Canad. Math. Bull. Vol. **63** (1), 2020 pp. 6–12
http://dx.doi.org/10.4153/S0008439519000122

A Remark on the Dixmier Conjecture

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Abstract. The Dixmier Conjecture says that every endomorphism of the (first) Weyl algebra A_1 (over a field of characteristic zero) is an automorphism, *i.e.*, if $PQ - QP = 1$ for some $P, Q \in A_1$, then $A_1 = K\langle P, Q\rangle$. The Weyl algebra A_1 is a $\mathbb{Z}$-graded algebra. We prove that the Dixmier Conjecture holds if the elements P and Q are sums of no more than two homogeneous elements of A_1 (there is no restriction on the total degrees of P and Q).

1 Introduction

In this paper, K is a field of characteristic zero and $K^* := K \smallsetminus \{0\}$. The algebra $A_1 := K\langle X, Y \mid [Y, X] = 1\rangle$ is called the *first Weyl algebra* where $[Y, X] = YX - XY$. The n-th tensor power of A_1,

$$A_n := A_1^{\otimes n} = \underbrace{A_1 \otimes \cdots \otimes A_1}_{n \text{ times}},$$

is called the *n-th Weyl algebra*. The algebra A_n is a simple Noetherian domain of Gel'fand–Kirillov dimension $\mathrm{GK}(A_n) = 2n$; it is canonically isomorphic to the algebra of polynomial differential operators $K\langle X_1, \dots, X_n, \partial_1, \dots, \partial_n\rangle$ (where $\partial_i = \frac{\partial}{\partial X_i}$) via $X_i \mapsto X_i$, $Y_i \mapsto \partial_i$ for $i = 1, \dots, n$.

In his seminal paper, Dixmier [9] found explicit generators for the group $G = \mathrm{Aut}_K(A_1)$ of K-automorphisms of the Weyl algebra A_1. Namely, the group G is generated by the obvious automorphisms:

$$(X, Y) \longmapsto (X, Y + \lambda X^n), \quad (X, Y) \longmapsto (X + \lambda Y^n, Y), \quad (X, Y) \longmapsto (\mu X, \mu^{-1} Y),$$

where $\lambda \in K$, $\mu \in K^*$, and $n \in \mathbb{N}_+ := \{1, 2, \dots\}$.

In [9], Dixmier posed six problems. The first problem of Dixmier (in the list) asks if *every endomorphism of the Weyl algebra A_1 is an automorphism*, i.e., given elements P, Q of A such that $[P, Q] = 1$, do they generate the algebra A_1? A similar problem, but for the n-th Weyl algebra, is called the *Dixmier Conjecture*. Problems three and six have been solved by Joseph [10], Problem four (in the case of homogeneous elements) and Problem five have been solved by Bavula [4].

The Dixmier Conjecture implies the Jacobian Conjecture [2], and the inverse implication is also true [8, 11]; a short proof is given in [6]; see also [1].

Received by the editors November 30, 2018; revised March 19, 2019.
Published online on Cambridge Core August 30, 2019.
Author V. V. B. was supported by Graduiertenkolleg "Experimentelle und konstruktive Algebra" of the German Research Foundation (DFG). Author V. L. was supported by Project II.6 of SFB-TRR 195 "Symbolic Tools in Mathematics and their Applications" of the DFG.
AMS subject classification: **16S50,** 16W20, 16S32, 16W50.
Keywords: Weyl algebra, Dixmier Conjecture, automorphism, endomorphism, a $\mathbb{Z}$-graded algebra.

In [5], it was shown that for each K-endomorphism $\phi: A_n \to A_n$, its image is very large, *i.e.*, the left A_{2n}-module ${}^{\phi}A_n{}^{\phi}$ is a holonomic A_{2n}-module, where for all a, $b \in A_n$ and $c \in {}^{\phi}A_n{}^{\phi}$, $a{\cdot}c{\cdot}b := \phi(a)c\phi(b)$. In particular, it has finite length with simple holonomic factors over A_{2n} (see [5] for details). To prove that the Dixmier Conjecture holds for the Weyl algebra A_n, it remains to show that the length is 1. Note that the Gel'fand–Kirillov dimension of a simple A_{2n}-module can be $2n, 2n+1, \dots, 4n-1$, and the last case is the generic case.

It was also shown [7] that every algebra endomorphism of the algebra $\mathbb{I}_1 = K\langle x, \partial, \int\rangle$ of polynomial integro-differential operators is an automorphism and it was conjectured that the same result holds for

$$\mathbb{I}_n := \mathbb{I}_1^{\otimes n} = K\Big\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \Big\rangle.$$

The Weyl algebra $A_1 = \bigoplus_{i\in\mathbb{Z}} A_{1,i}$ is a $\mathbb{Z}$-graded algebra ($A_{1,i}A_{1,j} \subseteq A_{1,i+j}$ for all i, $j \in \mathbb{Z}$) where $A_{1,0} = K[H]$, $H = YX$, and, for $i \geq 1$, $A_{1,i} = K[H]X^i$ and $A_{1,-i} = K[H]Y^i$. For a nonzero element a of A_1, the number of *nonzero homogeneous* components is called the *mass* of a, denoted by $m(a)$. For example, $m(\alpha X^i) = 1$ for all $\alpha \in K[H] \setminus \{0\}$ and $i \geq 1$. The aim of this paper is to prove the following theorem.

Theorem 1.1 *Let P, Q be elements of the first Weyl algebra A_1 with $m(P) \leq 2$ and $m(Q) \leq 2$. If $[P, Q] = 1$, then $P = \tau(Y)$ and $Q = \tau(X)$ for some automorphism $\tau \in \operatorname{Aut}_K(A_1)$.*

2 Proof of Theorem 1.1

The Weyl algebra is a generalized Weyl algebra. Let D be a ring with an automorphism σ and a central element a. The generalized Weyl algebra $A = D(\sigma, a)$ of degree 1, is the ring generated by D and two indeterminates X and Y subject to the relations [3]

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \text{ for all } \alpha \in D, YX = a, \text{ and } XY = \sigma(a).$$

The algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a $\mathbb{Z}$-graded algebra, where $A_n = Dv_n$, $v_n = X^n$ $(n > 0)$, $v_n = Y^{-n}$ $(n < 0)$, $v_0 = 1$. It follows from the defining relations that $v_n v_m = (n, m)v_{n+m}$ $= v_{n+m}{<}n, m{>}$, for some elements

$$(n, m) = \sigma^{-n-m}(<n, m>) \in D.$$

If $n > 0$ and $m > 0$, then

$$n \geq m : (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \qquad (-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$
$$n \leq m : (n, -m) = \sigma^n(a) \cdots \sigma(a), \qquad (-n, m) = \sigma^{-n+1}(a) \cdots a.$$

In other cases $(n, m) = 1$.

Let $K[H]$ be a polynomial ring in a variable H over the field K, $\sigma: H \to H - 1$ be the K-automorphism of the algebra $K[H]$, and $a = H$. The first Weyl algebra $A_1 = K\langle X, Y \mid YX - XY = 1\rangle$ is isomorphic to the generalized Weyl algebra $A_1 \simeq K[H](\sigma, H)$, $X \mapsto X$, $Y \mapsto Y$, $YX \mapsto H$. We identify both these algebras via this isomorphism, that is, $A_1 = K[H](\sigma, H)$ and $H = YX$.

If $n > 0$ and $m > 0$, then

$$\begin{aligned} n \geq m: (n, -m) &= (H-n)\cdots(H-n+m-1),\\ (-n, m) &= (H+n-1)\cdots(H+n-m),\\ n \leq m: (n, -m) &= (H-n)\cdots(H-1),\\ (-n, m) &= (H+n-1)\cdots H. \end{aligned}$$

In other cases $(n, m) = 1$.

The localization $B = S^{-1}A_1$ of the Weyl algebra A_1 at the Ore subset $S = K[H]\backslash\{0\}$ of A_1 is the *skew Laurent polynomial ring* $B = K(H)[X, X^{-1}; \sigma]$ with coefficients from the field $K(H) = S^{-1}K[H]$ of rational functions, where $\sigma \in \mathrm{Aut}_K K(H)$, and $\sigma(H) = H - 1$. The map $A_1 \to B$, $a \mapsto a/1$ is an algebra monomorphism. We identify the algebra A_1 with its image in the algebra B via $A_1 \to B$, $X \mapsto X$, $Y \mapsto HX^{-1}$. The algebra $B = \bigoplus_{i\in\mathbb{Z}} B_i$ is a $\mathbb{Z}$-graded algebra, where $B_i = K(H)X^i$. The algebra A_1 is a $\mathbb{Z}$-graded subalgebra of B.

A polynomial $f(H) = \lambda_n H^n + \lambda_{n-1}H^{n-1} + \cdots + \lambda_0 \in K[H]$ of degree n is called a *monic polynomial* if the leading coefficient λ_n of $f(H)$ is 1. A rational function $h \in K(H)$ is called a *monic rational function* if $h = f/g$ for some monic polynomials f, g. A homogeneous element $u = \alpha x^n$ of B is called *monic* if α is a monic rational function. We can extend the concept of degree of polynomial to the field of rational functions by the rule $\deg h = \deg f - \deg g$, where $h = f/g \in K[H]$. If $h_1, h_2 \in K(H)$, then $\deg h_1h_2 = \deg h_1 + \deg h_2$ and $\deg(h_1 + h_2) \leq \max\{\deg h_1, \deg h_2\}$. We denote by $\mathrm{sign}(n)$ and by $|n|$ the *sign* and the *absolute value* of $n \in \mathbb{Z}$, respectively.

Let A be an algebra and $a \in A$. The subalgebra of A,

$$C_A(a) = \{b \in A \mid ab = ba\},$$

is called the *centralizer* of the element a in A.

Proposition 2.1 (Centralizer of a Homogeneous Element of the Algebra B) [4, Proposition 2.1]

(i) *Let $u = \alpha X^n$ be a monic element of B_n with $n \neq 0$. Then the centralizer $C_B(u) = K[v, v^{-1}]$ is a Laurent polynomial ring for a unique element $v = \beta X^{\mathrm{sign}(n)s}$, where s is the least positive divisor of n for which there exists an element $\beta = \beta_s \in K(H)$, necessarily monic and uniquely defined, such that*

$$\begin{aligned} \beta\sigma^s(\beta)\sigma^{2s}(\beta)\cdots\sigma^{(n/s-1)s}(\beta) &= \alpha \quad \text{if } n > 0,\\ \beta\sigma^{-s}(\beta)\sigma^{-2s}(\beta)\cdots\sigma^{-(|n|/s-1)s}(\beta) &= \alpha \quad \text{if } n < 0. \end{aligned}$$

(ii) *Let $u \in K(H)\backslash K$. Then $C_B(u) = K(H)$.*

Let $A_{1,+} := K[H][X; \sigma]$ and $A_{1,-} := K[H][Y; \sigma^{-1}]$. The algebras $A_{1,+}$ and $A_{1,-}$ are (skew polynomial) subalgebras of A_1.

Lemma 2.2 ([4]) *If $u \in A_{1,\pm} \setminus \{0\}$, then $C_A(u) \subseteq A_{1,\pm}$.*

https://doi.org/10.4153/S0008439519000122 Published online by Cambridge University Press

The K-automorphism of the Weyl algebra A_1,

$$\xi: A_1 \longrightarrow A_1, \quad X \longmapsto Y, \quad Y \longmapsto -X,$$

reverses the $\mathbb{Z}$-grading of the Weyl algebra A_1, that is,

$$\xi(A_{1,i}) = A_{1,-i} \quad \text{for all } z \in \mathbb{Z}. \tag{2.1}$$

By the *degree* of an element of A_1 we mean its total degree with respect to the canonical generators X and Y of A_1. Let $A_{1,\leq i} := \{p \in A \mid \deg(p) \leq i\}$ for $i \in \mathbb{N}$. Then $\{A_{1,\leq i}\}_{i\in\mathbb{N}}$ is the standard filtration of the algebra A_1 associated with the generators X and Y. For all $i \in \mathbb{Z} \smallsetminus \{0\}$ and $f \in K[H] \smallsetminus K$,

$$\deg \sigma^i(f) = \deg f \text{ and } \deg(1-\sigma^i)(f) = \deg f - 1. \tag{2.2}$$

Proof of Theorem 1.1 Case 1. If $P, Q \in A_{1,\leq 1}$, then $P = \tau(Y)$ and $Q = \tau(X)$ for some $\tau \in \mathrm{Aut}_K(A_1)$. Clearly, $P = aY + bX + \lambda$ and $Q = cY + dX + \mu$ for some $a, b, c, d, \lambda, \mu \in K$. Then $1 = [P, Q] = ad - bc$. So the automorphism τ can be chosen of the form $\tau(Y) = aY + bX + \lambda$ and $\tau(X) = cY + dX + \mu$.

So, until the end of the proof we assume that at least one of the polynomials P or Q does not belong to the space $A_{1,\leq 1}$. In view of the relation $1 = [P, Q] = [-Q, P]$, we can assume that $P \notin A_{1,\leq 1}$. In view of (2.1), we can assume that the highest homogeneous part of P, say $P_p \in A_{1,p}$, satisfies the condition that $p \geq 2$. Since $m(P) \leq 2$, either $P = P_p$ (if $m(P) = 1$) or $P = P_r + P_p$ for some nonzero $P_r \in A_{1,r}$, where $r < p$.

Case 2. $(m(P), m(Q)) \neq (1,1)$. Suppose that $m(P) = m(Q) = 1$. We seek a contradiction. Then $P = \alpha X^p$ and $Q = \beta Y^p$ for some nonzero polynomials $\alpha, \beta \in K[H]$. Then

$$\begin{aligned} 1 = [P, Q] &= \alpha\sigma^p(\beta)(p, -p) - \beta\sigma^{-p}(\alpha)(-p, p) \\ &= \alpha\sigma^p(\beta)(p, -p) - \beta\sigma^{-p}(\alpha)\sigma^{-p}((p, -p)) \\ &= (1 - \sigma^{-p})(\alpha\sigma^p(\beta)(p, -p)). \end{aligned}$$

Since $p \geq 2$ (or $P \notin A_{1,\leq 1}$),

$$0 = \deg 1 = \deg(1 - \sigma^{-p})(\alpha\sigma^p(\beta)(p, -p)) = \deg\alpha + \deg\beta + \deg(p, -p) - 1$$

(by (2.2)) $\geq 0 + 0 + p - 1 \geq 2 - 1 = 1$, a contradiction.

Case 3. $(m(P), m(Q)) \neq (1,2)$. Suppose that $m(P) = 1$ and $m(Q) = 2$. Then $P = \alpha X^p$ for some $p \geq 2$ and $Q = Q_s + Q_q$, where $Q_s \in A_{1,s}$, $Q_q \in A_{1,q}$, and $s < q$. By Lemma 2.2, the equality $[P, Q] = 1$ implies that $[P, Q_s] = 1$ and $[P, Q_q] = 0$. By Case 2, this is not possible.

Case 4. Suppose that $m(P) = 2$ and $m(Q) = 1$. Then $P = P_r + P_p$ and $Q = Q_q$. By Lemma 2.2, the equality $[P, Q] = 1$ implies that $[P_p, Q_q] = 0$ and $[P_r, Q_q] = 1$. Then $q \geq 0$, by Lemma 2.2. The case $q = 0$ is not possible since then both $P_r, Q_q \in K[H]$ and this would contradict the equality $[P_r, Q_q] = 1$. Therefore, $q > 0$. Then $P_r = \beta Y^q$ and $Q_q = \alpha X^q$ for some nonzero elements $\beta, \alpha \in K[H]$. Then $-1 = [Q_q, P_r] = (1 - \sigma^{-q})(\alpha\sigma^p(\beta)(q, -q))$ implies that

$$0 = \deg(-1) = \deg(1 - \sigma^{-q})(\alpha\sigma^p(\beta)(q, -q)) = \deg\alpha + \deg\beta + q - 1,$$

by (2.2). Hence, $q = 1$, $\alpha, \beta \in K^*$, and $\beta = -\alpha^{-1}$. Then $P, Q \in A_{1,\leq 1}$, and, by Case 1, the pair (P, Q) is obtained from the pair (Y, X) by applying an automorphism of A_1.

https://doi.org/10.4153/S0008439519000122 Published online by Cambridge University Press

Case 5. $(m(P), m(Q)) \neq (2,2)$. Since $m(P) = m(Q) = 2$, we can write $P = P_r + P_p$ and $Q = Q_s + Q_q$ as sums of homogeneous elements, where $r < p$, $P_r \in A_{1,r}$, $P_p \in A_{1,p}$, and $s < q$, $Q_s \in A_{1,s}$, $Q_q \in A_{1,q}$. The equality $[P, Q] = 1$ implies that $[P_r, Q_s] = 0$ and $[P_p, Q_q] = 0$; see Lemma 2.2. By the same Lemma, the elements r and s have the same sign, *i.e.*, either $r < 0, s < 0$ or $r = s = 0$ or $r > 0, s > 0$, and also the elements p and q have the same sign. Since $p \geq 2$, we must have $q > 0$.

Suppose that $r \geq 0$, we seek a contradiction. Then $s \geq 0$ and so the elements P and Q are elements of the subring $A_{1,+} = \bigoplus_{i \geq 0} K[H]X^i$. Now

$$K[H] \ni 1 = [P, Q] \in [A_{1,+}, A_{1,+}] \subseteq \bigoplus_{i \geq 1} K[H]X^i,$$

a contradiction. Therefore, $r < 0$ and $s < 0$.

The equality $1 = [P, Q] = [P_r, Q_q] + [P_p, Q_s]$ and Lemma 2.2 imply that $r + q = 0$ and $p + s = 0$, that is, $r = -q$ and $s = -p$. So $P = P_{-q} + P_p$ and $Q = Q_{-p} + Q_q$. The elements P_p and P_{-q} are homogeneous elements of the Weyl algebra A_1. The Weyl algebra A_1 is a homogeneous subalgebra of the algebra $K(H)[X, X^{-1}; \sigma] = K(H)[Y, Y^{-1}; \sigma^{-1}]$, where $K(H)$ is the field of rational functions in the variable H and the automorphism σ of $K(H)$ is given by the rule $\sigma(H) = H - 1$. By [4, Proposition 2.1(1)], the centralizer $C_B(P_p)$ of the element P_p in B is a Laurent polynomial algebra

$$K[\alpha X^n, (\alpha X^n)^{-1}]$$

for some nonzero element $\alpha \in K(H)$ and $n \geq 1$. In general, $\alpha \notin K[H]$. Similarly, $C_B(P_{-q}) = K[\beta Y^m, (\beta Y^m)^{-1}]$, for some nonzero element $\beta \in K(H)$ and $m \geq 1$.

Since $[P_p, Q_q] = 0$, $Q_q \in C_B(P_p)$, and

$$P_p = \lambda(P_p)(\alpha X^n)^i = \lambda(P_p)\alpha\sigma^n(\alpha)\cdots\sigma^{n(i-1)}(\alpha)X^{ni} = \alpha_{n,i}X^p,$$
$$Q_q = \lambda(Q_q)(\alpha X^n)^j = \lambda(Q_q)\alpha\sigma^n(\alpha)\cdots\sigma^{n(j-1)}(\alpha)X^{nj} = \alpha'_{n,j}X^q,$$

for some nonzero scalars $\lambda(P_p), \lambda(Q_q) \in K^*$ and some $i \geq 1$ and $j \geq 1$, where,

$$\alpha_{n,i} = \lambda(P_p)\alpha\sigma^n(\alpha)\cdots\sigma^{n(i-1)}(\alpha) \in K[H], \quad p = ni,$$
$$\alpha'_{n,j} = \lambda(Q_q)\alpha\sigma^n(\alpha)\cdots\sigma^{n(j-1)}(\alpha) \in K[H], \quad q = nj.$$

Since $[P_{-p}, Q_{-p}] = 0$, $Q_{-p} \in C_B(P_{-q})$, and

$$P_{-q} = \lambda(P_{-q})(\beta Y^m)^s = \lambda(P_{-q})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(s-1)}(\beta)Y^{ms} = \beta_{m,s}Y^p,$$
$$Q_{-p} = \lambda(Q_{-p})(\beta Y^m)^t = \lambda(Q_{-p})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(t-1)}(\beta)Y^{mt} = \beta'_{m,t}Y^q,$$

for some nonzero scalars $\lambda(P_{-q}), \lambda(Q_{-p}) \in K^*$ and some $s \geq 1$ and $t \geq 1$, where,

$$\beta_{m,s} = \lambda(P_{-q})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(s-1)}(\beta) \in K[H], \; p = ms,$$
$$\beta'_{m,t} = \lambda(Q_{-p})\beta\sigma^{-m}(\beta)\cdots\sigma^{-m(t-1)}(\beta) \in K[H], \; q = mt.$$

Now

$$\begin{aligned} 1 = [P, Q] &= [P_p, Q_{-p}] + [P_{-q}, Q_q] = [\alpha_{n,i}X^p, \beta'_{m,t}Y^p] + [\beta_{m,s}Y^q, \alpha'_{n,j}X^q] \\ &= \alpha_{n,i}\sigma^p(\beta'_{m,t})(p, -p) - \beta'_{m,t}\sigma^{-p}(\alpha_{n,i})(-p, p) \\ &\quad + \beta_{m,s}\sigma^{-q}(\alpha'_{n,j})(-q, q) - \alpha'_{n,j}\sigma^q(\beta_{m,s})(q, -q). \end{aligned}$$

Using the equalities $(-p, p) = \sigma^{-p}((p, -p))$ and $(-q, q) = \sigma^{-q}((q, -q))$, the last equality above can be rewritten as follows

$$1 = (1 - \sigma^{-p})(a) + (1 - \sigma^{-q})(b), \tag{1-ab}$$

where $a = \alpha_{n,i}\sigma^p(\beta'_{m,t})(p, -p) \in K[H]$ and $b = \alpha'_{n,j}\sigma^q(\beta_{m,s})(q, -q) \in K[H]$.

Recall that $P = P_{-q} + P_p$, $Q = Q_{-p} + Q_q$,

$$p = mt = ni \geq 2 \quad \text{and} \quad q = ms = nj \geq 1. \tag{2-ab}$$

Suppose that $p = q$, and so $P = P_{-p} + P_p$, $Q = Q_{-p} + Q_p$. Then $Q = \lambda P_p$ for some $\lambda \in K^*$. Notice that

$$1 = [P, Q] = [P, Q - \lambda P], \quad m(P) = 2, \quad m(Q - \lambda P) = 1.$$

By Case 4, the pair $(P, Q - \lambda P)$ is obtained from the pair (Y, X) by applying an automorphism of the Weyl algebra A_1.

So, either $p < q$ or $p > q$. In view of (P, Q)-symmetry (i.e., $1 = [P, Q] = [-Q, P]$), it suffices to consider, say, the first case only. Since $p < q$, the equalities (2-ab) imply that $i < j$ and $t < s$. Then, using (2.2) and the fact that $\deg(p, -p) = p$ for all $p \geq 1$, we see that

$$\deg a = \deg \alpha_{n,i} + \deg \beta'_{m,t} + p - 1,$$
$$\deg b = \deg \alpha'_{n,j} + \deg \beta_{m,s} + q - 1.$$

Since $i < j$ and $t < s$, $\deg \alpha_{n,i} < \deg \alpha'_{n,j}$ and $\deg \beta'_{m,t} < \deg \beta_{m,s}$. In particular, $\deg a < \deg b$. This equality contradicts (1-ab) since, by (2.2), $0 = \deg 1 = \deg a - 1 - \deg b + 1 = \deg a - \deg b > 0$. This means that the cases $p < q$ and $p > q$ are impossible. The proof of the theorem is complete. ∎

Corollary 2.3 *Let P, Q be elements of the first Weyl algebra A_1 with $m(P) = 1$ or $m(Q) = 1$. If $[P, Q] = 1$, then $P = \tau(Y)$ and $Q = \tau(X)$ for some automorphism $\tau \in \mathrm{Aut}_K(A_1)$.*

Proof Without loss of generality we may assume $m(Q) = 1$ and $m(P) \geq 3$. That is, $Q = Q_q$ and $P = \sum_{i \in I} P_i$, where $I \subset \mathbb{Z}$ is a finite set, $q \in \mathbb{Z} \setminus \{0\}$ and the elements Q_q and P_i are homogeneous in A_1. By (2.1), we may assume that $q > 0$. Then $1 = [P, Q] = \sum_i [P_i, Q_q]$ implies that $-q \in I$, $[P_{-q}, Q_q] = 1$ and $[P_j, Q_q] = 0$ for all $j \in I$ such that $j \neq -q$. By Theorem 1.1,

$$q = 1, \quad Q_1 = \lambda X, \quad P_{-1} = \lambda^{-1} Y \quad \text{for some } \lambda \in K^*.$$

By Lemma 2.2, $C := P - P_{-1} \in C_A(X) = K[X]$. Then $P = \tau(Y)$ and $Q = \tau(X)$, where $\tau\colon A_1 \to A_1$, $X \mapsto \lambda X$, $Y \mapsto \lambda^{-1} Y + C$, is an automorphism. ∎

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