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CONTINUITY OF LIE DERIVATIONS ON BANACH ALGEBRAS

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The separating subspace of any Lie derivation on a semisimple Banach algebra A is contained in the centre of A.

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The Lie structure induced on a Banach algebra by the bracket $[a, b] = ab - ba$ is of lively interest for their intimate connections with the geometry of manifolds modeled on Banach spaces. Many mathematics have studied Lie derivations on associative rings [**1**, **5**] and Lie derivations on some Banach algebras [**2**, **7**, **8**].

A *Lie derivation* of a Banach algebra A is a linear map D from A into itself satisfying $D([a, b]) = [D(a), b] + [a, D(b)]$ for all $a, b \in A$. In this paper we study the continuity of a Lie derivation D on an arbitrary semisimple Banach algebra A. We measure the continuity of D by considering its separating subspace, which is defined as the subspace $\mathcal{S}(D)$ of those elements $a \in A$ for which there is a sequence $\{a_n\}$ in A satisfying $\lim a_n = 0$ and $\lim D(a_n) = a$. $\mathcal{S}(D)$ is easily checked to be a Lie ideal of A and the closed graph theorem shows that D is continuous if, and only if, $\mathcal{S}(D) = 0$.

Until further notice we assume that A is a unital semisimple complex Banach algebra and D stands for a Lie derivation of A. Let $\mathcal{Z}(A)$ denote the centre of A. For each $a \in A$ let ad$a$ denote the continuous linear operator ad$a(b) = [a, b]$ from A into itself. If P is a closed ideal of A we will denote by Q_P the quotient map from A onto A/P.

The next important result was essentially stated by M. P. Thomas and illustrate the typical sliding hump argument.

Lemma 1 [**10**, Proposition 1.3]. *Let X and Y be Banach spaces, $\{T_n\}$ a sequence of continuous linear operators from X into itself, and let $\{R_n\}$ be a sequence of continuous linear operators from Y into Banach spaces Y_n. If F is a linear operator from X into Y such that $R_nFT_1 \cdots T_m$ is continuous for $m > n$, then $R_nFT_1 \cdots T_n$ is continuous for sufficiently large n.*

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Lemma 2. *If P is a primitive ideal in A of infinite codimension, then $[A, \mathcal{S}(D)] \subset P$.*

Proof. Let us first observe that the extended centroid of A/P is $\mathbb{C}$ (see [6, Theorem 12]) and does not satisfy the standard polynomial identity S_4 (see [9, Theorem 7.1.14]).

We claim that there exist $\lambda \in \mathbb{C}$, a linear functional μ on A, and a functional ν on A such that

$$(D(a^2) - (Da)a - a(Da)) - (\lambda a^2 + \mu(a)a + \nu(a)) \in P$$

for all $a \in A$. Indeed, for every $a \in A$, we have

$$0 = D([a^2, a]) = [Da^2, a] + [a^2, Da] = [Da^2 - (Da)a - a(Da), a].$$

Consequently, the map q defined on A by $q(a) = Da^2 - (Da)a - a(Da)$ is a commuting trace of the bilinear map $B(a, b) = D(ab) - (Da)b - a(Db)$ on $A \times A$. The map q can be handled in the same way as in the proof of Theorem 1 in [1], the only difference being in the application of [1, Lemmas 1 and 2] to A/P instead of A.

We can now proceed as in the proof of [1, Theorem 4] in order to prove that the map d defined on A by $d(a) = D(a) + \lambda a + \frac{1}{2}\mu(a)$ satisfies

$$d(ab) - (d(a)b + ad(b)) \in P$$

for all $a, b \in A$ and therefore $Q_P d$ is a derivation from A to A/P. Indeed, the identity (4) in that proof becomes

$$d(ab + ba) - (d(a)b + d(b)a + ad(b) + bd(a) + \rho(a, b)) \in P$$

for all $a, b \in A$, for a suitable symmetric bilinear functional ρ on $A \times A$. The identity (7) now becomes

$$\rho(a, a)([ab, c] + [ba, c]) - \rho(a, b)[a^2, c] - \rho(a, a^2)[b, c] + (2\rho(a^2, b) - \rho(a, ab + ba))[a, c] \in P$$

for all $a, b, c \in A$. In particular $\rho(a, a)[a^2, c] - \rho(a, a^2)[a, c] \in P$ which gives $\rho(a, a)[[a^2, c], [a, c]] \in P$ for all $a, c \in A$. The arguments used in the proof of [1, Theorems 2 and 4] apply to this situation and it may be concluded that $\rho(a, b) = 0$ for all $a, b \in A$. Consequently, $d(a \cdot b) - (d(a) \cdot b + a \cdot d(b)) \in P$ for all $a, b \in A$, where $a \cdot b = \frac{1}{2}(ab + ba)$. By the same method as at the end of the proof of [1, Theorem 4] we get the relation $[a, b]r(d(ab) - d(a)b - ad(b))s[a, b] \in P$ for all $a, b, r, s \in A$, which yields the desired conclusion, since A/P is not commutative.

Our next goal is to prove that $\mathcal{S}(d) \subset P$. To this end we set an infinite-dimensional complex irreducible Banach left A-module X such that $P = \{a \in A: aX = 0\}$. We apply the construction in [4, Theorem 2.2] to get sequences $\{a_n\}$ in A and $\{x_n\}$ in X such that

$a_n \cdots a_1 x_n \neq 0$ and $a_{n+1}a_n \cdots a_1 x_n = 0$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we define the continuous linear operators $T_n(a) = aa_n$ from A into itself, $R_n(a+P) = ax_n$ from A/P into X, and $S_n(a+P) = aa_n + P$ from A/P into itself. It is a simple matter to verify that $R_n Q_P d T_1 \cdots T_m$ is continuous for $m > n$. Lemma 1 shows that $R_n Q_P d T_1 \cdots T_n$ is continuous for some $n \in \mathbb{N}$. On the other hand, it is immediate that $Q_P d T_1 \cdots T_n - S_1 \cdots S_n Q_P d$ is continuous and therefore $R_n S_1 \cdots S_n Q_P d$ is continuous. Consequently,

$$0 = R_n S_1 \cdots S_n \mathcal{S}(Q_P d) = \mathcal{S}(Q_P d) a_n \cdots a_1 x_n.$$

Since $\mathcal{S}(Q_P d)$ is easily seen to be a two-sided ideal of A/P, we see that

$$\mathcal{S}(Q_P d)X = \mathcal{S}(Q_P d)(A/P)a_n \cdots a_1 x_n \subset \mathcal{S}(Q_P d)a_n \cdots a_1 x_n = 0.$$

Hence $\mathcal{S}(Q_P d) = 0$ and therefore $\mathcal{S}(d) \subset P$.

For every $a \in A$, we have $\mathrm{ad}ad = \mathrm{ad}aD + \lambda \mathrm{ad}a$ and [9, Proposition 6.1.9(c)] shows that $\mathcal{S}(\mathrm{ad}aD) = \overline{\mathrm{ad}a(\mathcal{S}(D))}$ and $\mathcal{S}(\mathrm{ad}ad) = \overline{\mathrm{ad}a(\mathcal{S}(d))}$. Therefore

$$[a, \mathcal{S}(D)] \subset \overline{\mathrm{ad}a(\mathcal{S}(D))} = \mathcal{S}(\mathrm{ad}aD) = \mathcal{S}(\mathrm{ad}ad) = \overline{\mathrm{ad}a(\mathcal{S}(d))} \subset P$$

for all $a \in P$. □

Lemma 3. $\mathcal{S}(D) \subset \mathcal{Z}(A)$.

Proof. Suppose that the result fails. Then the set $\mathcal{P}$ of those primitive ideals P of A for which $[A, \mathcal{S}(D)] \not\subset P$ is non empty. According to Lemma 2 each $P \in \mathcal{P}$ has finite codimension and therefore A/P is simple. Moreover it is obvious that $\dim A/P > 1$. On account of [3, Theorem 2 and Corollary 1] $Q_P(\mathcal{S}(D)) = A/P$ for every $P \in \mathcal{P}$. Let I_0 denote the intersection of all the primitive ideals of A for which $[\mathcal{S}(D), A] \subset P$.

Take $P_1 \in \mathcal{P}$. Since $I_0 \not\subset P_1$ and A/P_1 is simple, it follows that $Q_{P_1}(I_0) = A/P_1$. Since A/P_1 is not commutative we can choose $a_1 \in I_0$ such that $[a_1, A] \not\subset P_1$.

Assume that $P_1, \cdots, P_n$ and $a_1, \cdots, a_n$ have been chosen satisfying the following conditions:

(i) $P_k \in \mathcal{P}$,

(ii) $a_k \in I_{k-1}$, where $I_k = I_0 \cap P_1 \cap \cdots \cap P_k$,

(iii) $\mathrm{ad}a_1 \cdots \mathrm{ad}a_k(A) \not\subset P_k$,

for $k = 1, \cdots, n$. We claim that there exists $P_{n+1} \in \mathcal{P}$ such that $I_n \not\subset P_{n+1}$ and $\mathrm{ad}a_1 \cdots \mathrm{ad}a_n(A) \not\subset P_{n+1}$. If the claim were false, there would be

$$\mathrm{ad}a_1 \cdots \mathrm{ad}a_n(A) \subset \bigcap_{P \in \mathcal{P}, I_n \not\subset P} P.$$

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Let $b_1 \in A$ such that $b_2 = \mathrm{ad}a_1 \cdots \mathrm{ad}a_n(b_1) \notin P_n$ and note that $b_2 \in \bigcap_{P \in \mathcal{P}, I_n \not\subset P} P$. Let X be a finite-dimensional irreducible Banach left A-module with $P_n = \{a \in A : aX = 0\}$. By the Jacobson density theorem it is a simple matter to show that there exists $b_3 \in A$ such that $\dim b_3 b_2 X = 1$ and $(b_3 b_2)^2 X = 0$. Set $c = b_3 b_2$. Since $\dim cX = 1$, there exists a linear functional f on A such that $(cac - f(a)c)X = 0$ for all $a \in A$. Consequently, we have

$$cac - f(a)c \in I_{n-1} \cap P_n \cap \left(\bigcap_{P \in \mathcal{P}, I_n \not\subset P} P \right) = \mathrm{Rad}(A) = 0.$$

Hence $\dim cAc < \infty$. Since $c^2 \in P_n$, we have

$$c^2 \in I_n \cap \left(\bigcap_{P \in \mathcal{P}, I_n \not\subset P} P \right) = \mathrm{Rad}(A) = 0.$$

Consequently, $(\mathrm{ad}c)^2 a = -2cac$ for all $a \in A$. Since $\dim(\mathrm{ad}c)^2(A) < \infty$, we see that $D(\mathrm{ad}c)^2$ is continuous. On the other hand, it is immediate that $D(\mathrm{ad}c)^2 - (\mathrm{ad}c)^2 D$ is continuous. Therefore $(\mathrm{ad}c)^2 D$ is continuous and hence $(\mathrm{ad}c)^2(\mathcal{S}(D)) = 0$. Thus $c\mathcal{S}(D)c = 0$ and

$$0 = Q_{P_n}(c) Q_{P_n}(\mathcal{S}(D)) Q_{P_n}(c) = Q_{P_n}(c)(A/P_n) Q_{P_n}(c) = 0.$$

From this it follows that $Q_{P_n}(c) = 0$, a contradiction. Choose P_{n+1} with the claimed properties. On account of [**1**, Corollary 1] the linear subspace of A/P_{n+1} generated by $\{\mathrm{ad}Q_{P_n}(a)(Q_{P_{n+1}}(A)) : a \in A\}$ equals A/P_{n+1}. Since $Q_{P_{n+1}}(I_n) = A/P_{n+1}$ and $\mathrm{ad}a_1 \cdots \mathrm{ad}a_n(A) \not\subset P_{n+1}$, we conclude that there is $a_{n+1} \in I_n$ such that $\mathrm{ad}a_1 \cdots \mathrm{ad}a_n \mathrm{ad}a_{n+1}(A) \not\subset P_{n+1}$.

Note that, for all $m, n \in \mathbb{N}$,

$$\begin{aligned} Q_{P_n} D \mathrm{ad}a_1 \cdots \mathrm{ad}a_m = {} & \sum_{k=1}^{m} \mathrm{ad}Q_{P_n}(a_1) \cdots \mathrm{ad}Q_{P_n}(D(a_k)) \cdots \mathrm{ad}Q_{P_n}(a_m) \\ & + \mathrm{ad}Q_{P_n}(a_1) \cdots \mathrm{ad}Q_{P_n}(a_m) Q_{P_n} D. \end{aligned}$$

Since $\sum_{k=1}^{m} \mathrm{ad}Q_{P_n}(a_1) \cdots \mathrm{ad}Q_{P_n}(D(a_k)) \cdots \mathrm{ad}Q_{P_n}(a_m)$ is continuous and $Q_{P_n}(a_m) = 0$ if $m > n$, Lemma 1 shows that $Q_{P_n} D \mathrm{ad}a_1 \cdots \mathrm{ad}a_n$ and therefore $\mathrm{ad}Q_{P_n}(a_1) \cdots \mathrm{ad}Q_{P_n}(a_n) Q_{P_n} D$ are continuous for some $n \in \mathbb{N}$. Accordingly, we have

$$0 = \mathrm{ad}Q_{P_n}(a_1) \cdots \mathrm{ad}Q_{P_n}(a_n) Q_{P_n}(\mathcal{S}(D)) = \mathrm{ad}Q_{P_n}(a_1) \cdots \mathrm{ad}Q_{P_n}(a_n)(A/P_n)$$

which contradicts the choice of a_n and P_n. □

Theorem. *Let D be a Lie derivation on a semisimple Banach algebra A. Then $\mathcal{S}(D) \subset \mathcal{Z}(A)$. Accordingly, D is continuous if $\mathcal{Z}(A) = 0$.*

Proof. If A is a complex Banach algebra without unit, then its unitization A_1 is a unital semisimple complex Banach algebra and we extend D to a Lie derivation on A_1 by defining $D_1(1)=0$. The preceding lemma shows that $\mathcal{S}(D_1) \subset \mathcal{Z}(A_1)$. On the other hand, we have $\mathcal{S}(D) \subset \mathcal{S}(D_1)$ and therefore $\mathcal{S}(D) \subset A \cap \mathcal{Z}(A_1) = \mathcal{Z}(A)$.

If A is a real Banach algebra, then we consider its complexification $A_{\mathbf{C}}$ and we extend D in the obvious way to a Lie derivation $D_{\mathbf{C}}$ of $A_{\mathbf{C}}$. From what has already been proved, we conclude that

$$\mathcal{S}(D) \subset A \cap \mathcal{S}(D_{\mathbf{C}}) \subset A \cap \mathcal{Z}(A_{\mathbf{C}}) = \mathcal{Z}(A). \qquad \square$$

Next we show a discontinuous derivation on a semisimple Banach algebra whose centre is $\mathbb{C}$.

Example. Let A be the Banach algebra of the Hilbert-Schmidt operators on an infinite-dimensional complex Hilbert space with an identity adjoined. A is semisimple, $\mathcal{Z}(A) = \mathbb{C}$, and $[a, b]$ is a trace class operator whenever a, b lie in A. Therefore $[A, A]$ is not closed in A and hence there exists a discontinuous linear functional f on A whose kernel contains $[A, A]$. The discontinuous linear operator D from A into itself defined by $D(a) = f(a)1$ is easily seen to be a Lie derivation of A.

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