APPROXIMATING FREDHOLM OPERATORS ON A NONSEPARABLE HILBERT SPACE

by RICHARD BOULDIN

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0. Abstract. This paper obtains a simple formula for the distance from a given operator to the set of invertible operators without requiring the underlying space to be separable. That formula is used to compute the distance to the Fredholm operators with a given index. These results require the further study of the concepts of essential nullity and essential deficiency, which permitted us to characterize the closure of the invertible operators. We also introduce a parameter called the modulus of Fredholmness.

1. Introduction. Extensive results related to the Calkin algebra and operator approximations have been proved for a (bounded linear) operator T on a *separable* Hilbert space. That literature is vast and the bibliography of **[1]** could be used as a starting point. Papers **[2]**, **[3]**, **[4]**, **[6]**, **[8]**, **[13]**, **[14]**, and **[15]** are relevant to this work. It is desirable to have these results on more general spaces, and an important step in that direction is the discovery of appropriate versions for Hilbert spaces that are not separable. There has been progress toward this goal, as indicated by **[5]**, **[7]**, **[9]**, and **[11]**. The purpose of this paper is to generalize and refine results in **[1]**, **[2]**, and **[7]** using some ideas introduced in **[2]**, **[5]**, and **[7]**.

Let $\mathscr{B}(H)$ denote the operators on the Hilbert space H, and define $\operatorname{nul} T(\operatorname{def} T)$ to be the cardinal number $\dim \ker T(\dim \ker T^*)$. Let $U|T|$ be the usual polar factorization of T, and let $E(\cdot)$ be the spectral measure for the non-negative operator $|T|$. In **[5]** we defined ess nul T by the equation

$$\operatorname{ess\,nul} T = \inf\{\dim E([0, \varepsilon))H : \varepsilon > 0\},$$

and by definition ess def T is ess nul T^*.

In **[5]** we showed that the closure of the invertible operators is the set of operators T such that $\operatorname{ess\,nul} T = \operatorname{ess\,def} T$. It is important to note that $\dim E([0, \varepsilon))H$ is a nondecreasing function of ε taking values in the cardinal numbers. Consequently, there is a positive γ such that $\dim E([0, \varepsilon))H = \operatorname{ess\,nul} T$ for $0 < \varepsilon \le \gamma$. (The key observation is that every cardinal number has a successor whether or not it has a predecessor.) It follows from Lemma 4 that ess nul T = nul T when T has closed range.

In **[7]** we defined the *modulus of invertibility*, denoted $\rho(T)$, by the equation

$$\rho(T) = \inf\{\lambda : \dim E((\lambda - \varepsilon, \lambda + \varepsilon))H = \dim H \text{ for } \varepsilon > 0\},$$

and we showed that the following formula is equivalent to the definition

$$\rho(T) = \inf\{\lambda: \dim E([0, \lambda + \varepsilon))H = \dim H \text{ for } \varepsilon > 0\}.$$

For the current work we need to define the *modulus of Fredholmness*, denoted by $\tau(T)$. Let $\beta(T) = \max\{\operatorname{ess\,nul} T, \operatorname{ess\,def} T, \aleph_0\}$; for simplicity we write β for $\beta(T)$. Define $\tau(T)$ by the equation

$$\tau(T) = \sup\{\lambda : \dim E([0, \lambda))H < \beta\},$$

where the supremum is understood to be 0 if there is no positive λ such that

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$\dim E([0, \lambda))H < \beta$. If T is Fredholm then $\beta = \aleph_0$ and $\tau(T) > 0$, and if T is not Fredholm then the preceding definition can be simplified by letting $\beta = \max\{\text{ess nul } T, \text{ess def } T\}$. These observations follow from Lemma 4.

Recall that the minimum modulus of the operator T, denoted by $m(T)$, is defined to be $\inf\{\|Tf\| : \|f\| = 1\}$. We have $m(T) > 0$ if and only if nul $T = 0$ and the range of T, denoted by TH, is closed. The *reduced minimum modulus* of T, denoted by $m_r(T)$ is defined to be $\inf\{\|Tf\| : \|f\| = 1 \text{ and } f \perp \ker T\}$. We have $m_r(T) > 0$ if and only if TH is closed. Let $\sigma_e(T)$ denote the set $\{z : T - zI \text{ is not a Fredholm operator}\}$, which is the essential spectrum of T. Define the *essential minimum modulus*, denoted by $m_e(T)$, to be $\inf\{\lambda : \lambda \in \sigma_e((T^*T)^{1/2})\}$. We have $m_e(T) > 0$ if and only if TH is closed and nul $T < \infty$. See **[2]** and its sources for more explanation of the assertions in this paragraph.

2. Distance to the invertible operators. In this section we obtain a formula for the distance from a given operator to the set of invertible operators without any assumption of separability. The formula obtained is analogous to the one obtained in **[5]** for the separable case. On a Hilbert space that is not required to be separable there are five different quantities that share the properties of the essential minimum modulus in a separable Hilbert space.

First we prove a lemma, which gives us one of the fundamental tools used in this research.

LEMMA 1. *Let $T = U|T|$ be the usual polar factorization, and let V be the isometry obtained by restricting U to the closure of the range of T^*, denoted by $(T^*H)^-$, and considering $(TH)^-$ to be the range of V. Let $E(.)$ and $F(.)$ denote the spectral measures of $|T|$ and $|T^*|$, respectively. If A denotes the restriction of $|T|$ to $(T^*H)^-$, i.e. $A = |T| \mid (T^*H)^-$, and $B = |T^*| \mid (TH)^-$, then $B = VAV^*$. It follows that $F(\mathcal{I}) = VE(\mathcal{I})V^*$ for any interval $\mathcal{I}$ contained in* $(0, \infty)$.

Proof. Note that $TT^* = U|T|^2U^* = U(T^*T)U^*$. Since the square root of an operator is the limit of polynomials in the operator, we see that $U|T|U^* = |T^*|$. It follows that $VAV^* = B$; note that

$$A = \int_{(0,\infty)} t\,dE(t), \qquad B = \int_{(0,\infty)} t\,dF(t).$$

It is routine to verify that

$$B = V\left(\int_{(0,\infty)} t\,dE(t)\right)V^* = \int_{(0,\infty)} t\,dVE(t)V^*.$$

Since the last equation characterizes $F(\cdot)$, we get the desired result.

Three of the operator parameters in a general Hilbert space, that share the properties that the essential minimum modulus $m_e(T)$ enjoys in a separable Hilbert space, are related in the next theorem. Each part of this theorem will be used subsequently.

To avoid tedious trivialities we assume henceforth that H is infinite dimensional.

THEOREM 2. (i) $\rho(T) \geq \tau(T) \geq m_e(T) \geq 0$.

(ii) $\tau(T) = \inf\{\lambda : \dim E((\lambda, \lambda + \varepsilon))H \geq \beta$ *for* $\varepsilon > 0\}$ *and* $\dim E((\tau(T), \tau(T) + \varepsilon))H \geq \beta$ *for* $\varepsilon > 0$.

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(iii) $\tau(T)=0$ *if and only if* ess nul $T \geq$ ess def T *and* ess nul $T \geq \aleph_0$.

(iv) *If* $\tau(T)=0=\tau(T^*)$, *then* ess nul $T=$ ess def T, *and consequently* T *belongs to the closure of the invertible operators, i.e.* $T \in \mathcal{G}^-$.

(v) *If both* $\tau(T)$ *and* $\tau(T^*)$ *are positive, then they are equal.*

Proof. The first inequalities follow from the observation that $\dim H \geq \beta \geq \aleph_0$.

In order to prove (ii) we let γ denote the infimum on the right side and assume that γ is positive. Choose x so that $0<x<\gamma$ and $\lambda \in [0,x]$; since there exists a positive ε such that $\dim E((\lambda,\lambda+\varepsilon))H<\beta$, we can find a finite cover of $(0,x]$, say $\{(\lambda_j,\lambda_j+\varepsilon_j): j=1,\ldots,n\}$, consisting of such intervals. The inequalities

$$\dim E((0,x])H \leq \sum_{j=1}^{n} \dim E((\lambda_j,\lambda_j+\varepsilon_j))H$$

and

$$\operatorname{nul}|T| = \dim E(\{0\})H \leq \text{ess nul } T$$

show that $\dim E([0,x))H<\beta$. On the other hand, if $x>\gamma$ then there exists some λ between γ and x such that $\dim E((\lambda,\lambda+\varepsilon))H \geq \beta$ for every positive ε. In particular, we may assume that $(\lambda,\lambda+\varepsilon) \subset (\gamma,x)$ and so $\dim E([0,x))H \geq \beta$. It now follows that γ is the least upper bound for the set $\{\lambda : \dim E([0,\lambda))H<\beta\}$. Thus, $\gamma=\tau(T)$. If $\gamma=0$, then $\beta=$ ess nul T and $\tau(T)=0$.

To prove the final assertion in (ii) we choose a positive δ. There exists some λ between $\tau(T)$ and $\tau(T)+\delta$ such that $\dim E((\lambda,\lambda+\varepsilon))H \geq \beta$ for every positive ε. We may assume that $(\lambda,\lambda+\varepsilon) \subset (\tau(T),\tau(T)+\delta)$ and consequently

$$\dim E((\tau(T),\tau(T)+\delta))H \geq \beta.$$

This proves (ii).

The quantity $\tau(T)$ is 0 if and only if there is no positive λ such that $\dim E([0,\lambda))H < \beta$. For all positive λ sufficiently small we have ess nul $T=\dim E([0,\lambda))H$; so it follows that ess nul $T \geq \beta$. This clearly gives the desired conclusion. The converse is straightforward.

If $\tau(T)=0=\tau(T^*)$, then it follows from part (iii) that ess nul $T=$ ess def T. The main result of **[5]** implies that $T \in \mathcal{G}^-$.

Part (v) follows routinely from Lemma 1 and part (ii) of this theorem.

The next lemma obtains the easier half of the formula that is stated in Theorem 7. The basic idea of the proof was first introduced in **[2]**.

Lemma 3. *If* ess nul $T \neq$ ess def T, *then* $\operatorname{dist}(T,\mathcal{G}) \leq \max\{\tau(T),\tau(T^*)\}$.

Proof. In view of part (iv) of Theorem 2, at least one of the quantities $\tau(T)$ and $\tau(T^*)$ is positive. Interchange T and T^*, if necessary, so that $\tau(T)>0$. Let μ denote $\tau(T)$, let a positive ε be given, and let UR be the polar factorization of T such that $\ker U=\ker R$. It follows from part (ii) of Theorem 2 that

$$\dim UE((0,\mu+\varepsilon))H = \dim E((0,\mu+\varepsilon))H \geq \beta.$$

Define $R(\varepsilon)$ to be R on $E([\mu+\varepsilon,\infty))H$ and to be μI on $E([0,\mu+\varepsilon))H$. Define $U(\varepsilon)$ to be U on $E([\mu+\varepsilon,\infty))H$ and on $E([0,\mu+\varepsilon))H$ let it be εV, where V is an isometry of

$E([0, \mu+\varepsilon))H$ onto $\operatorname{span}\{UE((0, \mu+\varepsilon))H, (TH)^{\perp}\}$. Since $UE((0,\infty))H = U(RH)^{-} = (TH)^{-}$, it is clear that $U(\varepsilon)$ is one-to-one and onto; thus, $U(\varepsilon)$ is invertible. Clearly $R(\varepsilon)$ is invertible and

$$\begin{aligned}\|UR - U(\varepsilon)R(\varepsilon)\| \\ &= \|(UR - U(\varepsilon)R(\varepsilon) \mid E([0, \mu+\varepsilon))H\| \\ &\le \|(UR - U(\varepsilon)R) \mid E([0, \mu+\varepsilon))H\| + \|U(\varepsilon)R - U(\varepsilon)R(\varepsilon)\| \\ &\le \|U - U(\varepsilon)\|\, \|R \mid E([0, \mu+\varepsilon))H\| + \|U(\varepsilon) \mid E([0, \mu+\varepsilon))H\|\, \|R - R(\varepsilon)\| \\ &\le \mu + \varepsilon(1 + 3\mu + 2\varepsilon).\end{aligned}$$

This proves that $\operatorname{dist}(T, \mathscr{G}) \le \tau(T)$, and $\tau(T) = \max\{\tau(T), \tau(T^*)\}$ according to part (v) of Theorem 2.

The next lemma collects some elementary observations about operators with closed ranges. These facts will be used repeatedly in subsequent proofs.

LEMMA 4. *Either the ranges of all of the operators T, $|T|$, T^*, $|T^*|$ are closed or none of the ranges is closed. The range of $|T|$ is closed if and only if 0 is not an accumulation point of its spectrum $\sigma(|T|)$. If TH is closed, then* ess nul T = nul T.

Proof. It follows from the equations

$$\|Tf\|^2 = \langle Tf, Tf\rangle = \langle T^*Tf, f\rangle = \||T|f\|^2$$

that $\ker T = \ker |T|$ and the partial isometry U in the polar factorization $T = U|T|$ maps $|T|H$ isometrically onto TH. In particular, if TH is closed, then so is $|T|H$. It then follows that there is a positive δ such that

$$\delta^2 \|f\|^2 \le \||T|f\|^2 = \langle T^*Tf, f\rangle \le \|T^*TF\|, \tag{*}$$

for every unit vector $f \in (\ker|T|)^{\perp}$. Since $|T|^2 = T^*T$, we see that $(\ker|T|)^{\perp}$ contains $(\ker T^*T)^{\perp}$; so (*) implies that T^*TH is closed. Since $T^*TH = T^*H$, we know that T^* and $|T^*|$ have closed range. The inequality analogous to (*) that we get when we replace $|T|$ with $|T^*|$ shows that TH is closed. This proves that if the range of one of the operators T, $|T|$, T^* or $|T^*|$ is closed then so is each of the others.

Let $E(\cdot)$ be the spectral measure for $|T|$ and assume that 0 is not an accumulation point of $\sigma(|T|)$. Thus, there is some interval $(0, \varepsilon)$ that is disjoint from $\sigma(|T|)$. It follows that

$$\||T|f\| \ge \varepsilon\, \|f\|, \quad \text{for} \quad f \in E((0,\infty))H,$$

and, since $E((0,\infty))H = (\ker|T|)^{\perp}$, we may conclude that $|T|H$ is closed. If 0 is an accumulation point of $\sigma(|T|)$, then $\dim E((0,\varepsilon))H \ge \aleph_0$, for every positive ε. Clearly we have

$$\inf\{\||T|f\| : f \perp \ker(|T|)\} = 0$$

and $|T|H$ is not closed.

The final conclusion follows from the preceding paragraph.

We shall prove that an inequality between ess nul T and ess def T leads to an inequality between $\tau(T)$ and $\tau(T^*)$.

LEMMA 5. *If* ess nul $T <$ ess def T, *then* $\tau(T) \ge \tau(T^*)$.

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Proof. We deal with the case that TH is closed first. By Lemma 4 we know that T^*H is closed. It follows that $\text{ess nul } T = \text{nul } T$ and $\text{ess def } T = \text{def } T$. If $\text{def } T = \beta$, then $\tau(T^*) = 0$ and the desired inequality is clear. If $\beta > \text{def } T$, then

$$\aleph_0 = \beta > \text{ess def } T > \text{ess nul } T$$

and it follows from (iii) of Theorem 2 that both $\tau(T)$ and $\tau(T^*)$ are positive. According to (v) of Theorem 2, $\tau(T) = \tau(T^*)$ and we are done.

If TH is not closed, then it follows from the fact that 0 is an accumulation point of $\sigma(|T|)$, as proved in Lemma 4, that ess nul T is infinite; so $\text{ess nul } T \geq \aleph_0$. Since T^*H is not closed, we also have $\text{ess def } T \geq \aleph_0$ and $\beta(T) = \max\{\text{ess nul } T, \text{ess def } T\}$. Similarly $\beta(T^*) = \max\{\text{ess nul } T^*, \text{ess def } T^*\}$ and so $\beta(T^*) = \beta(T)$. It now follows from Lemma 1 and (ii) of Theorem 2 that $\tau(T^*) = \tau(T)$.

The next lemma proves the most difficult part of the formula given in Theorem 7. In the case of a separable Hilbert space this difficult argument was obviated by the following folklore theorem. If $\|A - B\| < m_e(A)$, then $\text{ind } A = \text{ind } B$, where ind T denotes the index of T; i.e. $\text{ind } T = \text{nul } T - \text{def } T$. There is no result on a general Hilbert space that is very similar to the folklore theorem. Because the range of a compact operator cannot contain a nonseparable subspace, the theory of compact perturbations is intrinsically inadequate.

LEMMA 6. *Suppose that* $\text{ess nul } T \neq \text{ess def } T$ *and let* μ *denote* $\max\{\tau(T), \tau(T^*)\}$. *If* $\|T - B\| < \mu$, *then* B *is not invertible.*

Proof. Let B be an operator such that $\|T - B\| < \mu$. Interchange T and T^*, if necessary, so that $\text{ess nul } T < \text{ess def } T$. It follows from Lemma 5 that $\mu = \tau(T)$ and this quantity must be positive in order to avoid contradicting $\text{ess nul } T \neq \text{ess def } T$, according to part (iv) of Theorem 2.

Choose λ between $\|T - B\|$ and μ, and let H_0 represent $E([\lambda, \infty))H$; let Q denote $E([\lambda, \infty))$, which is the orthogonal projection onto H_0. Let $U|T|$ be the polar factorization of T such that $\ker U = \ker |T| = \ker T$.

Denote the restriction of T to H_0 by $T \mid H_0$. We have

$$\|QU^*(T - B) \mid H_0\| \leq \|T - B\| < \lambda,$$

and

$$QU^*T \mid H_0 = QU^*U\,|T|\,\big|\, H_0 = |T|| \, H_0.$$

Since

$$m(|T| \,\big|\, H_0) \geq \lambda,$$

we can find an inverse for $|T| \,\big|\, H_0$, that we denote by C, such that $\|C\| = 1/m(|T| \,\big|\, H_0) \leq 1/\lambda$. Thus, we have

$$\|(I - CQU^*B) \,\big|\, H_0\| \leq \|C\| \, \|T - B\| < 1$$

and we conclude that $CQU^*B \mid H_0$ is invertible. Since C is an invertible map on H_0, it must be that $QU^*B \mid H_0$ is invertible. It follows that $UQU^*B \mid H_0$ is an invertible map of H_0 onto UH_0.

By Lemma 1 we know that $UE(\mathcal{I}) = F(\mathcal{I})U$ for any interval $\mathcal{I} \subset (0, \infty)$, where $F(\cdot)$ and $E(\cdot)$ are the spectral measures of $|T^*|$ and $|T|$, respectively. Recall that $UH = (TH)^- = (|T^*|\, H)^- = F((0, \infty))H$. Thus,

$$UH_0 = F([\lambda, \infty))UH = F([\lambda, \infty))H,$$

and we denote this last space by H_1.

Because of Lemma 1, we know that

$$\dim E((0, \varepsilon))H = \dim F((0, \varepsilon))H$$

for positive ε. We choose a positive number γ such that

$$\dim E([0, \varepsilon))H = \text{ess nul } T, \quad \text{and} \quad \dim F([0, \varepsilon))H = \text{ess def } T$$

provided $0 < \varepsilon \leq \gamma$. We consider the case that TH is not closed. It was shown in Lemma 4 that $|T|\,H$ (or $|T^*|\,H$) is closed if and only if 0 is not an accumulation point of the spectrum of $|T|$ (or $|T^*|$). The Banach closed range theorem implies that the range of T^* (and $|T|$) is closed if and only if the range of T (and $|T^*|$) is. Thus, if TH is not closed, then 0 is an accumulation point of the spectra of $|T|$ and $|T^*|$. It follows that ess nul T and ess def T are both infinite. We note that

$$\begin{aligned}\text{def } T + \dim F((0, \gamma))H = \dim F([0, \gamma))H = \text{ess def } T \\ > \text{ess nul } T = \dim E([0, \gamma))H = \text{nul } T + \dim E((0, \gamma))H.\end{aligned}$$

Since the sum of any set of cardinal numbers that includes an infinite cardinal is simply the maximum of the cardinals, we deduce that ess def T = def T and

$$\text{nul } T \leq \text{ess nul } T < \text{ess def } T = \text{def } T.$$

On the other hand, if TH is closed then 0 is not an accumulation point of the spectra of $|T|$ and $|T^*|$. It follows that ess nul T = nul T and ess def T = def T. Thus, we again obtain the formula

$$\text{nul } T \leq \text{ess nul } T < \text{ess def } T = \text{def } T.$$

Since UU^* is the orthogonal projection onto $(TH)^- = (|T^*|\,H)^-$, we deduce that $UU^* = F((0, \infty))$. Also, we see that

$$\begin{aligned}UQU^*B \mid H_0 &= F([\lambda, \infty))UU^*B \mid H_0 \\ &= F([\lambda, \infty))F((0, \infty))B \mid H_0 \\ &= F([\lambda, \infty))B \mid H_0.\end{aligned}$$

Thus, $F([\lambda, \infty))B \mid H_0$ is an invertible map of H_0 onto H_1, and $F([\lambda, \infty))$ is an invertible map of BH_0 onto H_1.

In order to obtain a contradiction, we assume that B is invertible. Clearly $BH_0 + B(H_0)^\perp \supset B(H_0 + (H_0)^\perp) = BH = H$, and $BH_0 \cap B(H_0)^\perp \neq \{0\}$ would contradict the fact that B is one-to-one. Thus, $H = BH_0 \oplus B(H_0)^\perp$. Clearly the quotient space H/BH_0 is isomorphic to both $B(H_0)^\perp$ and $(BH_0)^\perp$; thus, $\dim(BH_0)^\perp = \dim B(H_0)^\perp$. Since B is one-to-one,

$$\begin{aligned}\dim B(H_0)^\perp = \dim(H_0)^\perp &= \dim E([0, \lambda))H \\ &= \text{nul } T + \dim E((0, \lambda))H.\end{aligned}$$

Before we can conclude the proof of this lemma, we need to verify one more observation. Since $F([\lambda, \infty))$ maps BH_0 onto $H_1 = F([\lambda, \infty))H$, for any vector f there exists $g \in BH_0$ such that $F([\lambda, \infty))g = F([\lambda, \infty))f$ or $(f - g) \in \ker F([\lambda, \infty))$. It follows that $H \subset BH_0 + \ker F([\lambda, \infty))$. Since $F([\lambda, \infty)) \mid BH_0$ is one-to-one, the preceding sum must be a direct sum. The decompositions

$$BH_0 \oplus \ker F([\lambda,\infty)) = H = BH_0 \oplus (BH_0)^{\perp}$$

imply that $\dim(BH_0)^{\perp} = \operatorname{nul} F([\lambda,\infty))$.

Now we can conclude the proof by noting that

$$\begin{aligned}\dim B(H_0)^{\perp} &= \dim(BH_0)^{\perp}\\ &= \operatorname{nul} F([\lambda,\infty))\\ &= \dim F([0,\lambda))H\\ &= \operatorname{def} T + \dim F((0,\lambda))H\\ &= \operatorname{def} T + \dim E((0,\lambda))H.\end{aligned}$$

Since $\lambda < \mu = \tau(T)$ we know that $\dim E([0,\lambda))H < \beta$, and we have already shown that

$$\operatorname{nul} T \leq \operatorname{ess\,nul} T < \operatorname{ess\,def} T = \operatorname{def} T.$$

If $\beta = \aleph_0$, then $\operatorname{nul} T$ and $\dim E((0,\lambda))H$ are finite, and it follows that

$$\operatorname{def} T + \dim E((0,\lambda))H > \operatorname{nul} T + \dim E((0,\lambda))H,$$

which is a contradiction. If $\beta > \aleph_0$ then $\beta = \operatorname{ess\,def} T = \operatorname{def} T$, and it follows that

$$\operatorname{def} T + \dim E((0,\lambda))H = \beta > \operatorname{nul} T + \dim E((0,\lambda))H,$$

which is a contradiction. It follows that B is not invertible.

We are now able to establish a formula for the distance from a fixed operator to the set of invertible operators. This formula should be contrasted to Theorem 3 of **[2]** and Theorem 2.9 of **[9]**.

THEOREM 7. *If* $\operatorname{ess\,nul} T = \operatorname{ess\,def} T$, *then* $\operatorname{dist}(T,\mathcal{G}) = 0$. *If* $\operatorname{ess\,nul} T \neq \operatorname{ess\,def} T$, *then*

$$\operatorname{dist}(T,\mathcal{G}) = \max\{\tau(T),\tau(T^*)\}.$$

If both $\tau(T)$ *and* $\tau(T^*)$ *are positive, then they are equal.*

Proof. The first assertion follows from the main theorem of **[5]**, as reviewed in the introduction. The second statement follows from Lemma 3 and Lemma 6. The final sentence is part (v) of Theorem 2.

The formula in Theorem 7 is simpler than it may appear. By considering the case that T is Fredholm separately we get the following simple formula.

COROLLARY 8. *If* $T \in \mathcal{B}(H)$, *then*

$$\operatorname{dist}(T,\mathcal{G}) = \max\{\tau(T),\tau(T^*)\}$$

unless T is a Fredholm operator with $\operatorname{ind} T = 0$ *in which case the distance is zero.*

Proof. Given Theorem 7 we need only consider the case that $\operatorname{ess\,nul} T = \operatorname{ess\,def} T$, which we assume henceforth. If $\operatorname{ess\,nul} T$ is infinite, then $\operatorname{ess\,nul} T = \beta = \operatorname{ess\,def} T$, and

$\tau(T) = 0 = \tau(T^*)$. Thus,

$$\max\{\tau(T), \tau(T^*)\} = 0$$

and Theorem 7 implies that $\operatorname{dist}(T, \mathscr{G}) = 0$.

The remaining case requires that ess nul T = ess def T and this number is finite. It follows from Lemma 4 that either all of the operators T, T^*, $|T|$, $|T^*|$ have closed range or none does. We know that $|T|H$ (or $|T^*|H$) is closed if and only if 0 is not an accumulation point of the spectrum of $|T|$ (or $|T^*|$). It now follows that if ess nul T is finite then TH is closed. In the circumstances that TH and T^*H are closed, it follows that ess nul T = nul T and ess def T = def T. Thus, in this remaining case T is Fredholm with index equal to 0. It follows from Theorem 7 that $\operatorname{dist}(T, \mathscr{G}) = 0$.

3. Distance to the Fredholm operators. After the appearance of [**2**] there were a number of results, such as [**1**], [**4**], [**6**], [**14**] and [**15**], that suggested that the distance to the invertible operators was the key to computing many distances. In particular, we refine some devices used in [**4**] and [**14**] in order to compute the distance from a fixed operator to the Fredholm operators on a general Hilbert space. We begin with an essential lemma.

LEMMA 2. *If* nul T = def T, *then* ess nul T = ess def T.

Proof. Choose a positive δ sufficiently small that $\dim E([0, \lambda))H$ = ess nul T, and $\dim F([0, \lambda))H$ = ess def T provided $0 < \lambda \leq \delta$. It follows from Lemma 1 that

$$\dim E((0, \lambda))H = \dim F((0, \lambda))H.$$

We note that

$$\begin{aligned} \text{ess nul } T &= \dim E([0, \lambda))H \\ &= \text{nul } T + \dim E((0, \lambda))H \\ &= \text{def } T + \dim F((0, \lambda))H \\ &= \text{ess def } T. \end{aligned}$$

The converse to the preceding lemma is false. Let M_t be multiplication on the space of square integrable functions with domain $[0, 1]$, and let S be a unilateral shift with finite multiplicity. If $T = S \oplus M_t$ then both ess nul T and ess def T equal $\aleph_0$, but nul $T = 0$ and def T equals the multiplicity of S.

Let $\mathscr{T}_0$ denote the set of operators T such that nul T = def T. For any integer n let $\mathscr{I}_n$ consist of the operators T such that ind T = nul T − def $T = n$. (Thus, $T \in \mathscr{I}_n$ requires that nul T and def T be finite.) Let $\mathscr{F}_n$ denote the Fredholm operators T with index n—i.e. ind $T = n$.

The next theorem establishes some important relations among the preceding sets.

THEOREM 10. $(\mathscr{T}_0)^- = (\mathscr{I}_0)^- = (\mathscr{F}_0)^- = (\mathscr{G})^-$.

Proof. The containments $\mathscr{T}_0 \supset \mathscr{I}_0 \supset \mathscr{F}_0 \supset \mathscr{G}$ are clear. According to Lemma 9, ess nul T = ess def T whenever T belongs to $\mathscr{T}_0$. It follows from Theorem 7 that

$$\mathscr{T}_0 \subset (\mathscr{G})^-.$$

The desired formula now follows routinely.

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We shall use notation like $P\mathscr{G}$ to mean $\{PT : T \in G\}$.

Before we can obtain the desired formulas in Theorem 14, we must collect some elementary facts in the next three lemmas. These facts will help us to understand the effects of composing a given operator A with an isometry with finite deficiency.

LEMMA 11. *Let S be an isometry on H with deficiency n (an integer) and let P denote the orthogonal projection SS^*. If $PB = B$, then*
(i) $\operatorname{dist}(B, \mathscr{G}) = \operatorname{dist}(B, P\mathscr{G})$,
(ii) $\operatorname{dist}(B, \mathscr{F}_0) = \operatorname{dist}(B, \mathscr{F}_0 \cap P\mathscr{F}_0)$,
(iii) $\operatorname{dist}(B, \mathscr{I}_0) = \operatorname{dist}(B, \mathscr{I}_0 \cap P\mathscr{I}_0)$.

Proof. Choose $C \in \mathscr{G}$ and let Q denote $(I - P)$. In order to prove (i) we choose a vector $f \in H$, and note that

$$\|(B - C)f\|^2 = \|(B - PC)f\|^2 + \|QCf\|^2.$$

Thus, $\|B - C\| \geq \|B - PC\|$ and it follows that

$$\operatorname{dist}(B, \mathscr{G}) \geq \operatorname{dist}(B, P\mathscr{G}).$$

For $C \in \mathscr{G}$ define C_λ to be $PC + \lambda QC$ for $\lambda \in (0, 1]$. For any vector $f \in H$ we have

$$\|(B - C_\lambda)f\|^2 = \|(B - PC)f\|^2 + \lambda^2 \|QCf\|^2.$$

It follows that

$$(\|B - PC\|^2 + \lambda^2 \|QC\|^2)^{1/2} \geq \|B - C_\lambda\| \geq \|B - PC\|.$$

Thus,

$$\inf\{\|B - C_\lambda\| : 0 < \lambda \leq 1\} = \|B - PC\|.$$

It is routine to see that C_λ is one-to-one and onto; so $C_\lambda \in \mathscr{G}$. This argument shows that $\operatorname{dist}(B, \mathscr{G}) \leq \operatorname{dist}(B, P\mathscr{F})$, which proves (i).

Now we prove part (ii). It is readily verified that $P\mathscr{G} \subset \mathscr{F}_0$ and the containment $P\mathscr{G} \subset P\mathscr{F}_0$ is obvious. Thus, we have $P\mathscr{G} \subset \mathscr{F}_0 \cap P\mathscr{F}_0$ and

$$\operatorname{dist}(B, P\mathscr{G}) \geq \operatorname{dist}(B, \mathscr{F}_0 \cap P\mathscr{F}_0) \geq \operatorname{dist}(B, \mathscr{F}_0).$$

Since $\mathscr{F}_0$ and $\mathscr{G}$ have the same closure, according to Theorem 10, we know that $\operatorname{dist}(B, \mathscr{G}) = \operatorname{dist}(B, \mathscr{F}_0)$. Now it follows that $\operatorname{dist}(B, \mathscr{F}_0 \cap P\mathscr{F}_0) = \operatorname{dist}(B, \mathscr{F}_0)$.

The proof of part (iii) is analogous to the proof of part (ii).

LEMMA 12. *Let S be an isometry on H with deficiency n (an integer) and let P denote the orthogonal projection SS^*. Then*
(i) $S\mathscr{I}_n = \mathscr{I}_0 \cap P\mathscr{I}_0$,
(ii) $S\mathscr{F}_n = \mathscr{F}_0 \cap P\mathscr{F}_0$.

Proof. Because S maps H isometrically onto PH the deficiency of SA, for $A \in \mathscr{I}_n$, is $(n + \operatorname{def} A)$ while $\operatorname{nul} SA = \operatorname{nul} A$. Thus, SA belongs to $\mathscr{I}_0$ and since $PSA = SA$ we see that SA belongs to $P\mathscr{I}_0$. Thus, $S\mathscr{I}_n \subset \mathscr{I}_0 \cap P\mathscr{I}_0$. If the range of A is closed, then the range of SA is closed and $S\mathscr{F}_n \subset \mathscr{F}_0 \cap P\mathscr{F}_0$.

Take $B \in \mathscr{I}_0 \cap P\mathscr{I}_0$ and let $A = S^*B$. Since $PB = B$, def B and nul B are not less than n. Because S^* maps PH isometrically onto H, it follows that

$$\operatorname{def} A = \operatorname{def}(S^*B) = \operatorname{def} B - n.$$

Since S is an isometry, we get

$$\operatorname{nul} A = \operatorname{nul}(SA) = \operatorname{nul}(SS^*B) = \operatorname{nul} B$$

and so $\operatorname{ind} A = n$ or $A \in \mathcal{I}_n$. Clearly $SA = B$ and we have proved that $S\mathcal{I}_n = \mathcal{I}_0 \cap P\mathcal{I}_0$. The preceding argument shows that if $B \in \mathcal{F}_0 \cap P\mathcal{F}_0$ then $A \in \mathcal{F}_n$ and consequently $S\mathcal{F}_n = \mathcal{F}_0 \cap P\mathcal{F}_0$.

LEMMA 13. *Let S be an isometry with deficiency n (an integer) on H. Then*

(i) $\tau(SA) = \tau(A)$,
(ii) $\tau(A^*S^*) = \tau(A^*)$.

Proof. Since

$$|SA| = (A^*S^*SA)^{1/2} = (A^*A)^{1/2} = |A|,$$

we see that $|SA|$ and $|A|$ have the same spectral measure, which we denote $E(\cdot)$. It follows that ess nul SA = ess nul A.

Note that

$$|(SA)^*| = |A^*S^*| = (SAA^*S^*)^{1/2}.$$

Since S is an isometry of H onto PH, where P is the orthogonal projection SS^*, and S^* is the inverse map of PH onto H, we see that AA^* and $(SAA^*S) \mid PH$ have the same spectral measure, say $G_1(\cdot)$. If $G_2(\cdot)$ is the spectral measure for (SAA^*S^*) and ε is a positive number then

$$G_2([0, \varepsilon))H = G_1([0, \varepsilon))H \oplus (I - P)H.$$

It follows that

$$\text{ess def } A = \text{ess nul } |A^*| = \text{ess nul } |A^*S^*| - n.$$

If ess def A is infinite then

$$\text{ess def } A = \text{ess def } A + n = \text{ess def}(SA).$$

If ess def A is finite then it is irrelevant in the computation of the β that appears in the definition of $\tau(A)$. Thus,

$$\max\{\text{ess nul } A, \text{ ess def } A, \aleph_0\} = \max\{\text{ess nul } SA, \text{ ess def } SA, \aleph_0\};$$

β is the same whether it is computed for A or SA. This fact combined with the first paragraph shows that $\tau(SA) = \tau(A)$.

In order to prove (ii) we use several of the preceding observations. First, β is not altered by passing from A to SA or A^*S^*. Since β is infinite, it does not matter whether $G_1(\cdot)$ or $G_2(\cdot)$ is used to locate $\tau(A^*S^*)$. It follows that $\tau(A^*S^*) = \tau(A^*)$.

Our final theorem shows that the distance from a fixed operator to the operators with a given finite index (or the Fredholm operators with a given index) is computed the same way as the distance to the invertible operators.

THEOREM 14. *Let A belong to $\mathcal{B}(H)$, and let n represent an integer. If $A \notin \mathcal{I}_n$, then*
(i) $\operatorname{dist}(A, \mathcal{I}_n) = \max\{\tau(A), \tau(A^*)\}$,
(ii) $\operatorname{dist}(A, \mathcal{F}_n) = \max\{\tau(A), \tau(A^*)\}$.

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Proof. Let n be a nonnegative integer and let S be an isometry on H with deficiency n. (If $n=0$ we understand S to be I.) In order to prove (i) we define B by $B=SA$. For $C \in \mathcal{I}_n$ we have

$$\|B - SC\| = \|S(A - C)\| = \|A - C\|.$$

It now follows from Lemma 12 that

$$\operatorname{dist}(B, \mathcal{I}_0 \cap P\mathcal{I}_0) = \operatorname{dist}(A, \mathcal{I}_n).$$

With this and Lemma 11, we conclude that $\operatorname{dist}(A, \mathcal{I}_n) = \operatorname{dist}(B, \mathcal{I}_0)$. From Theorem 10 we get $\operatorname{dist}(A, \mathcal{I}_n) = \operatorname{dist}(B, \mathcal{G})$. Using Corollary 8 and Lemma 13 we find that

$$\begin{aligned} \operatorname{dist}(A, \mathcal{I}_n) &= \max\{\tau(B), \tau(B^*)\} \\ &= \max\{(\tau(SA), \tau(A^*S^*)\} \\ &= \max\{\tau(A), \tau(A^*)\} \end{aligned}$$

unless B is a Fredholm operator with index equal to 0. Using arguments from the proof of Lemma 12 it is routine to show that B is Fredholm with index 0 if and only if A is Fredholm with index n, and that is contrary to the hypothesis.

For negative n we apply the preceding result to A^* and $\mathcal{I}_n^* = \{C^* : C \in \mathcal{I}_n\} = \mathcal{I}_{-n}$.

The proof of (ii) is analogous step for step to the proof of (i) just presented.

The preceding methods seem to be seriously inadequate for computing the distance from a given operator to the set of semi-Fredholm operators with a given index. The complexities of arithmetic with infinite cardinal numbers prevent any simple conclusions about composite operators in terms of their factors.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GEORGIA
ATHENS
GEORGIA 30602
U.S.A.

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