]$ is an $\alpha_i + 1$-critical $R[x]$-module by [3, Proposition 9.3]. This makes the result any easy consequence of Lemma 1. □

The next result shows that there is a bijection between associated primes in the composition series for M and associated primes in the composition series for $M[x]$; see [2, Lemma 1.7(i)].

THEOREM 3. *If M is a module over a right noetherian ring R then*

$$\mathrm{Ass}_{R[x]} M[x] = \{P[x] \mid P \in \mathrm{Ass}_R M\}$$

Proof. We note that, by Lemma 1, it suffices to prove the assertion that every associated prime of $M[x]$ is the extension of an associated prime of M. For this, we first reduce to the case when M is finitely generated.

Suppose, for a moment, that the assertion is valid for every finitely generated R-module, and let $Q \in \mathrm{Ass}\, M[x]$. Then $Q = \mathrm{ass}\, U$ for a finitely generated uniform submodule U of $M[x]$. Let M_1 be an R-submodule of M generated by the coefficients of a finite set of $R[x]$-generators of U. Thus U is a uniform $R[x]$-submodule of $M_1[x]$ and so, since M_1 is finitely generated over R, $Q = Q_1[x]$ for some $Q_1 \in \mathrm{Ass}\, M_1$. But, plainly, $Q_1 \in \mathrm{Ass}\, M$.

Next, assuming that M is finitely generated, we reduce to the case when M

is primary. For this reduction, we momentarily assume that our assertion (concerning associated primes of the polynomial module being extensions) holds for all primary R-modules. But, by [2, Proposition 1.3], M has a primary decomposition, $\bigcap N_i = 0$ say, where the N_i are primary submodules of M belonging to its distinct associated primes. In particular M/N_i is a, say, P_i-primary R-module and so $M[x]/N_i[x] \simeq (M/N_i)[x]$ is a $P_i[x]$-primary $R[x]$-module.

Now note that, since M is finitely generated, there are only finitely many N_i and so, by [2, Lemma 1.4], the intersection, $\bigcap N_i = 0$, is irredundant. Thus the same is true of the intersection $\bigcap N_i[x] = 0$. Therefore this intersection is a primary decomposition of $M[x]$, by [2, Lemma 1.4]. Thus it follows that each associated prime of $M[x]$ is the extension of one of M.

So suppose M is P-primary, $P \in \operatorname{spec} R$. By [2, Proposition 1.2], M has an essential submodule E with annihilator P. But E is P-primary. Thus, by Lemma 1, $E[x]$ is $P[x]$-primary. Thus $M[x]$ is $P[x]$-primary, by [2, Proposition 1.2], since $E[x]$ is easily seen to be essential in $M[x]$. □

It might be of interest to the reader to note that this theorem remains valid under the weaker assumption that R has Krull dimension.

The following corollary contains the result about Macaulay modules promised at the start.

Corollary 4. *Let M be a module over a right noetherian ring.*

(i) *M is primary if and only if $M[x]$ is primary.*

(ii) *M is Macaulay if and only if $M[x]$ is Macaulay.*

Proof. The first part is immediate and the second follows from the fact, using [3, Theorem 9.2], that for an ideal I of a right noetherian ring R, $K \dim R[x]/I[x] = K \dim R/I + 1$. □

One of the main results of [2] is that any fully right bounded right noetherian ring has a primary decomposition.

Corollary 5. *If R is a right FBN ring then the ring $R[x]$ has a primary decomposition.*

Proof. Let $\operatorname{Ass} R = \{P_1, \ldots, P_n\}$. By [2, Corollary 2.4] R has a primary decomposition $\bigcap_{i=1}^{n} T_i = 0$ where T_i is a primary two-sided ideal belonging to P_i. But $\bigcap T_i[x] = 0$ and, by our previous results, $R[x]/T_i[x]$ is primary, $T_i[x]$ belongs to $P_i[x]$ and $\operatorname{Ass} R[x] = \{P_1[x], \ldots, P_n[x]\}$. □

It is not known whether (right) noetherian rings in general have primary decompositions. But also, vis-a-vis Corollary 5, the polynomial ring over an FBN ring need not be (one-sided) fully bounded. We demonstrate this with the following example of J. C. Robson.

Let R be the quotient division ring of the Weyl algebra $A_1 = \mathbf{C}[x, y]$, $\mathbf{C}$ the complex numbers, $xy - yx = 1$. Then the maximal right ideal $(z + y)R[z]$ of $R[z]$ is unbounded because the maximal ideals of $R[z]$ are generated by elements of $\mathbf{C}[z]$ having degree 1.

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