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JONES POLYNOMIALS OF PERIODIC KNOTS

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We calculate the Zulli's matrix of a periodic knot and give some necessary conditions for the Jones polynomial of a periodic knot, which are slightly different from Yokota's result.

1. INTRODUCTION

A knot $\widetilde{K}$ is said to *have period* $r > 1$, if there exists an orientation preserving homeomorphism f on S^3 of order r which preserves $\widetilde{K}$ with $\mathrm{Fix}(f) = \{x \in S^3 \mid f(x) = x\} \cong S^1$ and $\mathrm{Fix}(f) \cap \widetilde{K} = \emptyset$.

By the positive solution of the Smith conjecture, $\mathrm{Fix}(f)$ is unknotted. Let $\Sigma^3 = S^3/f$ be the quotient space under f. Since $\mathrm{Fix}(f)$ is unknotted, Σ^3 is again a 3-sphere, $\widetilde{K}/f$ is a knot in Σ^3 and S^3 is an r-fold cyclic covering space of Σ^3 branched along $\mathrm{Fix}(f)$. Let $\psi : S^3 \longrightarrow \Sigma^3$ be the covering projection map. Denote $\psi(\widetilde{K}) = K$ and call it the *factor knot* of $\widetilde{K}$. Note that K is a knot in the 3-sphere Σ^3, so we may assume that K is also a knot in S^3.

Notice that we may have knot diagrams $D(\widetilde{K})$ and $D(K)$ of $\widetilde{K}$ and K respectively, which satisfy the following commutative diagram

$$\begin{array}{ccc} (S^3, \widetilde{K}) & \xrightarrow{\ p\ } & (S^2, D(\widetilde{K})) \\ {\scriptstyle /f}\big\downarrow & & \big\downarrow{\scriptstyle /g} \\ (S^3, K) & \xrightarrow[\ q\ ]{} & (S^2, D(K)), \end{array}$$

where g is the restriction of f to S^2 and p and q are regular projections indicating the knot diagrams $D(K)$ and $D(\widetilde{K})$, respectively.

In this paper, we shall not distinguish the notations for a knot and its diagram, so K will represents a knot or its diagram. Notice that the knot diagram $\widetilde{K}$ consists of r periodic sections, each of which gives us the diagram K of the factor knot. Then we

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can enumerate the crossings of the knot diagram $\widetilde{K}$ as follows. Let $c_1, c_2, \cdots, c_n$ be the crossings in the first periodic section of $\widetilde{K}$. Then

$$g^k(c_i) = c_{i+rk}, \ (i = 1, 2, \cdots, n, \ k = 0, 1, \cdots, r-1)$$

represent all the crossings of $\widetilde{K}$. Here we can identify the knot diagram K with the factor of the first periodic section of the knot diagram $\widetilde{K}$ so that $c_1, c_2, \cdots, c_n$ also represent the crossings of K.

Now we introduce the definition of the Zulli's matrix of a knot diagram.

Given a knot diagram K with the crossings $c_1, c_2, \cdots c_n$, give any orientation to K. The *Zulli's matrix* of the knot diagram K is the $n \times n$ matrix $T = (T_{ij})$ over $\mathbf{Z}_2$ defined as follows. For $i \neq j$, T_{ij} is defined to be the number of times (mod 2) that a traveller passes through crossing c_i while making the following trip - the traveller begins on the overcrossing c_j with the given direction until he returns to the undercrossing c_j. For $i = j$, T_{ii} is defined as follows

$$T_{ii} = \begin{cases} 1 & \text{if the crossing sign of } c_i \text{ is } +1 \\ 0 & \text{otherwise.} \end{cases}$$

A *state* of the knot diagram K is defined to be a function S from the set of all crossings of K to $\{A, B\}$, that is, a choice, at each crossing c_i, of a label A or B. Let $\mathcal{S}(K)$ denote the set of the states of the knot diagram K. Let $S \in \mathcal{S}(K)$ be the state obtained from the state $AA \cdots A$ by exchanging the labels in the positions $c_{i_1}, c_{i_2}, \cdots, c_{i_m}$. Let us denote T_S to be the matrix obtained from the matrix T as follows.

$$\mathrm{ent}_{ij}(T_S) = \begin{cases} 1 - \mathrm{ent}_{ii}(T), & i = i_1, i_2, \cdots i_m, \\ \mathrm{ent}_{ij}(T), & \text{otherwise.} \end{cases}$$

Given a matrix T, let $n(T)$ denote the nullity of the matrix T.

For a knot K, the Jones polynomial $V_K(t)$ of K is obtained from the Kauffman polynomial

$$P_K(A) = (-A^{-3})^{w(K)} \sum_S A^{A(S)-B(S)} (-A^2 - A^{-2})^{\sharp(K|S)-1}$$

by putting $A = t^{-1/4}$, where $w(K)$ is the writhe of the knot K, $A(S), B(S)$ are the numbers of A, B values in the state S, respectively, and $\sharp(K|S)$ is the numbers of circles in the split-open diagram $K|S$. (See Kauffman [**1, 2**], Murasugi [**3**] and Zulli [**6**].) In [**6**], Zulli proved that $n(T_S) = \sharp(K|S) - 1$. We shall use the following notation

$$A^{A(S)-B(S)} = \mathrm{Coeff}(S) \qquad \text{and} \qquad d = -(A^2 + A^{-2}),$$

for simplicity. Then the above equation can be simplified to

$$P_K(A) = (-A^{-3})^{w(K)} \sum_S \mathrm{Coeff}(S) d^{n(T_S)}.$$

2. Main Results

Theorem 1. *Suppose that $\widetilde{K}$ is an r-periodic oriented knot diagram which has the factor knot diagram K and T is the Zulli's matrix of K. Then the Zulli's matrix of $\widetilde{K}$ is the blockwise circulant matrix $\widetilde{T}$ of the form*

$$\widetilde{T} = \begin{bmatrix} T_1 & T_r & \cdots & T_2 \\ T_2 & T_1 & \cdots & T_3 \\ \cdots & \cdots & \cdots & \cdots \\ T_{r-1} & T_{r-2} & \cdots & T_r \\ T_r & T_{r-1} & \cdots & T_1 \end{bmatrix} \tag{1}$$

where each T_k is an $n \times n$ matrix such that T_1 is symmetric, $T_k = {}^t(T_{r-k+2})$ the transposed matrix of T_{r-k+2} $(2 \leqslant k \leqslant r,\ r \geqslant 2)$ and

$$\sum_{k=1}^{r} \mathrm{ent}_{ij}(T_k) = \mathrm{ent}_{ij}(T),$$

where $\mathrm{ent}_{ij}(A)$ denotes the (i,j)-entry of a matrix A.

Proof: From the definition of the Zulli's matrix, it is obvious that T_k is the $n \times n$ matrix whose ij-entry $\mathrm{ent}_{ij}(T_k)$ $(j = 1, 2, \cdots, n)$ is the number of times (mod 2) passing through the crossing c_{i+rk} in the k-th periodic section from the overcrossing of c_j to the undercrossing of c_j along the orientation of the knot diagram K. Since $g^k(c_j) = c_{j+rk}$, $\mathrm{ent}_{ij}(\widetilde{T}) = \mathrm{ent}_{i+rk\ j+rk}(\widetilde{T})$. Thus $\widetilde{T}$ has the form in (1). Since $\widetilde{T}$ is symmetric, T_1 is symmetric and $T_k = {}^t(T_{r-k+2})$. Since K and $\widetilde{K}$ are knots and $g^k(c_j) = c_{j+rk}$, for each $j = 1, 2, \cdots, rn$,

$$\sum_{i=1}^{rn} \mathrm{ent}_{ij}(\widetilde{T}) = \sum_{i=1}^{n} \mathrm{ent}_{ij}(T), \text{ for } i = 1, 2, \cdots, n.$$

∎

Theorem 2 *For an odd prime r, let $\widetilde{K}$ be an r-periodic knot with a factor knot K and f be the periodic map on S^3 realising the r-periodic knot $\widetilde{K}$.*

(1) If $\mathrm{lk}(\mathrm{Fix}(f), \widetilde{K}) \equiv 1 \pmod 2$, then

$$P_{\widetilde{K}}(A) \equiv [P_K(A)]^r \quad (\mathrm{mod}\ r,\ \lambda_r(A)).$$

(2) If $\mathrm{lk}(\mathrm{Fix}(f), \widetilde{K}) \equiv 0 \pmod 2$, then

$$P_{\widetilde{K}}(A) \equiv d^{r-1}[P_K(A)]^r \quad (\mathrm{mod}\ r,\ \lambda_r(A))$$

Here, $\lambda_r(A)$ is the polynomial defined by $\lambda_r(A) = A^{8r} - A^{4(r+1)} - A^{4(r-1)} + 1$.

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3. Lemmas and the Proof of Theorem 2

Lemma 1 *For an odd prime r, let $\widetilde{K}$ be an r-periodic knot with a factor knot K and f be the periodic map on S^3 realising the r-periodic knot $\widetilde{K}$. Then*

$$w(\widetilde{K}) = rw(K).$$

Proof: Since $\widetilde{K}$ consists of r periodic sections each of which gives us the diagram K in the quotient, Lemma 1 immediately follows from the definition of $w(K)$. ∎

Lemma 2 *For an odd prime r, let $\widetilde{K}$ be an r-periodic knot with a factor knot K and f be the periodic map on S^3 realising the r-periodic knot $\widetilde{K}$. Let $\widetilde{S} = S_1 S_2 \cdots S_r \in \mathcal{S}(\widetilde{K})$, where S_i is the state in the i-th periodic section of the knot diagram $\widetilde{K}$. If $S_i \neq S_j$ for some i, j, then*

$$\sum_{\widetilde{S} \in \mathcal{S}(\widetilde{K}), S_i \neq S_j} \mathrm{Coeff}(\widetilde{S}) d^{n(\widetilde{T}_{\widetilde{S}})} \equiv 0 \pmod r.$$

Proof: Given the state $\widetilde{S} = S_1 S_2 \cdots S_r \in \mathcal{S}(\widetilde{K})$ as above, let

$$\begin{array}{llll} \widetilde{S}_1 & = S_1 S_2 \cdots S_{r-1} S_r & = & \widetilde{S} \\ \widetilde{S}_2 & = S_2 S_3 \cdots S_r S_1 & & \\ \cdots & \cdots \quad \cdots & \cdots & \\ \widetilde{S}_r & = S_r S_1 \cdots S_{r-2} S_{r-1}. & & \end{array}$$

$S_i \neq S_j$ as the states of the factor knot K for some i, j, $\widetilde{S}_1, \widetilde{S}_2, \cdots, \widetilde{S}_r$ are all distinct. We have

$$n(\widetilde{T}_{\widetilde{S}_1}) = n(\widetilde{T}_{\widetilde{S}_2}) = \cdots = n(\widetilde{T}_{\widetilde{S}_r})$$

and

$$\mathrm{Coeff}(\widetilde{S}_1) = \mathrm{Coeff}(\widetilde{S}_2) = \cdots = \mathrm{Coeff}(\widetilde{S}_r).$$

Hence we have

$$\sum_{\widetilde{S} \in \mathcal{S}(\widetilde{K}), S_i \neq S_j} \mathrm{Coeff}(\widetilde{S}) d^{n(\widetilde{T}_{\widetilde{S}})} \equiv 0 \pmod r.$$

∎

Lemma 3 *For an odd prime r, let $\widetilde{K}$ be an r-periodic knot with a factor knot K and f be the periodic map on S^3 realising the r-periodic knot $\widetilde{K}$. Let $\widetilde{S} = SS \cdots S \in \mathcal{S}(\widetilde{K})$ with $S \in \mathcal{S}(K)$. Then*

$$n(\widetilde{T}_{\widetilde{S}}) \equiv n(T_S) \pmod{r-1}.$$

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PROOF: Let $\widetilde{S} = SS\cdots S \in \mathcal{S}(\widetilde{K})$ be a state of $\widetilde{K}$ and let $D_1, D_2, \cdots, D_k$ be the circles in the diagram $\widetilde{K}|\widetilde{S}$. For each D_i, $1 \leqslant i \leqslant k$, define an $rn \times 1$ matrix $\widetilde{R}_i = \widetilde{R}_i(\widetilde{S})$ by setting

$$\mathrm{ent}_{j1}(\widetilde{R}_i) = \begin{cases} 1, & \text{if } D_i \text{ passes through the crossing } c_j, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f(\widetilde{K}|\widetilde{S}) = \widetilde{K}|\widetilde{S}$, for each $i = 1, 2, \cdots, k$, either the circles D_i are all the same ; $D_i = g(D_i) = g^2(D_i) = \cdots = g^{r-1}(D_i)$ or the circles $D_i, g(D_i), g^2(D_i), \cdots, g^{r-1}(D_i)$ are all distinct circles in the diagram $\widetilde{K}|\widetilde{S}$.

Let

$$\widetilde{R}_i = \begin{bmatrix} R_{i1} \\ R_{i2} \\ \cdots \\ R_{ir} \end{bmatrix}$$

where R_{ij} is the $n \times 1$ matrix such that $\mathrm{ent}_{l1}(R_{ij}) = \mathrm{ent}_{r(j-1)+l,1}(\widetilde{R}_i)$, for $1 \leqslant l \leqslant n$.

The matrix $\widetilde{R}_i$ is said to be *Type I* if $R_{i1} = R_{i2} = \cdots R_{ir}$. Otherwise we say that $\widetilde{R}_i$ is of *type II*. Note that if $\widetilde{R}_i$ is of Type II, then

$$\begin{bmatrix} R_{i1} \\ R_{i2} \\ \cdots \\ R_{ir} \end{bmatrix}, \begin{bmatrix} R_{i2} \\ R_{i3} \\ \cdots \\ R_{i1} \end{bmatrix}, \cdots, \begin{bmatrix} R_{ir} \\ R_{i1} \\ \cdots \\ R_{ir-1} \end{bmatrix}$$

are all distinct.

Let

$$\begin{aligned} k_1 &= \text{the number of the matrices of type I,} \\ k_2 &= \text{the number of the matrices of type II.} \end{aligned} \tag{2}$$

Notice that $k = k_1 + k_2$ and that r divides k_2.
For each $\widetilde{R}_i$ $(i = 1, 2, \cdots, k)$, define an $n \times 1$ matrix $R_i = R_i(\widetilde{S})$ by setting

$$R_i = \sum_{j=1}^{r} R_{ij}.$$

Then, for a fixed $\widetilde{S} \in \mathcal{S}(\widetilde{K})$, the cardinality $\left|\{R_i \mid i = 1, 2, \cdots k\}\right| = k_1 + k_2/r$.
Zulli showed in [6] that $\{\widetilde{R}_1, \widetilde{R}_2, \cdots, \widetilde{R}_{k-1}\}$ forms a basis for $\ker(\widetilde{T}_{\widetilde{S}})$, where $\ker(\widetilde{T}_{\widetilde{S}})$ denotes the kernel of the matrix of $\widetilde{T}_{\widetilde{S}}$ over $\mathbf{Z}_2$. Now, we claim that the set $\{R_i \mid i \neq k\}$ is a basis for $\ker(T_{\widetilde{S}})$.

By Theorem 1,

$$\widetilde{T}_{\widetilde{S}} = \begin{bmatrix} T_1 & T_r & \cdots & T_2 \\ T_2 & T_1 & \cdots & T_3 \\ \cdots & \cdots & \cdots & \cdots \\ T_{r-1} & T_{r-2} & \cdots & T_r \\ T_r & T_{r-1} & \cdots & T_1 \end{bmatrix}$$

with $T_S = \sum_{i=1}^{r} T_i$, and

$$T_S R_i = (T_1 + T_2 + \cdots + T_r)(R_{i1} + R_{i2} + \cdots + R_{ir}) = 0,$$

for

$$\begin{bmatrix} T_1 & T_r & \cdots & T_2 \\ T_2 & T_1 & \cdots & T_3 \\ \cdots & \cdots & \cdots & \cdots \\ T_{r-1} & T_{r-2} & \cdots & T_r \\ T_r & T_{r-1} & \cdots & T_1 \end{bmatrix} \begin{bmatrix} R_{i1} \\ R_{i2} \\ \cdots \\ R_{ir-1} \\ R_{ir} \end{bmatrix} = 0.$$

Thus, $R_i \in \ker(T_S)$

To show that the set $\{R_i \mid i \neq k\}$ generates $\ker(T_S)$, assume that $T_S R = 0$ for some $n \times n$ matrix R. Then

$$\begin{bmatrix} T_1 & T_r & \cdots & T_2 \\ T_2 & T_1 & \cdots & T_3 \\ \cdots & \cdots & \cdots & \cdots \\ T_{r-1} & T_{r-2} & \cdots & T_r \\ T_r & T_{r-1} & \cdots & T_1 \end{bmatrix} \begin{bmatrix} R \\ R \\ \cdots \\ R \\ R \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{r} T_i R \\ \sum_{i=1}^{r} T_i R \\ \cdots \\ \sum_{i=1}^{r} T_i R \\ \sum_{i=1}^{r} T_i R \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{bmatrix} = 0.$$

Thus,

$$\begin{bmatrix} R \\ R \\ \cdots \\ R \\ R \end{bmatrix} \in \ker(\widetilde{T}_{\widetilde{S}}),$$

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and hence

$$\begin{bmatrix} R \\ R \\ \cdots \\ R \\ R \end{bmatrix} = c_1\widetilde{R}_1 + c_2\widetilde{R}_2 + \cdots + c_{k-1}\widetilde{R}_{k-1}$$

$$= \begin{bmatrix} c_1\widetilde{R}_{11} + c_2\widetilde{R}_{21} + \cdots + c_{k-1}\widetilde{R}_{k-1,1} \\ c_1\widetilde{R}_{12} + c_2\widetilde{R}_{22} + \cdots + c_{k-1}\widetilde{R}_{k-1,2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_1\widetilde{R}_{1r-1} + c_2\widetilde{R}_{2r-1} + \cdots + c_{k-1}\widetilde{R}_{k-1,r-1} \\ c_1\widetilde{R}_{1r} + c_2\widetilde{R}_{2r} + \cdots + c_{k-1}\widetilde{R}_{k-1,r} \end{bmatrix},$$

for some $c_1, c_2, \cdots, c_{k-1} \in \mathbf{Z}_2$. Thus

$$\begin{aligned} rR &= c_1 \sum_{i=1}^{r} R_{1i} + c_2 \sum_{i=1}^{r} R_{2i} + \cdots + c_{k-1} \sum_{i=1}^{r} R_{k-1,i} \\ &= c_1R_1 + c_2R_2 + \cdots + c_{k-1}R_{k-1}, \end{aligned}$$

so that $R = \sum_{i\neq k} c_iR_i$, in $\mathbf{Z}_2$.

Finally, we want to show that the set $\{R_i \mid i \neq k\}$ is linearly independent. To do this, assume that $c_1R_1 + c_2R_2 + \cdots + c_{k-1}R_{k-1} = 0$ with $c_1, c_2, \cdots, c_{k-1} \in \mathbf{Z}_2$. If the matrix $\widetilde{R}_k$ is of type I, then clearly,

$$c_1\widetilde{R}_1 + c_2\widetilde{R}_2 + \cdots + c_{k-1}\widetilde{R}_{k-1} = 0,$$

and all the r matrices obtained from $\widetilde{R}_i$ have the same coefficient.

If the matrix $\widetilde{R}_k$ is of type II, then $\sum c_i\widetilde{R}_i = 0$, where the sum runs over all matices and the r matrices obtained from $\widetilde{R}_k$ have coefficient 0 and the other r matrices obtained from $\widetilde{R}_i$ $(i \neq k)$ have the same coefficient c_i. Thus, in each case, we have $c_1 = c_2 = \cdots = c_{k-1} = 0$.

We have proved $n(T_S) = k_1 + k_2/r - 1$, and hence

$$\begin{aligned} n(\widetilde{T}_{\widetilde{S}}) &= k_1 + k_2 - 1 \\ &= rk_1 + k_2 - r - (r-1)(k_1-1) \\ &= r(k_1 + k_2/r - 1) - (r-1)(k_1-1) \\ &= rn(T_S) - (r-1)(k_1-1) \\ &= n(T_S) \pmod{r-1}. \end{aligned}$$

□

Lemma 4 *For an odd prime r, let $\widetilde{K}$ be an r-periodic knot with a factor knot K and f be the periodic map on S^3 realising the r-periodic knot $\widetilde{K}$. Let k_1 be the integer defined in* (2). *Then*

$$\operatorname{lk}\big(\operatorname{Fix}(f), \widetilde{K}\big) \equiv k_1 \pmod 2.$$

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PROOF: If D is a circle in $\widetilde{K}|\widetilde{S}$ such that $g(D) \neq D$, then clearly $D \cap \{(s,\theta) \mid s > 0, \theta = (2\pi/s)t\}$ is even, for $t = 1, 2, \cdots, r$. ∎

Now we are going to prove Theorem 2.

PROOF OF THEOREM 2: First, assume that

$$\mathrm{lk}\big(\mathrm{Fix}(f), \widetilde{K}\big) \equiv 1 \pmod 2.$$

If $\widetilde{S} = SS\cdots S \in \mathcal{S}(\widetilde{K})$ with $S \in \mathcal{S}(K)$, then, by Lemmas 3 and 4, k_1 is odd and

$$n(\widetilde{T}_{\widetilde{S}}) = rn(T_S) - 2(r-1)k, \text{ for some } k \in \mathbf{Z}.$$

Clearly, $\mathrm{Coeff}(\widetilde{S}) = \big(\mathrm{Coeff}(S)\big)^r$ and

$$\begin{aligned} d^{n(\widetilde{T}_{\widetilde{S}})} &= d^{rn(T_S)-2(r-1)k} \\ &\equiv (d^{n(T_S)})^r \pmod{r,\ d^{2(r-1)}-1} \\ &\equiv \big(d^{n(T_S)}\big)^r \pmod{r,\ \lambda_r(A)}, \end{aligned}$$

for

$$\begin{aligned} d^{2(r-1)} - 1 &= d^{-2}(d^{2r} - d^2) \\ &= d^{-2}\big[(A^2 + A^{-2})^{2r} - (A^2 + A^{-2})^2\big] \\ &\equiv d^{-2}\big[(A^{2r} + A^{-2r})^2 - (A^2 + A^{-2})^2\big] \pmod r \\ &= d^{-2}(A^{4r} - A^4 - A^{-4} + A^{-4r}) \\ &= d^{-2}A^{-4r}\lambda_r(A). \end{aligned}$$

Thus

$$\sum_{\widetilde{S}=SS\cdots S\in\mathcal{S}(\widetilde{K})} \mathrm{Coeff}(\widetilde{S})d^{n(\widetilde{K}_{\widetilde{S}})} \equiv \sum_{S\in\mathcal{S}(K)} \big(\mathrm{Coeff}(S)d^{n(T_S)}\big)^r \pmod{r,\ \lambda_r(A)},$$

and hence

$$\begin{aligned} \langle\widetilde{K}\rangle &= \sum_{\widetilde{S}\in\mathcal{S}(\widetilde{K})} \mathrm{Coeff}(\widetilde{S})d^{n(\widetilde{K}_{\widetilde{S}})} \\ &= \sum_{\widetilde{S}\in\mathcal{S}(\widetilde{K}), S_i\neq S_j} \mathrm{Coeff}(\widetilde{S})d^{n(\widetilde{T}_{\widetilde{S}})} + \sum_{\widetilde{S}=SS\cdots S\in\mathcal{S}(\widetilde{K})} \mathrm{Coeff}(\widetilde{S})d^{n(\widetilde{K}_{\widetilde{S}})} \\ &\equiv \sum_{S\in\mathcal{S}(K)} \big(\mathrm{Coeff}(S)^r d^{n(T_S)}\big)^r \pmod{r,\ \lambda_r(A)}, \text{ by Lemma 2} \\ &\equiv \Big(\sum_{S\in\mathcal{S}(K)} \mathrm{Coeff}(S)d^{n(T_S)}\Big)^r \pmod{r,\ \lambda_r(A)} \\ &= \langle K\rangle^r. \end{aligned}$$

By Lemma 1,

$$\begin{aligned} P_{\widetilde{K}}(A) &= (-A^{-3})^{w(\widetilde{K})}\langle\widetilde{K}\rangle \\ &\equiv \big((-A^{-3})^{rw(K)}\langle K\rangle\big)^r \pmod{r,\ \lambda_r(A)} \\ &= \big[P_K(A)\big]^r \pmod{r,\ \lambda_r(A)}. \end{aligned}$$

Next, assume that

$$\mathrm{lk}\big(\mathrm{Fix}(f), \widetilde{K}\big) \equiv 0 \pmod 2.$$

If $\widetilde{S} = SS\cdots S \in \mathcal{S}(\widetilde{K})$ with $S \in \mathcal{S}(K)$, then, by Lemmas 3 and 4, k_1 is even, so

$$\begin{aligned} n(\widetilde{T}_{\widetilde{S}}) &= rn(T_S) - (r-1)(2k-1), \text{ for some } k \in \mathbf{Z} \\ &= rn(T_S) + (r-1) - 2(r-1)(k-1). \end{aligned}$$

Thus

$$\begin{aligned} d^{n(\widetilde{T}_{\widetilde{S}})} &= d^{rn(T_S)+(r-1)-2(r-1)(k-1)} \\ &= d^{r-1} d^{rn(T_S)-2(r-1)(k-1)} \\ &\equiv d^{r-1}(d^{n(T_S)})^r \pmod{r,\ d^{2(r-1)}-1} \\ &\equiv d^{r-1}(d^{n(T_S)})^r \pmod{r,\ \lambda_r(A)}, \end{aligned}$$

and hence

$$\begin{aligned} \langle \widetilde{K} \rangle &= \sum_{\widetilde{S}\in\mathcal{S}(\widetilde{K})} \mathrm{Coeff}(\widetilde{S}) d^{n(\widetilde{K}_{\widetilde{S}})} \\ &= \sum_{\widetilde{S}\in\mathcal{S}(\widetilde{K}), S_i\neq S_j} \mathrm{Coeff}(\widetilde{S}) d^{n(\widetilde{T}_{\widetilde{S}})} + \sum_{\widetilde{S}=SS\cdots S\in\mathcal{S}(\widetilde{K})} \mathrm{Coeff}(\widetilde{S}) d^{n(\widetilde{K}_{\widetilde{S}})} \\ &\equiv \sum_{S\in\mathcal{S}(K)} \big(\mathrm{Coeff}(S)^r d^{r-1} d^{n(T_S)}\big)^r \pmod{r,\ \lambda_r(A)}, \text{ by Lemma 2} \\ &\equiv d^{r-1}\Big(\sum_{S\in\mathcal{S}(K)} \mathrm{Coeff}(S) d^{n(T_S)}\Big)^r \pmod{r,\ \lambda_r(A)} \\ &= d^{r-1}\big(\langle K \rangle\big)^r \end{aligned}$$

and by Lemma 1,

$$\begin{aligned} P_{\widetilde{K}}(A) &= (-A^{-3})^{w(\widetilde{K})}\langle \widetilde{K} \rangle \\ &\equiv (-A^{-3})^{rw(K)} d^{r-1} \langle K \rangle^r \pmod{r,\ \lambda_r(A)} \\ &= d^{r-1}\big[P_K(A)\big]^r \pmod{r,\ \lambda_r(A)}. \end{aligned}$$

∎

COROLLARY *For an odd prime r, let $\widetilde{K}$ be an r-periodic knot with a factor knot K and f the periodic map on S^3 realising the r-periodic knot $\widetilde{K}$.*

(1) If $\mathrm{lk}\big(\mathrm{Fix}(f), \widetilde{K}\big) \equiv 1 \pmod 2$, *then*

$$V_{\widetilde{K}}(t) \equiv \big[V_K(t)\big]^r \pmod{r,\ \xi_r(t)}.$$

(2) If $\mathrm{lk}\big(\mathrm{Fix}(f), \widetilde{K}\big) \equiv 0 \pmod 2$, *then*

$$V_{\widetilde{K}}(t) \equiv d^{r-1}\big[V_K(t)\big]^r \pmod{r,\ \xi_r(t)}.$$

Here, $\xi_r(t) = t^{2r} - t^{r+1} - t^{r-1} + 1.$

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