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COMPLEX CLIFFORD ANALYSIS AND DOMAINS OF HOLOMORPHY

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Abstract

Integrals related to Cauchy's integral formula and Huygens' principle are used to establish a link between domains of holomorphy in n complex variables and cells of harmonicity in one higher dimension. These integrals enable us to determine domains to which analytic functions on real analytic surfaces in $\mathbf{R}^{n+1}$ may be extended to solutions to a Dirac equation.

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1. Introduction

The use of Clifford algebras in the study of a number of aspects of mathematical analysis has steadily increased in recent years (see for example [1, 4, 6, 7, 8]). In this paper we use the Cauchy integral formula arising in Clifford analysis [3] to introduce a fundamental link between domains of holomorphy, in n complex variables, and cells of harmonicity in $(n+1)$ complex variables.

For each domain of holomorphy, $\mathscr{U}$, we use the Cauchy integral to deduce a Cauchy Kowalewski extension, to a cell of harmonicity, for each holomorphic function on $\mathscr{U}$. Each extension is a solution to the Dirac operator in $\mathbb{C}^{n+1}$. Our approach enables us to show that the Cauchy Kowalewski extension of an analytic function defined on a real, n-dimensional, analytic surface $\mathbb{R}^{n+1}$, described by Sommen [15], is determined by the functions' holomorphic extension, and vice versa. For the special case where the extensions are defined on a strip in $\mathbb{R}^{n+1}$ this result has previously been deduced, using different techniques, in [14]. An advantage in the approach used here is that we are able to establish a topological isomorphism between the Clifford module of holomorphic functions defined on $\mathscr{U}$ and the Clifford module of their Cauchy Kowalewski extensions. From this we obtain an approximation theorem over domains of holomorphy. We extend some of the results presented here to higher order iterates of the Dirac operator.

2. Preliminaries

Consider the complex, 2^{n+1}-dimensional Clifford algebra $A_{n+1}(\mathbb{C})$, with basis elements $1, e_1, \ldots, e_{n+1}, \ldots, e_{j_1}, \ldots, e_{j_r}, \ldots, e_1 \ldots e_{n+1}$ where $e_i e_j + e_j e_i = 2\delta_{ij}$ the Kronecker delta, and $j_1 < \cdots < j_r$ with $1 \leq r \leq n+1$. We denote the complex space spanned by $e_1, \ldots, e_{n+1}$ by $\mathbb{C}^{n+1}$. A vector $z_1 e_1 + \cdots + z_{n+1} e_{n+1} \in \mathbb{C}^{n+1}$ is denoted by $\mathbf{z}$ and we set $\{\mathbf{z}: \mathbf{z}^2 = 0\} = N(\mathbf{0})$. For each $\mathbf{z}_1 \in \mathbb{C}^{n+1}$ we denote the set $\{\mathbf{z} \in \mathbb{C}^{n+1}: (\mathbf{z} - \mathbf{z}_1)^2 = 0\}$ by $N(\mathbf{z}_1)$. The subspace of $\mathbb{C}^{n+1}$ spanned by $e_2, \ldots, e_{n+1}$ is denoted by $\mathbb{C}^n$. The norm of a vector $Z = z_0 + z_1 e_1 + \cdots + z_{1 \cdots n+1} e_1 \cdots e_{n+1}$ is $\|Z\| = (|z_0|^2 + |z_1|^2 + \cdots + |z_{1\cdots n+1}|^2)^{1/2}$. Moreover $\|Z_1 Z_2\| \leq 2^{n+1} \|Z_1\| \, \|Z_2\|$ for each $Z_1, Z_2 \in A_{n+1}(\mathbb{C})$.

Suppose that $\mathbf{z}_1, \ldots, \mathbf{z}_p \in \mathbb{C}^{n+1}$ and we denote $\mathbf{z}_1 \cdots \mathbf{z}_p$ by a. Following [9] we denote $\mathbf{z}_p \cdots \mathbf{z}_1$ be $\tilde{a}$. Consider the Clifford matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a = a_1 \cdots a_{p_1}$, $b = b_1 \cdots b_{p_2}$, $c = c_1 \cdots c_{p_3}$, $d = d_1 \cdots d_{p_4}$ with $a_{j_1}, b_{j_2}, c_{j_3}, d_{j_4} \in \mathbb{C}^{n+1}$ and $1 \leq j_k \leq p_k$, with $k \in \{1, 2, 3, 4\}$, $a\tilde{c}, \tilde{c}d, d\tilde{b}\tilde{b}a \in \mathbb{C}^{n+1}$ and $a\tilde{d} - b\tilde{c} \in \mathbb{C} \backslash \{0\}$. Then in [11] we show that $(a\mathbf{z} + b)(c\mathbf{z} + d)^{-1}$ is a well defined Möbius transformation in $\mathbb{C}^{n+1}$. This extends a result described over $\mathbb{R}^{n+1}$ in [1]. An example of such a matrix is $\left(\begin{smallmatrix} 1 & -e_1 \\ -e_1 & 1 \end{smallmatrix}\right)$. The corresponding Möbius transformation is the Cayley map $(\mathbf{z} - e_1)(-e_1 \mathbf{z} + 1)^{-1}$. The Clifford matrix $\left(\begin{smallmatrix} 1 & e_1 \\ -e_1 & 1 \end{smallmatrix}\right)$ gives the Möbius transformation $(\mathbf{z} + e_1)(-e_1 \mathbf{z} - 1)^{-1}$. On restricting these transformations to the sets $\mathbb{C}^n \backslash \{\mathbf{z} \in \mathbf{C}^n: \mathbf{z}^2 = 1\}$ we obtain holomorphic charts for the complex, n-dimensional sphere $S^n_{\mathbb{C}} = \{\mathbf{z} \in \mathbb{C}^{n+1}: \mathbf{z}^2 = 1\}$.

DEFINITION 1 [3]. For U a domain in $\mathbb{R}^{n+1}$, the real space spanned by $e_1, \ldots, e_{n+1}$, a C^1 function $f: U \to A_{n+1}(\mathbb{C})$ is called a *left regular*, or *left monogenic*, function if for each $\mathbf{x} \in U$ we have that $\sum_{j=1}^{n+1} e_j \partial f(\mathbf{x})/\partial x_j = 0$.

A similar definition may be given for right regular functions.

We denote the set of left regular functions defined on U by $M_l(U, A_{n+1}(\mathbb{C}))$, and the set of right regular functions defined on U by $M_r(U, A_{n+1}(\mathbb{C}))$. The set $M_l(U, A_{n+1}(\mathbb{C}))$ is a right $A_{n+1}(\mathbb{C})$ module while $M_r(U, A_{n+1}(\mathbb{C}))$ is a left $A_{n+1}(\mathbb{C})$ module.

Following [3, 7, 13] we have that for ech domain $U' \subseteq \mathbb{R}^n$, the space spanned by $e_2, \ldots, e_{n+1}$, and for each real analytic function $g: U' \to A_{n+1}(\mathbb{C})$, there is a domain $U'_g \subseteq \mathbb{R}^{n+1}$, with $U' \subseteq U'_g$, and a left regular function $f_g: U' \to A_{n+1}(\mathbb{C})$ with $f_g|_{U'} = g$. On a suitable subdomain of U'_g we have that $f_g(\mathbf{x}) = \exp(-x_1 e_1 \partial_{\vec{\mathbf{x}}})g(\vec{\mathbf{x}})$, where $\partial_{\vec{\mathbf{x}}} = \sum_{j=2}^{n+1} e_j \partial/\partial x_j$ and $\vec{\mathbf{x}} = \sum_{j=2}^{n+1} x_j e_j \in U'$.

DEFINITION 2. The function f_g is called the *left regular Cauchy Kowalewski extension* of g.

If two analytic functions g_1, g_2 are defined on U' then it is not in general the case that $U'_{g_1} = U'_{g_2}$; for example let $g_1(\mathbf{x}) = (1 + \|\mathbf{x}\|)$ and $g_2(\mathbf{x}) = (1 + \|\mathbf{x}\|)^{-1}$ on $\mathbb{R}^n$.

For each left regular function $f: U \to A_{n+1}(\mathbb{C})$ we have the Cauchy integral formula [3]

$$f(\mathbf{x}_0) = \frac{1}{\omega_n} \int_{\partial M} G(\mathbf{x} - \mathbf{x}_0) D\mathbf{x} f(\mathbf{x}), \tag{1}$$

where $\mathbf{x}_0 \in \mathring{M}$, with M a real, $(n+1)$-dimensional, compact submanifold of U, ω_n the surface area of the unit sphere in $\mathbb{R}^{n+1}$, $G(\mathbf{x}) = \mathbf{x}|\mathbf{x}|^{-n+1}$ and $D\mathbf{x} = \sum_{j=1}^{n+1} e_j (-1)^j \, d\hat{x}_j$.

It may be deduced from (1) that the left regular Cauchy Kowalewski extension of g is unique.

When $n + 1 = 2k$ we have $G(\mathbf{x}) = -\mathbf{x}(x_1^2 + \cdots + x_{n+1}^2)^{-k}$ so the formula (1) has a unique holomorphic continuation to

$$f^+(\mathbf{z}_0) = \frac{1}{\omega_n} \int_{\partial M} G^+(\mathbf{x} - \mathbf{z}_0) D\mathbf{x} f(\mathbf{x}), \tag{2}$$

where $\mathbf{z}_0$ lies in the component M^+ of $\mathbb{C}^{n+1} \backslash \bigcup_{\mathbf{x} \in \partial M} N(\mathbf{x})$ which contains $\mathring{M}$, and $G^+(\mathbf{z}) = -\mathbf{z}(z_1^2 + \cdots + z_{n+1}^2)^{-k}$. The domain M^+ is an example of a cell of harmonicity and has previously been discussed in [2, 12] and elsewhere.

If $\mathbf{z}_0 \in M^+ \backslash \mathring{M}$ then it is straightforward to deduce that $N(\mathbf{z}_0) \cap M$ is an $(n-1)$ dimensional sphere, $S^{n-1}(\mathbf{z}_0)$. By Stokes' theorem and the Cauchy

Riemann equations we have that the integral (2) is equal to

$$\frac{1}{\omega_n}\int_{\Sigma(\mathbf{z}_0)} G^+(\mathbf{z}-\mathbf{z}_0)D\mathbf{z}f^+(\mathbf{z}), \tag{3}$$

where $\Sigma(\mathbf{z}_0)$ is an S^1 fibration of $S^{n-1}(\mathbf{z}_0)$, where for each $\mathbf{x}\in S^{n-1}(\mathbf{z}_0)$ the S^1 fibre lies in the complex hyperplane $\mathbb{C}_{\mathbf{x}}$ which contains $\mathbf{x}$ and its antipodal point. Moreover, $\Sigma(\mathbf{z}_0)\subseteq M^+$, $D\mathbf{z}=\sum_{j=1}^{n+1}(-1)^j e_j\, d\hat{z}_j$, and the S^1 fibre at $\mathbf{x}$ does not surround the antipodal point of $\mathbf{x}$ within $\mathbb{C}_{\mathbf{x}}$.

From the residue calculus the integral (3) may be partially evaluated to reveal [4] an analogue of the Huygens' principle for the wave equation in even dimensions described by Garabedian in [5, Chapter 6].

When $n = 2k$ the function $G(\mathbf{x})$ does not have a unique holomorphic continuation to $\mathbb{C}^{n+1}\backslash N(\mathbf{0})$. However, we have for each real $(n+1)$-dimensional manifold $M\subseteq\mathbb{R}^{n+1}$, the set $\{\mathbf{z}\in M^+ : \lambda\mathbf{x}+(1-\lambda)\mathbf{z}\in M^+$ for $\mathbf{x}\in N(\mathbf{z})\cap M$ and $\lambda\in[0,1]\}$ is a subdomain of M^+. We denote this subdomain by NM^+, and it is straightforward to deduce that its fundamental group $\pi_1(NM^+)$ is isomorphic to the fundamental group $\pi_1(M)$. Consequently, each left regular function $f: U\subseteq\mathbb{R}^{n+1}\to A_{n+1}(\mathbb{C})$, with $M\subseteq U$, has a unique holomorphic continuation to $f^+: NM^+\to A_{n+1}(\mathbb{C})$. Moreover when M is compact

$$\begin{aligned} f^+(\mathbf{z}_0) &= \frac{1}{\omega_n}\int_{\Sigma(\mathbf{z}_0)} G^+(\mathbf{z}-\mathbf{z}_0)D\mathbf{z}f^+(\mathbf{z}) \\ &= \frac{1}{\omega_n}\int_{D(\mathbf{z}_0,\Sigma(\mathbf{z}_0))} G(\mathbf{x}-\mathbf{z}_0)D\mathbf{x}f(\mathbf{x}), \end{aligned} \tag{4}$$

where $\mathbf{z}_0\in NM^+\backslash\mathring{M}$ and $D\mathbf{z}_0, \Sigma(\mathbf{z}_0)$ is an n-dimensional disc in $\mathbb{R}^n$ with boundary the inner sphere of $\Sigma(\mathbf{z}_0)\cap\mathbf{R}^{n+1}$.

DEFINITION 3. For U^+ a domain in $\mathbb{C}^{n+1}$ and $f^+: U^+\to A_n(\mathbb{C})$ a holomorphic function we say that f^+ is a *complete left regular* function if for each $\mathbf{z}\in U^+$ we have $\sum_{j=1}^{n+1} e_j\partial f(\mathbf{z})/\partial z_j = 0$.

A similar definition can be given for complex right regular functions.

3. Domains of holomorphy in $\mathbb{C}^n$

We may generalize the integral (3) as follows:

THEOREM 1. *Suppose that U is a domain in $\mathbb{R}^{n+1}$, that $n+1=2k$, and $M\subseteq U$ is a compact $(n+1)$-dimensional manifold. Suppose that $\mathbf{z}_0\in M^+\backslash M$*

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and $\Gamma(\mathbf{z}_0)$ is the real $(n+1)$-dimensional submanifold of $\mathbb{C}^{n+1}$ with boundary $\Sigma(\mathbf{z}_0)$, and for each $\mathbf{x} \in U \cap N(\mathbf{z}_0)$ the set $\Gamma(\mathbf{z}_0) \cap \mathbb{C}_{\mathbf{x}}$ is the union of two discs, one centered at $\mathbf{x}$ and the other at its antipodal point. Then for each $\mathbf{x} \in M^+$ for which $\Gamma(\mathbf{z}_0) \cap N(\mathbf{z}_1) \neq \emptyset$ and $\Sigma(\mathbf{z}_0) \cap N(\mathbf{z}_1) = \emptyset$, we have

$$f^+(\mathbf{z}_1) = \frac{1}{\omega_n} \int_{\Sigma(\mathbf{z}_0)} G^+(\mathbf{z} - \mathbf{z}_1) D\mathbf{z} f^+(\mathbf{z}). \tag{5}$$

In fact it is straightforward to show that when $\mathbf{z}_1 \notin \overset{\circ}{M}$ the set $\Gamma(\mathbf{z}_0) \cap N(\mathbf{z}_1)$ is an $(n-1)$-dimensional manifold homomorphic to S^{n-1} and for each $\mathbf{x} \in N(\mathbf{z}_0) \cap M$ there is a point in $\Gamma(\mathbf{z}_0) \cap N(\mathbf{z}_1)$ lying in $\mathbb{C}_{\mathbf{x}}$. Consequently, we may apply the residue calculus to partially evaluate the right hand side of (5) to obtain an integral over the manifold $N(\mathbf{z}_1) \cap M$.

It is also straightforward to deduce, in the above notation,

PROPOSITION 1. *The set $\{\mathbf{z}_1 \in M^+ : \Gamma(\mathbf{z}_0) \cap N(\mathbf{z}_1) \neq \emptyset$ and $\Sigma(\mathbf{z}_0) \cap N(\mathbf{z}_1) = \emptyset\}$ is an open subset of M^+.*

We denote the open set appearing in Proposition 1 by $U(\Gamma(\mathbf{z}_0))$.

Suppose now that $\Sigma'(\mathbf{z}_0)$ is an n-cycle in $\mathbb{C}^{n+1}$ which is homologous within $\mathbb{C}^{n+1+1} \backslash N(\mathbf{z}_0)$ to $\Sigma(\mathbf{z}_0)$. So $\Sigma'(\mathbf{z}_0)$ is also a fibration of $N(\mathbf{z}_0) \cap M$ and for each $\mathbf{z} \in N(\mathbf{z}_0) \cap M$ this homological equivalence restricts to a homological equivalent between the fibres of $\Sigma'(\mathbf{z}_0)$ and $\Sigma(\mathbf{z}_0)$ within $\mathbb{C}_{\mathbf{x}} \backslash \{\mathbf{x}, \mathbf{y}\}$, where $\mathbf{y}$ is the antipodal point of $\mathbf{x}$ in $N(\mathbf{z}_0) \cap M$. Suppose that this homological equivalent also holds within $M^+ \backslash N(\mathbf{z}_0)$. Then, using the same notation as in Theorem 1 we have

THEOREM 2. *Suppose that $\mathbf{z}_0 \in M^+ \backslash \overset{\circ}{M}$ and $\Gamma(\mathbf{z}_0)$ is the real $(n+1)$-dimensional manifold lying in $\mathbb{C}^{n+1}$, with boundary $\Sigma'(\mathbf{z}_0)$ and for each $\mathbf{x} \in M \cap N(\mathbf{z}_0)$ the set $\Gamma'(\mathbf{z}_0) \cap \mathbb{C}_{\mathbf{x}}$ is the union of the closure of two domains in $\mathbb{C}_{\mathbf{x}}$. Then for each $\mathbf{z}_1 \in M^+$ for which $\Gamma'(\mathbf{z}_0) \cap N(\mathbf{z}_1) \neq \emptyset$ and $\Sigma'(\mathbf{z}_0) \cap N(\mathbf{z}_1) = \emptyset$ we have*

$$f^+(\mathbf{z}_1) = \frac{1}{\omega_n} \int_{\Sigma'(\mathbf{z}_0)} G^+(\mathbf{z} - \mathbf{z}_1) D\mathbf{z} f^+(\mathbf{z}). \tag{6}$$

Again we have that the set $\{\mathbf{z}_1 \in M^+ : \Gamma'(\mathbf{z}_0) \cap N(\mathbf{z}_1) \neq \emptyset$ and $\Sigma'(\mathbf{z}_0) \cap N(\mathbf{z}_0) \cap N(\mathbf{z}_1) = \emptyset\}$ is an open subset of M^+. We denote this open subset by $U(\Gamma'(\mathbf{z}_0))$.

The integrals (5) and (6) only depend on the values of f^+ on a complex, n-dimensional subspace of $\mathbb{C}^{n+1}$. Consequently, we have the following constructions:

PROPOSITION 2. *Suppose that* U' *is a domain in* $\mathbb{C}^n$, *the complex subspace of* $\mathbb{C}^{n+1}$ *spanned by the vectors* $e_2, \dots, e_{n+1}$, *and* $g\colon U' \to A_{n+1}(\mathbb{C})$ *is a holomorphic function. Suppose also that* $\mathbf{z}_0 \in \mathbb{C}^{n+1}\backslash\mathbb{C}^n$ *with* $\Sigma(\mathbf{z}_0) \subseteq U'$. *Then the integral*

$$\frac{1}{\omega_n}\int_{\Sigma(\mathbf{z}_0)} G^+(\mathbf{z}-\mathbf{z}_1)D\mathbf{z}g(\mathbf{z})$$

defines a complex left regular function $g^{\#}_{\mathbf{z}_0}(\mathbf{z}_1)$ *on* $U(\Gamma(\mathbf{z}_0))$. *Furthermore, for each* $\Sigma'(\mathbf{z}_0) \subseteq U'$ *we have that the integral*

$$\frac{1}{\omega_n}\int_{\Sigma(\mathbf{z}_0)} G^+(\mathbf{z}-\mathbf{z}_0)D\mathbf{z}g(\mathbf{z})$$

defines a complex left regular function $g^{\#1}_{\mathbf{z}_0}(\mathbf{z}_0)$ *on* $U(\Gamma'(\mathbf{z}_0))$, *with* $g^{\#1}_{\mathbf{z}_0}(\mathbf{z}_1) = g^{\#}_{\mathbf{z}_0}(\mathbf{z}_1)$ *for each* $\mathbf{z}_1 \in U(\Gamma(\mathbf{z}_0)) \cap U(\Gamma'(\mathbf{z}_0))$.

Using the same notation as in from the Cauchy Riemann equations we have Proposition 2,

PROPOSITION 3. *Suppose that* $\mathbf{z}_0, \mathbf{z}_2 \in \mathbb{C}^{n+1}\backslash C^n$, *with* $\Sigma'(\mathbf{z}_0)$ *and* $\Sigma'(\mathbf{z}_2) \subseteq U'$. *Then we have for each* $\mathbf{z}_1 \in U(\Gamma'(\mathbf{z}_0)) \cap U(\Gamma'(\mathbf{z}_2))$

$$\frac{1}{\omega_n}\int_{\Sigma'(\mathbf{z}_0)} G^+(\mathbf{z}_0-\mathbf{z}_1)D\mathbf{z}g(\mathbf{z}) = \frac{1}{\omega_n}\int_{\Sigma'(\mathbf{z}_2)} G^+(\mathbf{z}-\mathbf{z}_1)D\mathbf{z}g(\mathbf{z}).$$

As a consequence of Proposition 3 we have constructed a complex left regular function

$$g^{\#}\colon U(\Gamma'(\mathbf{z}_0)) \cup U(\Gamma'(\mathbf{z}_2)) \to A_{n+1}(\mathbb{C}),$$

with

$$g^{\#}|U(\Gamma'(\mathbf{z}_0)) = g^{\#}_{\mathbf{z}_0}, \quad \text{and} \quad g^{\#}|U(\Gamma'(\mathbf{z}_2)) = g^{\#}_{\mathbf{z}_2}.$$

PROPOSITION 4. *Suppose that* $\lambda\colon [0, 1) \to \mathbb{C}^{n+1}\backslash\mathbb{C}^n$ *with* $\lim_{t\to 1}\lambda(t) = \mathbf{z}(1) \in U'$ *and* $\Sigma'(\lambda(t)) \subseteq U'$ *for each* $t \in [0, 1)$. *Then*

$$\lim_{t\to 1} g^{\#}(\lambda(t)) = g(\mathbf{z}(1)).$$

OUTLINE OF PROOF. From the Huygens principle we have

$$\lim_{t\to 1}\frac{1}{\omega_n}\int_{\Sigma(\lambda(t))} G^+(\mathbf{z}-\lambda(t))D\mathbf{z}(g(\mathbf{z}) - g(\mathbf{z}(1))) = 0.$$

COROLLARY. *The function* $g^{\#}(z)$ *is the complex left regular Cauchy Kowalewski extension of* g.

It is easy to adapt Propositions 2, 3, and 4, and the corollary to Proposition 4 to obtain

THEOREM 3. *Suppose that* $\mathscr{U} \subseteq \mathbb{C}^n$ *is a domain of holomorphy, with* $n+1 = 2k$. *Then each holomorphic function* $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ *has a unique continuation to a complex left regular function* $g^{\#}: H(\mathscr{U}) \to A_{n+1}(\mathbb{C})$, *where* $H(\mathscr{U}) = \{\mathbf{z} \in \mathbb{C}^{n+1}\backslash\mathbb{C}^n : \Sigma''(\mathbf{z}) \subseteq \mathscr{U}$ *where* $\Sigma''(\mathbf{z})$ *is an* S^1 *fibration of* $N(\mathbf{z}) \cap (c(\mathbf{z})\mathbb{R}^n + \mathbf{z}_1(\mathbf{z}))$, *with* $c(\mathbf{z}) \in S^1 \subset \mathbb{C}$, *and* $\mathbf{z}_1(z) \in \mathscr{U}\} \cup \mathscr{U}$.

We denote the $A_{n+1}(\mathbb{C})$ module of $A_{n+1}(\mathbb{C})$ valued holomorphic functions defined on $\mathscr{U}$ by $\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C}))$. By regarding this module strictly as a right module we have from Theorem 3

PROPOSITION 5. *The right* $A_{n+1}(\mathbb{C})$ *modules*

$$\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C})) \quad \textit{and} \quad M_l(H(\mathscr{U}), A_{n+1}(\mathbb{C}))$$

are isomorphic.

We may extend Proposition 5 as follows:

THEOREM 4. *The right* $A_{n+1}(\mathbb{C})$ *modules*

$$\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C})) \quad \textit{and} \quad M_l(H(\mathscr{U}), A_{n+1}(\mathbb{C}))$$

are topologically isomorphic Fréchet modules.

PROOF. for the domain of holomorphy $\mathscr{U}$ consider a sequence $\{U_i\}_{i=1}^{\infty}$ of subdomains, with $U_i \subseteq U_{i+1}$ and $\bigcup_{i=1}^{\infty} U_i = \mathscr{U}$. Then we have that $H(U_i) \subseteq H(\mathscr{U})$, $\overline{H(U_i)} \subseteq H(U_{i+1})$ and $\bigcup_{i=1}^{\infty} H(U_i) = H(\mathscr{U})$. Also $U_i \subsetneq H(U_i)$.

For each $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ we define $p_i(g) = \sup_{\mathbf{z} \in U_i} \|g(\mathbf{z})\|$. The family of functions $\{p_i: \mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C})) \to \mathbb{R}^+ \cup \{0\}\}_{i=1}^{\infty}$ defines a system of norms on $\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C}))$ which endows $\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C}))$ with a Fréchet topology. We denote this Fréchet module by $(\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C})), P)$.

For each $g^{\#}: H(\mathscr{U}) \to A_{n+1}(\mathbb{C})$ we define $q_i(g^{\#}) = \sup_{\mathbf{z} \in H(U_i)} \|g^{\#}(\mathbf{z})\|$. The family of functions $\{q_i: M_l(H(\mathscr{U}), A_{n+1}(\mathbb{C})) \to \mathbb{R}^+ \cup \{0\}\}_{i=1}^{\infty}$ defines a system of norms on $M_l(H(\mathscr{U}), A_{n+1}(\mathbb{C}))$ which endows this module with a Fréchet topology. We denote the Fréchet module by

$$(M_l(H(\mathscr{U}), A_{n+1}(\mathbb{C})), Q).$$

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For each $g^{\#}(H(\mathscr{U}), A_{n+1}(\mathbb{C}))$ we have that $p_i(g^{\#}) \leq q_i(g^{\#})$. By placing $\mathscr{U} = \bigcup_{\mathbf{z}\in H(\mathscr{U})\backslash\mathscr{U}} \Sigma''(\mathbf{z})$ we have from (3) that for each i there is a positive constant C_i such that $q_{i+1}(g\#) \leq C_i p_i(g^{\#})$. Consequently, the two Fréchet modules $(\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C})), P)$ and $(M_l(H(\mathscr{U}), A_{n+1}(\mathbb{C})), Q)$ are topology isomorphic.

From the above proof we may straightforwardly obtain the following approximation theorem:

THEOREM 5. *Suppose that $\mathscr{U}'$, is a subdomain of $\mathscr{U}$, with $\overline{\mathscr{U}'} \subseteq \mathscr{U}$ and $g\colon H(\mathscr{U}') \to A_{n+1}(\mathbb{C})$, $h\colon H(\mathscr{U}) \to A_{n+1}(\mathbb{C})$ are complex left regular functions with*

$$\sup_{\mathbf{z}\in\mathscr{U}'} \|g(\mathbf{z}) - h(\mathbf{z})\| < \varepsilon$$

for some $\varepsilon \in \mathbb{R}^+$. Then there is a constant $C(\mathscr{U}') \in \mathbb{R}^+$ with

$$\sup_{\mathbf{z}\in H(\mathscr{U}')} \|g(\mathbf{z}) - h(\mathbf{z})\| \leq C(\mathscr{U}')\varepsilon.$$

We now turn to the case where $n+1 = 2k+1$. Suppose now that $\mathscr{U}$ is a domain of holomorphy in $\mathbb{C}^n$. Then the set $\{\mathbf{z} \in \mathbb{C}^{n+1}\backslash\mathbb{C}^n : \Sigma''(\mathbf{z}) \subseteq \mathscr{U}$ and there is an n-dimensional disc $D(\mathbf{z})$ with boundary in $\Sigma''(\mathbf{z})$, and $D(\mathbf{z}) \subseteq \mathscr{U}\}\cup\mathscr{U}$ is denoted by $NH(\mathscr{U})$. Clearly $NH(\mathscr{U}) \subseteq H(\mathscr{U})$. It is also straightforward to deduce

PROPOSITION 6. *The set $NH(\mathscr{U})$ is an open subset of $H(\mathscr{U})$.*

Now using Proposition 2 we have

THEOREM 6. *Suppose that $\mathscr{U} \subseteq \mathbb{C}^n$ is a domain of holomorphy, with $n = 2k$. Then each holomorphic function $g\colon \mathscr{U} \to A_{n+1}(\mathbb{C})$ has a unique continuation to a complex left regular function $g^{\#}\colon NH(\mathscr{U}) \to A_{n+1}(\mathbb{C})$.*

On restricting the system of norms $\{q_i\}_{i=1}^{\infty}$, appearing in the proof of Theorem 4, to act on the open sets $NH(U_i)$ we may deduce

THEOREM 7. *When $n = 2k$ the right $A_{n+1}(\mathbb{C})$ modules $\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C}))$ and $M_l(NH(\mathscr{U}), A_{n+1}(\mathbb{C}))$ are topologically isomorphic Fréchet modules.*

It is now straightforward to deduce an analogue of Theorem 5 in odd dimensions.

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From Theorems 4 and 6 we have

THEOREM 8. *Suppose that* $U = \{\mathbf{x} \in \mathbb{R}^{n+1} = x_1 e_1 + \vec{\mathbf{x}}$ *where* $-r < x_1 < r$ *with* $r \in \mathbb{R}^+$ *and* $\vec{\mathbf{x}} \in \mathbb{R}^n\}$. *Suppose that* $\mathscr{U} = \{\vec{\mathbf{z}} \in \mathbb{C}^n : \vec{\mathbf{z}} = \vec{\mathbf{x}}' + i\vec{\mathbf{y}}$ *with* $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^n$ *and* $|\vec{\mathbf{y}}| < r\}$. *Then* $H(\mathscr{U}) = U^+$, *the component of* $\mathbb{C}^{n+1} \setminus \bigcup_{\mathbf{x} \in \partial(U)} N(\mathbf{x})$ *containing* U, *and the modules* $M_l(U, A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

Theorem 8 extends a result given in [14].
We also have, from Theorems 4 and 6,

PROPOSITION 7. *Suppose that* $\mathscr{U}$ *is the Lie ball*

$$L_n = \{\mathbf{z} \in \mathbb{C}^{n+1} : |z_1|^2 + \cdots + |z_{n+1}|^2 + [(|z_1|^2 + \cdots + |z_{n+1}|^2)^2 - |z_1^2 + \cdots + z_{n+1}^2|^2]^{1/2} < 1\}.$$

Then $H(\mathscr{U})$ *is the Lie ball in* $\mathbb{C}^{n+1}$, *and the modules* $M_l(\{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| < 1\}, A_n(\mathbb{C}))$ *and* $\mathscr{O}(L_n, A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

Theorem 7 and Proposition 7 are special cases of the following result:

THEOREM 9. *Suppose that the domain* $U \subseteq \mathbb{R}^{n+1}$ *is a normal neighbourhood of an open set* $W \subseteq \mathbb{R}^n$ *and for each* $x_1 e_1 + \vec{\mathbf{x}} \in \mathscr{U}$ *we have that* $\lambda x_1 e_1 + \vec{\mathbf{x}} \in \mathscr{U}$ *for* $\lambda \in (0, 1]$ *and* $\vec{\mathbf{x}} \in W$. *Suppose also that* U^+ *is the component of* $\mathbb{C}^{n+1} \setminus \bigcup_{\mathbf{x} \in \overline{U} \setminus U} N(\mathbf{x})$ *containing* U. *Then each component of* $U^+ \cap \mathbb{C}^n$ *is a domain of holomorphy, and the Fréchet modules* $M_l(U, A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(U^+ \cap \mathbb{C}^n, A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

When $n + 1 = 2k$ we have the following improvement to Theorem 9:

THEOREM 10. *Suppose that the domain* $U \subseteq \mathbb{R}^{n+1}$ *with* $n + 1 = 2k$. *Suppose also that* U^+ *is the component of* $\mathbb{C}^{n+1} \setminus \bigcup_{\mathbf{x} \in \overline{U} \setminus U} N(\mathbf{x})$ *containing* U, *and that for each* $x_1 e_1 + \vec{\mathbf{x}} \in U$ *we have that*

$$N(x_1 e_1 + \vec{\mathbf{x}}) \cap ((-\vec{\mathbf{x}}) + i\mathbb{R}^n) \subseteq U^+.$$

Then each component of $U^+ \cap \mathbb{C}^n$ *is a domain of holomorphy, and the Fréchet modules* $M_l(U, A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(U^+ \cap \mathbb{C}^n, A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

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4. Domains of holomorphy in $S^n_{\mathbb{C}}$ and $\mathbb{C}^n$

Following [10] we have that for each holomorphic function $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ the holomorphic function

$$g^{\#\#}(z_1 e_1 + \vec{\mathbf{z}}) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} z_1^{2m} \Delta_{n,\mathbb{C}}^m g(\vec{\mathbf{z}})$$

is a well defined holomorphic function on some neighbourhood in $\mathbb{C}^{n+1}$ of $\mathscr{U}$, where $\Delta_{n,\mathbb{C}} = \sum_{j=2}^{n+1} \partial^2/\partial z_j^2$.

Moreover, $\sum_{k=1}^{n} (\partial^2/\partial z_k^2) g^{\#\#}(\vec{\mathbf{z}}) = 0$, and $g^{\#\#}|_U = g$.

Definition 4. Suppose that $\mathscr{U} \subseteq \mathbb{C}^n$, is a domain of holomorphy and $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ is a holomorphic function. A holomorphic function $H_g: \mathscr{U}_g \to A_n(\mathbb{C})$, where $\mathscr{U}_g \subseteq \mathbb{C}^{n+1}$ and $\mathscr{U} \subset \mathscr{U}_g$, is called a *harmonic Cauchy Kowalewski extensions of* g if $H_g|_{\mathscr{U}} = g$ and $\Delta_{n+1,\mathbb{C}} H_g(\mathbf{z}) = 0$.

Whereas, the complex left regular (and right regular) Cauchy Kowalewski extension of g is unique it is not longer the case that each $g \in \mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C}))$ has a unique harmonic Cauchy Kowalewski extension, for example, the function $\frac{1}{2} g^{\#\#}(\mathbf{z}) + \frac{1}{2} g^{\#}(\mathbf{z})$ is a harmonic Cauchy Kowalewski extension of g, but $\frac{1}{2} g^{\#\#}(\mathbf{z}) + \frac{1}{2} g^{\#}(\mathbf{z}) \neq g^{\#\#}(\mathbf{z})$ for all $\mathbf{z}$.

For each harmonic function $h: U \to A_{n+1}(\mathbb{C})$ we have Green's formula

$$h(\mathbf{x}_0) = \frac{1}{\omega_n} \int_{\partial M} \left(G(\mathbf{x} - \mathbf{x}_0) D\mathbf{x} h(\mathbf{x}) + H(\mathbf{x} - \mathbf{z}_0) D\mathbf{x} \sum_{j=1}^{n+1} e_j \frac{\partial h}{\partial x_j}(\mathbf{x}) \right),$$

where

$$H(\mathbf{x}) = \frac{1}{(n-1)} \frac{1}{|\mathbf{x}|^{n-1}},$$

M is a real $(n+1)$-dimensional compact submanifold of $U \subseteq \mathbb{R}^{n+1}$ and $\mathbf{x}_0 \in \mathring{M}$.

When $n + 1 = 2k$ the function h has a unique holomorphic continuation [2] to a complex harmonic function

$$(7)\quad h^+(\mathbf{z}_0) = \frac{1}{\omega_n} \int_{\partial M} \left(G^+(\mathbf{x} - \mathbf{z}_0) D\mathbf{x} h(\mathbf{x}) + H^+(\mathbf{x} - \mathbf{z}_0) D\mathbf{x} \sum_{j=1}^{n+1} e_j \frac{\partial h}{\partial x_j}(\mathbf{x}) \right),$$

where $\mathbf{z}_0 \in M^+$ and, when $n + 1 = 2k + 1$, h has a unique holomorphic continuation to NM^+.

When $n+1=2k$ the integral (7) is equal to

$$\frac{1}{\omega_n}\int_{\sigma(\mathbf{z}_0)} G^+(\mathbf{z}-\mathbf{z}_1)D\mathbf{z}h^+(\mathbf{z}) + H^+(\mathbf{z}-\mathbf{z}_1)D\mathbf{z}\sum_{j=1}^{n+1} e_j\frac{\partial h^+}{\partial z_j}(\mathbf{z})$$

for $\mathbf{z}_1 \in U(\Gamma(\mathbf{z}_0))$, for some $\mathbf{z}_0 \in M^+$, where

$$H^+(\mathbf{z}) = (n-1)^{-1}(z_1^2 + \cdots + z_{n+1}^2)^{-(n+1)/2}.$$

Using the complex harmonic Cauchy Kowalewski extension of $g\colon \mathscr{U} \to A_{n+1}(\mathbb{C})$ described in the first sentence of this section we may deduce

Proposition 8. *Suppose that $g\colon \mathscr{U} \to A_{n+1}(\mathbb{C})$ is a holomorphic function, and $n+1=2k$. Suppose that $\mathbf{z}_0 \in \mathbb{C}^{n+1}\backslash\mathbb{C}^n$ with $\Sigma(\mathbf{z}_0) \subseteq \mathscr{U}$. Then*

$$\frac{1}{\omega_n}\int_{\Sigma(\mathbf{z}_0)} \left(G^+(\mathbf{z}-\mathbf{z}_1)D\mathbf{z}g(\mathbf{z}) + H^+(\mathbf{z}-\mathbf{z}_1)D\mathbf{z}\sum_{j=1}^{n+1} e_j\frac{\partial}{\partial_{\mathbf{z}_j}} g^{\#\#}(\mathbf{z})\right) \tag{8}$$

is a complex harmonic function.

By similar arguments to those used in the previous section we have that the function given by formula (8) is a continuation of the function $g^{\#\#}$. Consequently, we have

Theorem 11. *Suppose that $\mathscr{U} \subseteq \mathbb{C}^n$ is a domain of holomorphy, with $n+1=2k$. Then each holomorphic function $g\colon \mathscr{U} \to A_{n+1}(\mathbb{C})$ has a continuation to a complex harmonic function*

$$g^{\#\#}\colon H(\mathscr{U}) \to A_{n+1}(\mathbb{C}).$$

A similar formula to (4) may be deduced for complex harmonic functions in odd dimensions, so that we have the following theorem.

Theorem 11′. *Suppose that $\mathscr{U} \subseteq \mathbb{C}^n$ is a domain of holomorphy, with $n+1=2k+1$. Then each holomorphic function $g\colon \mathscr{U} \to A_{n+1}(\mathbb{C})$ has a continuation to a complex harmonic function $g^{\#\#}\colon NH(\mathscr{U}) \to A_{n+1}(\mathbb{C})$.*

The open set $H(\mathscr{U})$ is called a cell of harmonicity.

Proposition 9 [9]. *Suppose $(a\mathbf{z}+b)(c\mathbf{z}+d)^{-1}$ is a Möbius transformation in $\mathbb{C}^{n+1}$, with $n+1=2k$, and $f((a\mathbf{z}+b)(c\mathbf{z}+d)^{-1})$ is a complex left regular function with respect to the variable $\mathbf{w} = (a\mathbf{z}+b)(c\mathbf{z}_d)^{-1}$, while $g(\mathbf{w})$ is a complex harmonic function. Then $J_1(c\mathbf{z}+d)f(\mathbf{w})$ is a complex left regular*

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function with respect to the variable $\mathbf{z}$ *and* $J_2(c\mathbf{z}+d)g(\mathbf{w})$ *is complex harmonic with respect to the variable* $\mathbf{z}$, *where*

$$J_1(c\mathbf{z}+d) = (c\mathbf{z}\tilde{+}d)\{(c\mathbf{z}+d)(c\mathbf{z}\tilde{+}d)\}^{-k}$$

and

$$J_2(c\mathbf{z}+d) = \{(c\mathbf{z}+d)(c\mathbf{z}\tilde{+}d)\}^{-k}.$$

From [10] it is straightforward to observe that whenever $J_1(C\mathbf{z}+d)$ is defined it is invertible in the algebra $A_{n+1}(\mathbb{C})$. Clearly $J_2(c\mathbf{z}+d)$ is invertible, when defined. Using these facts we can use the Cayley map, the transform $(\mathbf{z}+e_1)(-e_1\mathbf{z}-1)^{-1}$, and the complex left regular, and complex harmonic Cauchy Kowalewski extensions from domains in $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ to deduce

PROPOSITION 10 [10]. *Suppose that* $\mathscr{U}$ *is a domain in* $S^n_{\mathbb{C}}$ *with* $n+1=2k$, *and* $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ *is a holomorphic function. Then there exists a domain* $\mathscr{U}_g$ *in* $\mathbb{C}$ *with* $U \subseteq \mathscr{U}_g$, *and there is a unique complex left regular function* $f_g: \mathscr{U}_g \to A_{n+1}(\mathbb{C})$ *and a complex harmonic function* $h_g: \mathscr{U}_g \to A_{n+1}(\mathbb{C})$ *such that*

$$f_g|_{\mathscr{U}} = h_g|_{\mathscr{U}} = g.$$

Using the Clifford matrices $(\mathbf{z}-e_1)(-e_1\mathbf{z}+1)^{-1}$ and $(\mathbf{z}+e_1)(-e_1\mathbf{z}-1)^{-1}$, and Theorem 3 we may characterize the domain $\mathscr{U}_g$. We may also characterize the domain $\mathscr{U}_g$ by more direct means. First we need

LEMMA 1. *Suppose that* $\mathbf{z} \in \mathbb{C}^{n+1} \backslash (N(\mathbf{0}) \cup S^n_{\mathbb{C}})$. *Then the set* $N(\mathbf{z}) \cap S^n_{\mathbb{C}}$ *contains a real manifold which is holomorphic to the sphere* S^{n-1}.

PROOF. As $\mathbf{z} \in \mathbb{C}^{n+1}\backslash N(\mathbf{0})$ there exists $z \in \mathbb{C}\backslash\{0\}$ and $\mathbf{w} \in S^n_{\mathbb{C}}$ such that $\mathbf{z} = z\mathbf{w}$. We may place $\mathbf{w} = i\mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$. As $\mathbf{w} \in S^{n+1}_{\mathbb{C}}$ we have that $\mathbf{xy}+\mathbf{yx} = \mathbf{0}$. If $\mathbf{y} \neq \mathbf{0}$ then we may choose $(n-1)$ vectors $\{e(\mathbf{z})_1, \dots, e(\mathbf{z})_{n-1}\}$ in $i\mathbb{R}^{n+1}$ such that

$$\mathbf{x}e(\mathbf{z})_j + e(\mathbf{z})_j\mathbf{x} = \mathbf{y}e(\mathbf{z})_j\mathbf{y} = e(\mathbf{z})_j e(\mathbf{z})_k + e(\mathbf{z})_j = \mathbf{0}$$

for $j \neq k$, and $e(\mathbf{z})_j^2 = 1$ for $j = 1, \dots, n-1$. By solving the simultaneous equations

$$(\mathbf{z}-\mathbf{z}(\mathbf{w}))^2 + \mathbf{z}(w^1)^2 + \sum_{j=1}^{n-1} \mathbf{z}_j^2 = 0,$$

$$(\mathbf{z}(w))^2 + (\mathbf{z}(\mathbf{w}^1)) + \sum_{j=1}^{n-1} \mathbf{z}_j^2 = 1,$$

where

$$\mathbf{w}^1 = \frac{|\mathbf{x}|}{|\mathbf{y}|}\mathbf{y} + i\frac{|\mathbf{y}|}{|\mathbf{x}|}\mathbf{x},$$

we have that $z(\mathbf{w}) = (z^2+1)/2$. On placing $1 - \mathbf{z}(\mathbf{w})^2 - (\mathbf{z} - \mathbf{z}(\mathbf{w}))^2/2 = y^2$ we have

$$\mathbf{z}(\mathbf{w})\mathbf{w} + y\mathbf{v} \in N(\mathbf{z}) \cap S^n_{\mathbb{C}},$$

for each $\mathbf{v}$ in the real space spanned by $\mathbf{w}'$, $e(\mathbf{z})_1, \dots, e(\mathbf{z})_{n-1}$, with $\mathbf{v}^2 = 1$. A similar result can be obtained by similar means when $\mathbf{y} = \mathbf{0}$.

For $\mathbf{z} \in \mathbb{C}^{n+1} \backslash (N(\mathbf{0}) \cup S^n_{\mathbb{C}})$ we denote the $(n-1)$-dimensional manifold constructed in Lemma 1 by $\Psi(\mathbf{z})$. As $\Psi(\mathbf{z})$ is a real $(n-1)$-dimensional submanifold of the complex, n-dimensional manifold $S^n_{\mathbb{C}}$ we have that for each $\mathbf{z}_1 \in \Psi(\mathbf{z})$ there is a complex 1-dimensional submanifold $U_{\mathbf{z}_1}$ of $S^n_{\mathbb{C}}$, containing $\mathbf{z}_1$, with

$$U_{\mathbf{z}_1} \cap N(\mathbf{z}) = \{\mathbf{z}_1\}.$$

Consequently, we may construct a real, n-dimensional manifold $\Theta(\mathbf{z})$ lying in $S^n_{\mathbb{C}}$ which is an S^1 fibration of $\Psi(\mathbf{z})$. Moreover, for each $\mathbf{z}_1 \in \Psi(\mathbf{z})$ the S^1 fibre lies in $U_{\mathbf{z}_1}$, and in contractible within $U_{\mathbf{z}_1}$ to $\mathbf{z}_1$.

The manifold $\Theta(\mathbf{z})$ is the boundary of a real n-dimensional manifold $\chi(\mathbf{z})$, with $\chi(\mathbf{z}) \cap U_{\mathbf{z}_1}$ homeomorphic to a disc.

By perturbing the point $\mathbf{z}$ it is now straightforward to show

LEMMA 2. *There is an open subset $U(\mathbf{z})$ of $\mathbb{C}^{n+1} \backslash N(\mathbf{0})$ such that for each $\mathbf{z}^1 \in U(\mathbf{z})$ the set $N(\mathbf{z}^1) \cap \mathring{\chi}(\mathbf{z})$ is a manifold homeomorphic to S^n and $N(\mathbf{z}^1) \cap \mathring{\chi}(\mathbf{z}) \cap U_{\mathbf{z}_1}$ consists of precisely one point.*

From Lemma 2 it is now possible to repeat arguments used in the previous section of this paper to show

THEOREM 12. *Suppose that $\mathscr{U} \subseteq S^n_{\mathbb{C}}$ is a domain of holomorphy, with $n+1 = 2k$. Then each holomorphic function $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ has a unique continuation to a complex left regular function*

$$g^{\#}: H(\mathscr{U}) \to A_{n+1}(\mathbb{C}),$$

where $H(\mathscr{U}) = \{\mathbf{z} \in \mathbb{C}^{n+1} \backslash (N(\mathbf{0}) \cup S^n_{\mathbb{C}}) : \Psi(\mathbf{z}) \subset \mathscr{U}\} \cup \mathscr{U}$.

Now by analogous arguments to those used to establish Theorem 4 we have, for $\mathscr{U} \subseteq S^n_{\mathbb{C}}$, THEOREM 13. *The right* $A_{n+1}(\mathbb{C})$ *modules*

$$\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C})) \quad and \quad M_l(H(\mathscr{U}), A_{n+1}(\mathbb{C}))$$

are topologically isomorphic Fréchet modules.

We also have, from Proposition 10 and Theorem 3,

PROPOSITION 11. *For* $\mathscr{U} \subseteq S^n_{\mathbb{C}}$, *a domain of holomorphy,* $n+1 = 2k$, *and* $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ *a holomorphic function, there exists a complex harmonic Cauchy Kowalewski extension of* g *to* $H(\mathscr{U})$.

In [11] we illustrate that for each complex left regular function $f: U_{\mathbb{C}} \subseteq \mathbb{C}^{n+1} \to A_{n+1}(\mathbb{C})$, the function $\mathbf{z}^k f(\mathbf{z})$ is annihilated by the operation D^{k+1}, where $D = \sum_{j=1}^{n+1} e_j \partial/\partial z_j$. Also, in [11] we illustrate that for each complex harmonic function $h: U_{\mathbb{C}} \to A_{n+1}(\mathbb{C})$ we have that $\mathbf{z}^k h(\mathbf{z})$ is annihilated by the operator D^{k+2}. From this we have, from Theorem 12 and Proposition 11,

PROPOSITION 12. *For* $\mathscr{U} \subseteq S^n_{\mathbb{C}}$ *a domain of holomorphy,* $n+1 = 2k$, *and* $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ *a holomorphic function, there exists for each* $q \in N^+$ *a holomorphic function*

$$g_q: H(\mathscr{U}) \to A_{n+1}(\mathbb{C})$$

such that $D^q g_q(\mathbf{z}) = 0$ *and* $g_q|_U = g$.

DEFINITION 5. A holomorphic function $g: U_{\mathbb{C}} \to A_{n+1}(\mathbb{C})$ is called complex **q**-left regular if $D^q g(\mathbf{z}) = 0$.

In [11] we show that if $f((a\mathbf{z}+b)(c\mathbf{z}+d)^{-1})$ is a complex q-left regular function with respect to the variable $(a\mathbf{z}+b)(c\mathbf{z}+d)^{-1}$, with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a Clifford matrix, then $J_q(c\mathbf{z}+d)f((a\mathbf{z}+b)(c\mathbf{z}+d)^{-1})$ is a complex k-left regular function with respect to the variable $\mathbf{z}$, where

$$J_q(c\mathbf{z}+d) = \begin{cases} (c\tilde{\mathbf{z}+d})((c\mathbf{z}+d)(c\tilde{\mathbf{z}+d}))^{-n/2+(q-1)}/2 & \text{if } q \text{ is odd}, \\ ((c\mathbf{z}+d)(c\tilde{\mathbf{z}+d}))^{-n/2+q/2} & \text{if } q \text{ is even}. \end{cases}$$

It now follows from Proposition 12 that we have the following theorem.

THEOREM 14. *Suppose that* $\mathscr{U}$ *is a domain of holomorphy in* $\mathbb{C}^n$, *with* $n = 2k-1$, *and* $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ *is a holomorphic function. Then there is a complex* q*-left regular function*

$$g_q: H(\mathscr{U}) \to A_{n+1}(\mathbb{C})$$

such that $g_q|_{\mathscr{U}} = g$.

The construction given in Lemma 1 does not depend on n being odd, nor does the statement immediately preceding Theorem 14 depend on n being odd. Consequently, we may for each domain $\mathscr{U} \subseteq S^n_{\mathbb{C}}$ construct the domain $NH(\mathscr{U})$ in $\mathbb{C}^{n+1}\backslash N(\mathbf{0})$, and we have

THEOREM 15. *For* $\mathscr{U} \subseteq S^n_{\mathbb{C}}$, *a domain of holomorphy,* $n = 2k$, *and* $g: \mathscr{U} \to A_{n+1}(\mathbb{C})$ *a holomorphic function, there exists for each* $q \in N^+$ *a complex* q*-left regular function* $g_q: NH(\mathscr{U}) \to A_{n+1}(\mathbb{C})$ *such that* $g_q|_{\mathscr{U}} = g$. *Moreover, for* $\mathscr{U}^1 \subseteq \mathbb{C}^n$, *and* $g^1: \mathscr{U}^1 \to A_{n+1}(\mathbb{C})$ *a holomorphic function, there exists a complex* q*-left regular function* $g^1_q: NH(\mathscr{U}^1) \to A_{n+1}(\mathbb{C})$ *such that* $g^1_q|_{\mathscr{U}^1} = g^1$.

As a consequence of Theorems 14 and 15 we have that the functions

$$\operatorname{Exp} \overrightarrow{\mathbf{z}} = \sum_{j=0}^{\infty} \frac{1}{j!}(z_2 e_2 + \cdots + z_{n+1} e_n)^j,$$

$$\operatorname{Exp}(\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{z}}) = \sum_{j=0}^{\infty} \frac{1}{j!}(x_2 z_2 e_2 + \cdots + x_{n+1} z_{n+1} e_{n+1})^j$$

and $e^{i(x_2 z_2 + \cdots + x_{n+1} z_{n+1})}$ all have complex k-left regular Cauchy Kowalewski extensions to $\mathbb{C}^{n+1}$. It follows that many of the existing results on transformation analysis associated to the monogenic Cauchy Kowalewski extensions of these function [3, 13, 14] also hold for these other more general extensions.

5. Holomorphic extensions of analytic functions on n-dimensional manifolds in $\mathbb{R}^{n+1}$

In this section we shall assume that M is a real analytic n-dimensional manifold lying in $\mathbb{R}^{n+1}$. As M is analytic there exists a holomorphic, complex n-dimensional manifold $\mathbb{C}M \subseteq \mathbb{C}^{n+1}$, with $M \subseteq \mathbb{C}M$.

Suppose that M is such a manifold. Then for each $\mathbf{x} \in M$ we may choose a unit normal vector $n(\mathbf{x})$ to M at $\mathbf{x}$. We may also, for each $\mathbf{x} \in M$, choose a real n-dimensional analytic submanifold $M(\mathbf{x})$ of $\mathbb{C}M$ with $\mathbf{x} \in M(\mathbf{x})$ and with tangent space $iTM_{\mathbf{x}}$ at $\mathbf{x}$, where $TM_{\mathbf{x}}$ is the tangent space of M at $\mathbf{x}$. We now have that for each $\lambda \in \mathbb{R}\backslash\{0\}$ the set $N(\mathbf{x}+\lambda n(\mathbf{x})) \cap iTM_{\mathbf{x}}$ is an $(n-1)$-dimensional sphere. It is now straightforward to adopt technical arguments given in [12] to establish the following.

LEMMA 3. *For each* $\mathbf{x} \in M$ *there exists* $\lambda(\mathbf{x})$ *and* $\Theta(\mathbf{x}) \in \mathbb{R}^+$ *such that for each* $\lambda \in (0, \lambda(\mathbf{x})) \cup (-\Theta(\mathbf{x}), 0)$ *we have that the set* $N(\mathbf{x}+\lambda n(\mathbf{x})) \cap M(\mathbf{x})$ *is a smooth manifold homeomorphic to* S^{n-1}.

We denote this manifold by $X(\mathbf{y})$ where $\mathbf{y} = \mathbf{x} + \lambda n(\mathbf{x})$; $X(\mathbf{y})$ is a real submanifold of $\mathbb{C}M$. Consequently, for each $\mathbf{z} \in X(\mathbf{y})$ there is a one dimensional complex submanifold $U_{\mathbf{z}}$ of $\mathbb{C}M$ which contains $\mathbf{z}$ and $(TU_{\mathbf{z}})_{\mathbf{z}}$ is the complex space in $T\mathbb{C}M_{\mathbf{z}}$ which does not contain any vector from $\mathbb{C}TM(x)_{\mathbf{z}}$, the complexification of the tangent space $TM(\mathbf{x})_{\mathbf{z}}$. We may now choose a fibration $Y(\mathbf{y})$ of $X(\mathbf{y})$ with fibre $U_{\mathbf{z}}$ at each $\mathbf{z} \in X(\mathbf{y})$. We may also choose as S^1 fibration $Z(\mathbf{y})$ of $X(\mathbf{y})$ where each fibre is the boundary of $U_{\mathbf{z}}$. Moreover, we may choose each fibre $U_{\mathbf{z}}$ so that it is contractible within itself to $\mathbf{z}$. We now have, by similar arguments to those outlined in the previous two sections,

PROPOSITION 13. *Within* $\mathbb{C}^{n+1}$ *there is an open set* $W(\mathbf{y})$ *containing* $\mathbf{y}$ *and such that for each holomorphic function* $g: \mathscr{U}_M \to A_{n+1}(\mathbb{C})$, *with* $\mathscr{U}_M$ *a complex, open submanifold of* $\mathbb{C}M$ *with* $Y(\mathbf{y}) \subseteq \mathscr{U}_M$, *we have*

$$\frac{1}{\omega_n} \int_{Z(y)} G^+(\mathbf{x} - \mathbf{z}_0) D\mathbf{z} g(\mathbf{z})$$

is a complex left regular function on $W(\mathbf{y})$, *when* $n+1 = 2k$.

It is also straightforward to use the Cauchy Riemann equations to adapt Proposition 3 and 4 to obtain, from Proposition 13,

THEOREM 16. *Suppose that* $h: M \to A_{n+1}(\mathbb{C})$ *is an analytic function with* $n+1 = 2k$. *Then there exists a neighbourhood* $M(h)$ *of* M *in* $\mathbb{R}^{n+1}$ *and a left regular function* $f_h: M(h) \to A_{n+1}(\mathbb{C})$ *such that* $f_h|_M = h$.

Theorem 16 was established in [15] using different techniques. However, our approach gives other information. We have

THEOREM 17. *Suppose that* $h: M \to A_{n+1}(\mathbb{C})$ *is an analytic function with* $n+1 = 2k$. *Then* h *has a Cauchy Kowalewski extension to a left regular function* f_k *on the domain*

$\{\mathbf{y} \in \mathbb{R}^{n+1} : \mathbf{y} = \mathbf{x} + \lambda n(\mathbf{x})$, *where* $\mathbf{x} \in M$ *and* $\lambda = 0$ *or* $N(\mathbf{y}) \cap M(\mathbf{x}) = X(\mathbf{y})$ *with* $X(\mathbf{y}) \subseteq \mathscr{U}_h$, *the domain of holomorphy in* $\mathbb{C}M$ *to which* h *holomorphically extends*$\}$.

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Note that the domain in $\mathbb{R}^{n+1}$ described in Theorem 17 need not necessarily be a normal neighbourhood of M.

We denote the domain in $\mathbb{R}^{n+1}$ appearing in Theorem 17 by $V(\mathscr{U}_h)$. By similar arguments to those used in the first section of this paper we have the following generalization of Theorem 10.

THEOREM 18. *Suppose that* $n + 1 = 2k$ *and* $\mathscr{U}$ *is a domain of holomorphy lying in* $\mathbb{C}M$. *Suppose that* $\mathscr{O}(\mathscr{U}, A_{n+1}(\mathbb{C}))$ *is the right* $A_{n+1}(\mathbb{C})$ *module of holomorphic functions defined on* $\mathscr{U}$. *Then the Fréchet modules* $M_l(V(\mathscr{U}), A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(\mathscr{U}', A_{n+1}(\mathbb{C}))$ *are topologically isomorphic, where* $\mathscr{U}'$ *is the maximal subdomain of* $\mathscr{U}$ *for which* $N(\mathbf{z}) \cap \mathscr{U}'$ *contains a manifold homeomorphic to* S^{n-1} *for* $\mathbf{z} \in \mathbb{C}^{n+1} \backslash V(\mathscr{U}')$.

For $\mathscr{U}$ a domain of holomorphy lying in $\mathbb{C}M$ we denote the set

$\{\mathbf{y} \in \mathbb{R}^{n+1} : \mathbf{y} = \mathbf{x} + \lambda n(\mathbf{x})$, where $\mathbf{x} \in M$ and $\lambda = 0$ or $N(\mathbf{y}) \cap M(\mathbf{x}) = X(\mathbf{y})$, with $X(\mathbf{y}) \subset \mathscr{U}$ and $X(\mathbf{y})$ is the boundary of a real n-dimensional submanifold of $M(\mathbf{x})\}$

by $NV(\mathscr{U})$.

We now have, from similar arguments to those used in the previous two sections,

THEOREM 19. *Suppose that* $n = 2k$. *Then the Fréchet modules* $M_l(NV(\mathscr{U}'), A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(\mathscr{U}', A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

We now turn to Cauchy Kowalewski extensions, of analytic functions defined on M, which satisfy a finite iterate of the Euclidean Dirac operator. First we have

PROPOSITION 14. *Suppose that* $h: M \to (\mathbb{C})$ *is an analytic function,* $\mathbf{0} \notin M$ *and* $f_{k,h}(\mathbf{x})$ *is the left regular Cauchy Kowalewski extension of* $\mathbf{x}^{-k+1} h(\mathbf{x})$ *to* $M(h)$. *Then* $\mathbf{x}^{k-1} f_{k,h}(\mathbf{x})$ *is annihilated by the operator* $(\sum_{j=1}^{n+1} e_j \partial/\partial x_j)^k$ *and* $\mathbf{x}^{k-1} f_{k,h}(\mathbf{x})|_M = h(\mathbf{x})$, *for* $k \in N^+$.

LEMMA 4. *Suppose that* $\mathbf{0} \in M$. *Then there is an* $\mathbf{x}_0 \in \mathbb{R}^{n+1}$ *such that* $\mathbf{0} \notin M^1 = M + x_0$.

As the operator $(\sum_{j=1}^{n+1} e_j \partial/\partial z_j)^k$ is a constant coefficient operator it now follows from Proposition 14 and Lemma 4 that we have the following theorem.

THEOREM 20. *Suppose that* $h: M \to A_{n+1}(\mathbb{C})$ *is an analytic function. Then for each* $k \in N^+$ *there is a function* $h_k: M(h) \to A_{n+1}(\mathbb{C})$ *such that* $h_k|_M = h$ *and* $(\sum_{j=1}^{n+1} e_j\partial/\partial x_j)^k h_k(\mathbf{x}) = 0$.

6. Real, n-dimensional, analytic manifolds in $\mathbb{C}^{n+1}$

DEFINITION 6. A real n-dimensional, analytic manifold M lying in $\mathbb{C}^{n+1}$, with the property that for each $\mathbf{z} \in M$ we have

$$TM_{\mathbf{z}} \cap N(\mathbf{z}) = \{\mathbf{z}\},$$

is called a *restricted analytic manifold.*

Examples of such manifolds include each real n-dimensional, analytic manifold in $\mathbb{R}^{n+1}$, and the rotation of such a manifold within $\mathbb{C}^{n+1}$ via an element of the complex orthogonal group $O(\mathbb{C}^{n+1}) = \{(a_{ij}) = (a_{ij})(a_{ij})^T = (\delta_{ij})$, with $a_{ij} \in \mathbb{C}$, $1 \leq i, j \leq n\}$.

It is fairly straightforward to extend the results of the previous sections to special types of restricted analytic manifolds, and associated domains of holomorphy. First we note that for each restricted manifold M there is a complex n-dimensional manifold $\mathbb{C}M$ with $M \subseteq \mathbb{C}M$. Consequently, for each $\mathbf{z} \in M$ and each $c \in S^1 \subset \mathbb{C}$, we have that there is a real analytic manifold $M(\mathbf{z}, c) \subset \mathbb{C}M$, with $\mathbf{z} \in M(\mathbf{z}, c)$ and $TM(\mathbf{z}, c)_{\mathbf{z}} = cTM_{\mathbf{z}}$.

DEFINITION 6. Suppose M is a restricted analytic manifold, such that for each $\mathbf{z} \in M$ there is a vector $m(\mathbf{z}) \in \mathbb{C}^{n+1}$ such that

(i) the minimal real vector space in $\mathbb{C}^{n+1}$ containing $TM_{\mathbf{z}}$ and $m(\mathbf{z})$ is $(n+1)$-dimensional, and

(ii) for each vector $v(\mathbf{z})$ in this space we have that $v(\mathbf{z})^2 = 0$ if and only if $v(\mathbf{z}) = \mathbf{0}$.

Then M is called a *normally restricted manifold*.

We now have

LEMMA 5. *For each* $\lambda \in \mathbb{R}\backslash\{0\}$ *and each* $\mathbf{z} \in M$ *the set* $N(\mathbf{z} + \lambda c m(\mathbf{z})) \cap TM(\mathbf{z}, c)_{\mathbf{z}}$ *is an* $(n-1)$*-dimensional manifold homeomorphic to the sphere* S^{n-1}.

We now have the following generalization of Lemma 3.

LEMMA 6. *Suppose that* M *is a normally restricted analytic manifold. Then for each* $\mathbf{z} \in M$ *and each* $c \in S^1$ *there exists* $\lambda(\mathbf{z}, c)$, $\Theta(\mathbf{z}, c) \in \mathbb{R}^+$ *such* *that for each* $\lambda \in (0, \lambda(\mathbf{z}, c)) \cup (-\Theta(\mathbf{z}, c), 0)$ *the set* $N(\mathbf{z}+\lambda c m(\mathbf{z})) \cap M(\mathbf{z}, c)$ *is a smooth manifold homeomorphic to* S^{n-1}.

We denote this $(n-1)$-dimensional manifold by $X(\mathbf{z}^1)$ where $\mathbf{z}^1 = \mathbf{z} + \lambda c m(\mathbf{z})$. Then $X(\mathbf{z}^1)$ is a real submanifold of $\mathbb{C}M$. Consequently, for each $\mathbf{z}'' \in X(\mathbf{z}^1)$ there is a one-dimensional complex submanifold $U_{\mathbf{z}''}$ of $\mathbb{C}M$ which contains $\mathbf{z}''$ and $(TU_{\mathbf{z}''})_{\mathbf{z}''}$ is the complex subspace of $T\mathbb{C}M_{\mathbf{z}''}$ which does not contain any vector from $\mathbb{C}TM(\mathbf{z}, c)_{\mathbf{z}''}$. We may also choose an S^1 fibration $Z(\mathbf{z}')$ of $X(\mathbf{z}')$ where each fibre is the boundary of $U_{\mathbf{z}'}$. Moreover, we may choose each fibre $U_{\mathbf{z}'}$ so that it is contractible within itself to $\mathbf{z}'$. We not have the following generalization of Proposition 13.

PROPOSITION 15. *Suppose that* $n+1 = 2k$ *and* M *is a normally restricted analytic manifold lying in* $\mathbb{C}^{n+1}$. *Then we have an open set* $W(\mathbf{z}^1)$ *containing* $\mathbf{z}^1$, *such that for each open set* $\mathscr{U}_M \subseteq \mathbb{C}M$, *with* $X(\mathbf{z}') \subseteq \mathscr{U}_M$, *and each holomorphic function* $g: \mathscr{U}_M \to A_{n+1}(\mathbb{C})$, *the function*

$$\frac{1}{\omega_n} \int_{Z(\mathbf{z}^1)} G^+(\mathbf{z}'' - \mathbf{z}_0) D\mathbf{z}'' g(\mathbf{z}'')$$

in complex left regular on $W(\mathbf{z}^1)$.

Proposition 15 holds for each $\mathbf{z}' = \mathbf{z} + \lambda c m(\mathbf{z})$. Moreover, we have

PROPOSITION 16.

$$\lim_{\mathbf{z}_0 \to \mathbf{z} \in M} \frac{1}{\omega_n} \int_{Z(\mathbf{z}')} G^+(\mathbf{z}'' - \mathbf{z}_0) D\mathbf{z}'' g(\mathbf{z}'') = g(\mathbf{z}).$$

Consequently, we have

THEOREM 21. *Suppose that* M *is a normally restricted analytic manifold lying in* $\mathbb{C}^{n+1}$ *with* $n+1 = 2k$. *Then for each holomorphic function* $g: \mathscr{U}_M \supseteq M \to A_{n+1}(\mathbf{x})$, *there is a complex left regular function*

$$g^+: V'_M(\mathscr{M}) \to A_{n+1}(\mathbb{C})$$

such that $g^+|_{\mathscr{U}} = g$, *where*

$$V'_M(\mathscr{U}_M) = \{\mathbf{z}' \in \mathbb{C}^{n+1} : \mathbf{z}' = \mathbf{z} + \lambda c m(\mathbf{z}), \ \text{where } \mathbf{z} \in M, \lambda = 0 \\ \text{or } N(\mathbf{z}') \cap M(\mathbf{z}, c) = X(\mathbf{z})\}.$$

For M a normally restricted analytic manifold and $\mathscr{U}$ a domain of holomorphy lying in $\mathbb{C}M$ we denote the set

$$\{\mathbf{z}' \in \mathbb{C}^{n+1} : \mathbf{z}' = \mathbf{z} + \lambda c m(\mathbf{z}), \text{ where } \mathbf{z} \in M, \ \lambda = 0 \text{ or } N(\mathbf{z}') \cap M(\mathbf{z}, c) = X(\mathbf{z}') \text{ with a manifold } D(\mathbf{z}) \subseteq M(\mathbf{z}, c), \text{ homeomorphic to an } n\text{-dimensional disc, and } \partial D(\mathbf{z}) \subset X(\mathbf{z}')\}$$

by $NV'_M(\mathscr{U})$.

We now have the following theorem.

THEOREM 22. (i) $n+1 = 2k$ *the Fréchet modules* $M_l(V'_M(\mathscr{U})A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(\mathscr{U} \cap V'_M(\mathscr{U}), A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

(ii) *For* $n = 2k$ *the Fréchet modules* $M_l(NV'_M(\mathscr{U}), A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(\mathscr{U} \cap NV'_M(\mathscr{U}), A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

We can go further than Theorem 22. First we need

DEFINITION 7. A complex n-dimensional manifold $\mathbb{C}M$ lying in $\mathbb{C}^{n+1}$ is called a *restricted complex manifold* if for each $\mathbf{z} \in \mathbb{C}M$ there is a normally restricted analytic manifold $K(\mathbf{z}) \subset \mathbb{C}M$ with $\mathbf{z} \in I(\mathbf{z})$.

For a restricted complex manifold $\mathbb{C}M$ we denote the domain

$$\bigcup_{\mathbf{z} \in \mathbb{C}M} V'_{K(\mathbf{z})}(\mathbb{C}M)$$

by $H(\mathbb{C}M)$, and we denote the domain

$$\bigcup_{\mathbf{z} \in \mathbb{C}M} NV'_{K(\mathbf{z})}(\mathbb{C}M)$$

by $NH(\mathbb{C}M)$.

Suppose that $\mathbb{C}M$ is a restricted complex manifold. Then we have

THEOREM 23. (i) *For* $n+1 = 2k$ *the Fréchet modules* $M_l(H(\mathbb{C}M), A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(\mathbb{C}M, A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

(ii) *For* $n = 2k$ *the Fréchet modules* $M_l(NH(\mathbb{C}M), A_{n+1}(\mathbb{C}))$ *and* $\mathscr{O}(\mathbb{C}M, A_{n+1}(\mathbb{C}))$ *are topologically isomorphic.*

Examples of restricted complex manifolds include $\mathbb{C}^n$, $S^n_{\mathbb{C}}$ and

$$\{\mathbf{z}_1 e_1 + \cdots + \mathbf{z}_{n+1} e_{n+1} \in \mathbb{C}^{n+1} : \mathbf{z}_1^2 a_1 + \cdots + \mathbf{z}_{n+1}^2 a_{n+1} = b, \quad \text{where } a_1, \ldots, a_{n+1}, \ b \in \mathbb{C}\backslash\{0\}\}.$$

A complex submanifold of $\mathbb{C}^{n+1}$ which is not a restricted complex manifold is $N(\mathbf{0})\backslash\{\mathbf{0}\}$.

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