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NORM INEQUALITIES RELATING ONE-SIDED SINGULAR INTEGRALS AND THE ONE-SIDED MAXIMAL FUNCTION

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Abstract

In this paper we prove that if a weight w satisfies the C_q^+ condition, then the $L^p(w)$ norm of a one-sided singular integral is bounded by the $L^p(w)$ norm of the one-sided Hardy-Littlewood maximal function, for $1 < p < q < \infty$.

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1. Introduction

One-sided singular integrals were defined by Aimar, Forzani and Martín-Reyes in [AFM] as singular integrals T^+f whose kernel has support on $(-\infty, 0)$. In the same paper they proved that a weight w satisfies $\int |T^+f|^p w \leq C \int |f|^p w$, for all $f \in L^p(w)$ if, the weight satisfies the one-sided A_p^+ condition, introduced by Sawyer [S1], that characterizes the boundedness of the one-sided Hardy-Littlewood maximal operator $M^+f(x) = \sup_{h>0} h^{-1} \int_x^{x+h} |f|$.

A crucial step in the proof, is the fact that if $w \in A_\infty^+$, then

$$\int |T^+f|^r w \leq C \int [M^+f]^r w, \tag{1.1}$$

for any $1 < r$. We recall definitions of the A_p^+ classes: $w \in A_p^+$, $1 < p$ if there exists a constant C such that for all $a < b < c$

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$$(A_p^+) \qquad \int_a^b w\left(\int_b^c w^{1-p'}\right)^{p-1} \le C(c-a)^p,$$

where $p + p' = pp'$. A weight w is in A_∞^+ if there exist positive constants C and ϵ such that for any $a < b < c$ and any measurable set $E \subset (a, b)$,

$$(A_\infty^+) \qquad \frac{\int_E w}{\int_a^c w} \le C\left(\frac{|E|}{c-b}\right)^\epsilon .$$

These definitions and many properties of A_p^+ and A_∞^+ can be found in [MPT]. A natural question arises. Can we find conditions weaker than A_∞^+ that are sufficient for (1.1)? In [S2] Sawyer considered the following condition, introduced first by Muckenhoupt in [Mu].

There exists two positive constants C and ϵ such that for every interval $I \in \mathbb{R}$ and every measurable subset $E \subset I$ we have

$$(C_p) \qquad \int_E w \le C\left(\frac{|E|}{|I|}\right)^\epsilon \int [M\chi_I]^p w < \infty,$$

where M is the Hardy-Littlewood maximal operator. Sawyer proved that for a standard singular integral Tf, C_q is suficient for

$$(1.2) \qquad \int |Tf|^p w \le C \int [Mf]^p w$$

provided $q > p$. He does not require $\int[M\chi_I]^p w < \infty$. Observe that if $\int[M\chi_I]^q w = \infty$ for some I, then $\int[M\chi_J]^q w = \infty$ for every interval J. Then for every $f \ge 0$ and $p \le q$ we have that $\int[Mf]^p w = \infty$. In this paper we introduce a one-sided version of this condition C_p^+, and prove that if $q > p$, then

$$\int |T^+ f|^p w \le C \int [M^+ f]^p w.$$

The definition of C_p^+ is as follows.

Definition. A weight w satisfies C_p^+ if there exist $\epsilon > 0$ and $C > 0$, so that for any $a < b < c$, with $c - b < b - a$, and any measurable set $E \subseteq (a, b)$, the following holds

$$(C_p^+) \qquad \int_E w \le C\left(\frac{|E|}{(c-b)}\right)^\epsilon \int_{\mathbb{R}} [M^+ \chi_{(a,c)}]^p w < \infty.$$

Observe that if $w \in A_\infty^+$ then $w \in \bigcap_{p>1} C_p^+$. We give examples of weights that satisfy C_p^+ condition for all $p > 1$ but they do not satisfy A_∞^+ condition.

The class of one-sided singular integrals is a subclass of the standard singular integrals and our theorem says that for this subclass we can obtain a more precise result. On one hand, we obtain a smaller right hand side, with M^+f instead of Mf. On the other hand, the condition C_p^+ is different from C_p. These facts make the proof more complicated than in the standard case although it follows the same lines as the paper by Sawyer.

Now we recall the definition of one-sided singular integrals studied in [AFM]. We say that a function k in $L^1_{\mathrm{loc}}(\mathbb{R} - \{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(a) There exists a finite constant B_1 such that

$$\left| \int_{\epsilon<|x|<N} k(x)\,dx \right| \le B_1$$

for all ϵ and all N with $0 < \epsilon < N$. Furthermore $\lim_{\epsilon\to 0^+} \int_{\epsilon<|x|<N} k(x)\,dx$ exists.

(b) There exists a finite constant B_2 such that

$$|k(x)| \le B_2/|x|$$

for all $x \neq 0$.

(c) There exists a finite constant B_3 such that

$$|k(x-y) - k(x)| \le B_3|y||x|^{-2}$$

for all x and y with $|x| > 2|y| > 0$.

A one-sided singular integral is

$$T^+f(x) = \lim_{\epsilon\to 0} \int_{x+\epsilon}^{\infty} k(x-y)f(y)\,dy,$$

where k is a Calderón–Zygmund kernel, with support in $\mathbb{R}^-$. We also define

$$T^{*+}f(x) = \sup_{\epsilon>0} \left| \int_{x+\epsilon}^{\infty} k(x-y)f(y)\,dy, \right|.$$

Examples of such kernels are given in [AFM].

We end this section with some notation. A weight w is a non-negative, locally integrable function. If E is a measurable set, $w(E)$ denotes the integral of w over E. Throughout the paper the letter C represents a positive constant that may change from time to time.

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2. Statement and proof of the result

THEOREM 1. *Let T^+f be a one-sided singular integral, $1 < p < q < \infty$ and assume that w satisfies C_q^+, then*

$$\int_{\mathbb{R}} |T^+f|^p w \le C \int_{\mathbb{R}} [M^+f]^p w$$

for all f such that the right hand side is finite.

REMARK. If $w(x) = e^x$ then $w \in A_1^+ \subset A_\infty^+ \subset C_p^+$, $p > 1$. But $\int [M\chi_I]^p w = \infty$, and therefore $w \notin C_p$, $p > 1$.

The proof is based on a series of lemmas that we now state and prove.

LEMMA 1. *Let us assume that w satisfies C_q^+, $1 < q < \infty$, then for any $\delta > 0$ there exists $C(\delta)$ such that for any disjoint family of intervals $\{J_j\}$ contained in $I = (a, b)$ we have*

(i) $$\int_I \sum_j [M^+\chi_{J_j}]^q w \le C(\delta) w(I) + \delta \int_{\mathbb{R}} [M^+\chi_I]^q w$$

and

(ii) $$\int_{\mathbb{R}} \sum_j [M^+\chi_{J_j}]^q w \le C \int_{\mathbb{R}} [M^+\chi_I]^q w.$$

PROOF. First, we claim that (i) implies (ii). Indeed,

$$\begin{aligned}
\int_{\mathbb{R}} \Big(\sum_j [M^+\chi_{J_j}]^q\Big) w &= \int_I \Big(\sum_j [M^+\chi_{J_j}]^q\Big) w + \int_{(-\infty,a)} \Big(\sum_j [M^+\chi_{J_j}]^q\Big) w \\
&\le C(\delta) w(I) + \delta \int_{\mathbb{R}} [M^+\chi_I]^q w + \int_{(-\infty,a)} \frac{\sum_j |J_j|^q}{(b-x)^q} w \\
&\le C(\delta) w(I) + \delta \int_{\mathbb{R}} [M^+\chi_I]^q w + \int_{(-\infty,a)} \frac{|I|^q}{(b-x)^q} w \\
&\le C(\delta) w(I) + (\delta + 1) \int_{\mathbb{R}} [M^+\chi_I]^q w \\
&\le 2C(\delta) \int_{\mathbb{R}} [M^+\chi_I]^q w + (\delta + 1) \int_{\mathbb{R}} [M^+\chi_I]^q w.
\end{aligned}$$

To prove (i) we use the fact that there exists $\alpha > 0$ such that for every $\lambda > 0$ we have

(2.1) $$|E_\lambda| = \left|\left\{x : \sum_j [M^+\chi_{J_j}]^q(x) > \lambda\right\}\right| \le Ce^{-\alpha\lambda}|I|$$

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(for details see [FeSt]). We define a sequence of points as follows: $x_0 = a$ and for $i \in \mathbb{N}$, $x_i - x_{i-1} = b - x_i$ and consider the sets $E_\lambda^i = E_\lambda \cap (x_i, x_{i+1})$. For $x \in (x_i, x_{i+1})$ we may assume that J_j in $\sum_j |M^+\chi_{J_j}|^q(x)$ are all contained in (x_i, b). It follows from (2.1) that

$$|E_\lambda^i| \le Ce^{-\alpha\lambda}(b - x_i) = Ce^{-\alpha\lambda}(x_{i+2} - x_{i+1}).$$

If we now use condition C_q^+ for the set E_λ^i and the points x_i, x_{i+1}, x_{i+2} we get

$$w(E_\lambda^i) \le Ce^{-\alpha\lambda\epsilon} \int [M^+\chi_{(x_i,x_{i+2})}]^q w.$$

It is easy to see that $\sum_{i>1} M^+\chi_{(x_i,x_{i+2})} \le CM^+\chi_I$ and adding up we get

$$w(E_\lambda \cap I) \le Ce^{-\alpha\lambda\epsilon} \int [M^+\chi_I]^q w.$$

Therefore,

$$\begin{aligned}
\int_I \sum_j [M^+\chi_{J_j}]^q w &= \int_0^{\lambda_0} \int_{E_\lambda \cap I} w\, d\lambda + \int_{\lambda_0}^{\infty} \int_{E_\lambda \cap I} w\, d\lambda \\
&\le \lambda_0 w(I) + \int_{\lambda_0}^{\infty} w(E_\lambda \cap I)\, d\lambda \\
&\le \lambda_0 w(I) + C \int_{\lambda_0}^{\infty} e^{-\alpha\lambda\epsilon}\, d\lambda \int [M^+\chi_I]^q w \\
&\le C(\delta) w(I) + \delta \int [M^+\chi_I]^q w,
\end{aligned}$$

if we choose λ_0 big enough. □

For the next lemma we need to define a new operator, $M_{p,q}^+$. Let f be a nonnegative measurable function. Let us consider

$$\Omega_k = \left\{x : f(x) > 2^k\right\} = \bigcup_i I_i^k,$$

where I_i^k are the connected components of Ω_k. Then

$$\left[M_{p,q}^+ f(x)\right]^p = \sum_{k,i} 2^{pk}\left[M^+\chi_{I_i^k}(x)\right]^q.$$

LEMMA 2. *Let $1 < p < q < \infty$, $w \in C_q^+$, and f non-negative, bounded and of compact support. Then*

$$\int [M_{p,q}^+(M^+ f)] w \le C \int [M^+ f]^p w.$$

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PROOF. Let $\Omega_k = \{x : M^+f(x) > 2^k\} = \bigcup_j I_j^k$, where I_j^k are the connected components of Ω_k. Let $N \geq 1$, note that $\Omega_k \subseteq \Omega_{k-N}$ for all k. Given a connected component of Ω_{k-N}, I_i^{k-N} we estimate $|\Omega_k \cap I_i^{k-N}|$. First, we put $f = g + h$ with $g = f\chi_{I_i^{k-N}}$. Observe that if $x \in I_i^{k-N} = (a, b)$, then $M^+h(x) \leq M^+f(b) \leq 2^{k-N}$. So if $x \in \Omega_k \cap I_i^{k-N}$, then

$$M^+g(x) \geq M^+f - M^+h \geq 2^k - 2^{k-N} \geq \frac{1}{2}2^k.$$

Now using the fact that the operator M^+ is of weak type $(1, 1)$ with respect to Lebesgue measure we get

$$\begin{aligned} |\Omega_k \cap I_i^{k-N}| &\leq \left|\left\{x : M^+g(x) \geq \frac{1}{2}2^k\right\}\right| \leq C2^{-k}\int g \\ &= C2^{-k}\int_{I_i^{k-N}} f \leq C2^{-k}|I_i^{k-N}|\,M^+f(a) \leq C2^{-N}|I_i^{k-N}|. \end{aligned} \tag{2.2}$$

Let $S(k) = 2^{kp}\sum_j \int [M^+\chi_{I_j^k}]^q w$ and $S(k, N, i) = 2^{kp}\sum_{j:I_j^k \subseteq I_i^{k-N}} \int [M^+\chi_{I_j^k}]^q w$. Then

$$S(k, N, i) = 2^{kp}\sum_{j:I_j^k \subseteq I_i^{k-N}} \int_{I_i^{k-N}} [M^+\chi_{I_j^k}]^q w + 2^{kp}\sum_{j:I_j^k \subseteq I_i^{k-N}} \int_{(I_i^{k-N})^c} [M^+\chi_{I_j^k}]^q w = \mathrm{I} + \mathrm{II}.$$

By Lemma 1

$$\mathrm{I} \leq C(\delta)2^{kp}w(I_i^{k-N}) + \delta 2^{kp}\int [M^+\chi_{I_i^{k-N}}]^q w,$$

where $\delta > 0$ is chosen later. Now, by (2.2)

$$\begin{aligned} \mathrm{II} &\leq C2^{kp}\int_{-\infty}^a \frac{\sum |I_j^k|^q}{(b-x)^q} w \leq C2^{kp}\int_{-\infty}^a \frac{(C2^{-N}|I_i^{k-N}|)^q}{(b-x)^q} w \\ &\leq C2^{N(p-q)}2^{p(k-N)}\int [M^+\chi_{I_i^{k-N}}]^q w. \end{aligned}$$

So we get

$$S(k) = \sum_i S(k, N, i) \leq C(\delta)2^{kp}\sum_i w(I_i^{k-N}) + [\delta 2^{Np} + C2^{N(p-q)}]S(k-N).$$

As $p < q$, we can choose δ small and N big enough such that

$$S(k) \leq C(\delta)2^{kp}w(\Omega_{k-N}) + \frac{1}{2}S(k-N).$$

Now

$$S_M = \sum_{k \leq M} S(k) \leq \frac{1}{2}S_M + C\int [M^+f]^p w,$$

for all M. If we prove that under the assumptions on f, we have $S_M < \infty$, we are finished. Let us suppose that supp $f \subset I = (a, b)$. There exists L such that $2^L < 1/(b-a)\int_a^b f \leq 2^{L+1}$.

If $k \geq L+1$, then $\Omega_k \subset I^- \cup I$, where $I^- = (2a-b, a)$. Indeed, if $x < 2a-b$, then

$$M^+f(x) = \sup_{h>a-x>b-a} \frac{1}{h}\int_x^{x+h} \leq \frac{1}{b-a}\int_a^b f \leq 2^{L+1}.$$

If I_j^k are the connected components of Ω_k, using Lemma 1 and since $q > p$, we have

$$\sum_{k=L+1}^{M}\sum_j 2^{kp}\int [M^+\chi_{I_j^k}]^q w \leq \sum_{k=L+1}^{M} 2^{kp}\int [M^+\chi_{I^-\cup I}]^q w \leq C\int [M^+\chi_I]^p w < \infty.$$

If $k \leq L$ we can show again that $\Omega_k \subset 2^{L-k+2}(I^-) \cup I$, where $2^n(I^-) = (c_n, a)$, with $(a - c_n) = 2^n(b-a)$. Then by Lemma 1 we have

$$\sum_{k\leq L}\sum_j 2^{kp}\int [M^+\chi_{I_j^k}]^q w \leq C\sum_{k\leq L} 2^{kp}\int [M^+\chi_{2^{L-k+2}(I^-)\cup I}]^q w.$$

Now its easy to see, using $p < q$, that

$$\sum_{k\leq L} 2^{kp}[M^+\chi_{2^{L-k+2}(I^-)\cup I}(x)]^q \leq C2^{Lp}[M^+\chi_I(x)]^p < \infty. \qquad \square$$

LEMMA 3. *Let $1 < p < q < \infty$, $w \in C_q^+$ and let f be a non-negative bounded function with compact support. Then*

$$\int [M_{p,q}^+(T^{*+}f)]^p w \leq C\left[\int [T^{*+}f]^p w + \int [M^+f]^p w\right].$$

PROOF. Let $\Omega_k = \{x : T^{*+}f(x) > 2^k\} = \bigcup_j I_j^k$, where I_j^k are the connected components of Ω_k. Observe that in the proof of the 'good lambda inequality' in [AFM, Lemma 2.7], what they really show is

$$\text{(2.3)}\quad |\{x \in I_i^{k-N} : T^{*+}f(x) > 2^k\}| \leq C2^{-N}|I_i^{k-N}| \quad \text{if } I_i^{k-N} \not\subseteq \{x : M^+f(x) > 2^{k-N}\}.$$

Let $O_k = \{x : M^+f(x) > 2^k\} = \bigcup_j J_j^k$, where J_j^k are the connected components of O_k. For each I_i^{k-N} we have two cases

(i) $I_i^{k-N} \subseteq O_{k-N}$,
(ii) $I_i^{k-N} \not\subseteq O_{k-N}$.

Case (1). There exists l_i such that $I_i^{k-N} \subseteq J_{l_i}^{k-N}$.
Case (2). (2.3) implies

$$\sum_{j:I_j^k \subseteq I_i^{k-N}} |I_j^k| = |\{x \in I_i^{k-N} : T^{*+}f(x) > 2^k\}| \leq C2^{-N}|I_i^{k-N}|. \tag{2.4}$$

Let $S(k) = 2^{kp}\sum_j \int [M^+\chi_{I_j^k}]^q w$ and $S(k,N,i) = 2^{kp}\sum_{j:I_j^k\subseteq I_i^{k-N}} \int [M^+\chi_{I_j^k}]^q w$. Then

$$S(k,N,i) = 2^{kp}\sum_{j:I_j^k\subseteq I_i^{k-N}} \int_{I_i^{k-N}} [M^+\chi_{I_j^k}]^q w + 2^{kp}\sum_{j:I_j^k\subseteq I_i^{k-N}} \int_{(I_i^{k-N})^c} [M^+\chi_{I_j^k}]^q w = \mathrm{I} + \mathrm{II}.$$

By Lemma 1 we have that

$$\mathrm{I} \leq C(\delta)2^{kp}w(I_i^{k-N}) + \delta 2^{kp}\int [M^+\chi_{I_i^{k-N}}]^q w,$$

where $\delta > 0$. We denote $(a_i^{k-N}, b_i^{k-N}) = I_i^{k-N}$, then by (2.4) we obtain

$$\begin{aligned}\mathrm{II} &\leq C2^{kp}\int_{-\infty}^{a_i^{k-N}} \frac{\sum_{j:I_j^k\subseteq I_i^{k-N}} |I_j^k|^q}{(b_i^{k-N}-x)^q} w \leq C2^{kp}\int_{-\infty}^{a_i^{k-N}} \frac{(C2^{-N}|I_i^{k-N}|)^q}{(b_i^{k-N}-x)^q} w \\ &\leq C2^{kp-Nq}\int [M^+\chi_{I_i^{k-N}}]^q w.\end{aligned}$$

Adding I and II we get

$$S(k,N,i) \leq C(\delta)2^{kp}w(I_i^{k-N}) + (\delta + C2^{-Nq})2^{kp}\int [M^+\chi_{I_i^{k-N}}]^q w.$$

Then

$$S(k) = \sum_{\substack{i:I_i^{k-N} \\ \text{is in case (1)}}} S(k,N,i) + \sum_{\substack{i:I_i^{k-N} \\ \text{is in case (2)}}} S(k,N,i) = \mathrm{III} + \mathrm{IV}.$$

For III we observe that I_j^k is contained in exatly one J_l^{k-N} and by Lemma 1 we have

$$\begin{aligned}\mathrm{III} &= \sum_{i:I_i^{k-N}\subseteq J_{l_i}^{k-N}} S(k,N,i) = \sum_{i:I_i^{k-N}\subseteq J_{l_i}^{k-N}}\ \sum_{j:I_j^k\subseteq I_i^{k-N}} 2^{kp}\int [M^+\chi_{I_j^k}]^q w \\ &\leq \sum_l \sum_{j:I_j^k\subseteq J_{l_i}^{k-N}} 2^{kp}\int [M^+\chi_{I_j^k}]^q w \leq C\sum_l 2^{kp}\int [M^+\chi_{J_l^{k-N}}]^q w.\end{aligned}$$

To estimate IV we observe that

$$\mathrm{IV} \leq C(\delta)2^{kp}\sum_i w(I_i^{k-N}) + (\delta + C2^{-Nq})2^{kp}\sum_i \int [M^+\chi_{I_i^{k-N}}]^q w$$

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$$\leq C2^{kp}w(\Omega_{k-N}) + \frac{1}{2}S(k-N),$$

choosing δ small and N big enough. Combining III and IV we get

$$S(k) \leq \frac{1}{2}S(k-N) + C2^{kp}w(\Omega_{k-N}) + C2^{kp}\sum_l \int [M^+\chi_{J_l^{k-N}}]^q w.$$

Using Lemma 2

$$\begin{aligned} S_M = \sum_{k\leq M} S_k &\leq \frac{1}{2}S_M + C\int [T^{*+}f]^p w + C\int [M^+_{p,q}(M^+f)]^p w \\ &\leq \frac{1}{2}S_M + C\left(\int [T^{*+}f]^p w + \int [M^+f]^p w\right), \end{aligned}$$

and since $S_M < \infty$ (see Lemma 2), we get

$$\int [M^+_{p,q}(T^{*+}f)]^p w \leq C\left(\int [T^{*+}f]^p w + \int [M^+f]^p w\right). \qquad \square$$

PROOF OF THEOREM 1. First we observe that $|T^+f| \leq T^{*+}f$, so it is enough to prove the theorem for T^{*+}. Let f be a non-negative bounded function with compact support.

Let $\Omega_k = \{x : T^{*+}f(x) > 2^k\} = \bigcup_j J_j^k$ where J_j^k, are the connected components of Ω_k. Let us fix $(a, b) = J_j^k$. We partition (a, b) as follows. Let $x_0 = a$, and we choose x_{i+1} such that $x_{i+1} - x_i = b - x_{i+1}$ and we let $I_i^k = (x_i, x_{i+1})$. By 'the good lambda inequality' in [AFM, Lemma 2.7] we have that

$$|E_i^k| = |\{x \in I_i^k : T^{*+}f(x) > 2^{k+1}, M^+f(x) \leq \gamma 2^k\}| \leq C\gamma |I_i^k| \quad \text{for } 0 < \gamma < 1.$$

From C_q^+ condition we have

$$w(E_i^k) \leq C\gamma^\epsilon \int [M^+\chi_{I_i^k \cup I_{i+1}^k}]^q w.$$

Summing over all i and using Lemma 1 we infer that

$$\begin{aligned} &w\left(\{x \in J_j^k : T^{*+}f(x) > 2^{k+1}, M^+f(x) \leq \gamma 2^k\}\right) \\ &\qquad \leq C\gamma^\epsilon \sum_i \int [M^+\chi_{I_{i,j}^k \cup I_{i+1,j}^k}]^q w \leq C\gamma^\epsilon \int [M^+\chi_{J_j^k}]^q w. \end{aligned}$$

Now, summing over all j we have that

$$w\left(\{x \in \Omega_k : T^{*+}f(x) > 2^{k+1}, M^+f(x) \leq \gamma 2^k\}\right) \leq C\gamma^\epsilon \sum_j \int [M^+\chi_{J_j^k}]^q w.$$

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Then by Lemma 3,

$$\begin{aligned}\int (T^{*+}f)^p w &= \sum_k \int_{\Omega_k - \Omega_{k+1}} (T^{*+}f)^p w \le 2^p \sum_k 2^{kp} w(\Omega_k) \\ &= C \sum_k 2^{kp} \Big[w\big(\{x \in \Omega_k : T^{*+}f > 2^{k+1}, M^+ f \le \gamma 2^k\}\big) \\ &\qquad\qquad + w\big(\{x \in \Omega_k : T^{*+}f > 2^{k+1}, M^+ f > \gamma 2^k\}\big)\Big] \\ &\le \sum_{j,k} \Big(C\gamma^{\epsilon} 2^{kp} \int [M^+ \chi_{J_j^k}]^q w \Big) \\ &\quad + C \sum_k 2^{kp} w(\{x \in \Omega_k : M^+ f(x) > \gamma 2^k\}) \\ &\le C\gamma^{\epsilon} \Big[\int [T^{*+}f]^p w + \int [M^+ f]^p w \Big] + C \int [M^+ f]^p w.\end{aligned}$$

Finally we prove that under the assumptions on f, we have that $\int [T^{*+}f]^p w < \infty$, and choosing γ small enough we finish the proof. To see that $\int [T^{*+}f]^p w < \infty$, let supp $f \subset I = (a, b)$ and $I^- = (2a - b, a)$. If $x < 2a - b$, then $T^{*+}f(x) \le CM^+ f(x)$, so

$$\int_{-\infty}^{2a-b} [T^{*+}f]^p w \le \int_{-\infty}^{2a-b} [M^+ f]^p w < \infty.$$

Since $T^{*+}f$ is a singular integral and f is bounded, it is known that $\int_{I^- \cup I} e^{\alpha T^{*+}f} < \infty$ for some $\alpha > 0$. Thus

$$|E_\lambda| = |\{x \in I^- \cup I : T^{*+}f(x) > \lambda\}| \le Ce^{-\lambda\alpha} |I^- \cup I|$$

for all $\lambda > 0$. Applying the C_q^+ condition to the set E_λ and the points $2a - b$, b, $2b - a$, we get

$$w(E_\lambda) \le Ce^{-\lambda\alpha\epsilon} \int [M^+ \chi_{I^- \cup I \cup I^+}]^q w,$$

where $I^+ = (b, 2b - a)$. Integrating with respect to λ, using that $p < q$, and proceeding as in the final step of the proof of Lemma 1, we have

$$\int_{I^- \cup I} [T^{*+}f]^p w \le C \int [M^+ \chi_{I^- \cup I \cup I^+}]^p w < \infty. \qquad \square$$

As observed in the introduction $A_\infty^+ \subseteq \bigcap_{p>1} C_p^+$. We now show that the inclusion is proper.

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PROPOSITION 1. *Let $w \in A_\infty$, then $w\chi_{(-\infty,0)} \in \bigcap_{p>1} C_p^+$.*

PROOF. First we observe that $w\chi_{(-\infty,0)} \notin A_\infty^+$. Let us consider $a < b < c$ such that $c - b < b - a$ and E a mesurable set such that $E \subset (a, b)$. We have several cases

(i) $a < b < c < 0$. In this case there is nothing to prove because

$$A_\infty \Longrightarrow A_\infty^+ \Longrightarrow \bigcap_{p>1} C_p^+.$$

(ii) $a < b < 0 < c$. There exist $\epsilon > 0$ and $C > 0$ such that

$$\begin{aligned} w\chi_{(-\infty,0)}(E) = w(E) &\le C\left(\frac{|E|}{b-a}\right)^\epsilon w(a,b) \le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_a^b [M^+\chi_{(a,b)}]^p w \\ &\le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,b)}]^p w \le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w. \end{aligned}$$

(iii) $a < 0 < b < c$, and $b \le -2a$. Suppose that $E \subseteq (a, 0)$. Note that since $b - a \le -3a$,

$$\begin{aligned} w\chi_{(-\infty,0)}(E) = w(E) &\le C\left(\frac{|E|}{0-a}\right)^\epsilon w(a,0) \le C\left(\frac{|E|}{b-a}\right)^\epsilon \int_a^0 [M^+\chi_{(a,0)}]^p w \\ &\le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w. \end{aligned}$$

If $E \not\subseteq (a, 0)$, then

$$\begin{aligned} w\chi_{(-\infty,0)}(E) = w(E \cap (-\infty, 0)) &\le C\left(\frac{|E \cap (-\infty,0)|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w \\ &\le C\left(\frac{|E|}{c-b}\right)^\epsilon \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w. \end{aligned}$$

(iv) $a < 0 < b < c$ and $b > -2a$.

$$w\chi_{(-\infty,0)}(E) \le w(E) \le C\left(\frac{|E|}{b-a}\right)^\epsilon w(a,b) \le C\left(\frac{|E|}{c-b}\right)^\epsilon w(a,b).$$

If we prove that $w(a,b) \le C\int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w$, we have finished the proof. Using that w satisfies the doubling condition and that $b > -2a$ if and only if $a + b > b/2$ we have

$$\begin{aligned} \int_{-\infty}^0 [M^+\chi_{(a,c)}]^p w &\ge \int_{-\infty}^0 [M^+\chi_{(a,b)}]^p w \ge \int_{-b}^a [M^+\chi_{(a,b)}]^p w = \int_{-b}^a \left(\frac{b-a}{b-x}\right)^p w \\ &\ge \int_{-b}^a \left(\frac{1}{2}\right)^p w \ge \frac{C}{2^p} w(-b,a) \ge C w(a,b). \qquad \square \end{aligned}$$

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