7

Closed time path effective action for gauge theories

In this chapter we treat out-of-equilibrium behavior of gauge fields, particularly of the nonabelian kind. This is a broad topic, so we will only discuss some specific points.

Overall, we may distinguish two sets of features that make problems involving gauge fields different from those where only "matter" fields are present. On the one hand, there are "technical" differences associated with the fact that problems involving gauge fields usually abound with massless degrees of freedom. An important example is the so-called "hard thermal loop" problem, which is discussed in Chapter 10. We also consider "technical" difficulties associated with a particular symmetry breaking pattern or with the property of confinement, which clearly has a strong impact on the nonequilibrium phenomenology of QCD. Because of the rich variety of behavior, these problems are best treated on a case by case basis. In Chapter 14, for example, we give a brief account of nonequilibrium phenomena in relativistic heavy ion collisions.

On the other hand, there is an intrinsic difference between gauge and nongauge theories, coming from the fact that the "natural" description of the former in terms of spacetime fields is redundant. For example, the most efficient description of the Maxwell field is in terms of the potential 4-vector, but many different 4-vectors describe the same physical electromagnetic field. There is an intrinsic ambiguity in the equations of motion of the theory, which do not determine the evolution completely. At the same time there are restrictions on our freedom to choose Cauchy data for the physical fields; we say that the theory is "constrained."

In the quantum theory, the redundancy in the field variables is reflected in the fact that the "naive" Hilbert space of the theory is overlarge. The constraints of the classical theory become restrictions that the physical states must satisfy; these restrictions ensure that physical states respond to the physical part of the redundant field operators, but are impervious to the gauge part.

However, to get rid of ambiguities and constraints by reducing the theory to operators associated with measurable observables acting on physical states is, if at all possible, overwhelmingly inconvenient. These difficulties can be dealt with by formulating the theory in terms of the redundant, but natural, field variables. The subject of this chapter is to explore how nonequilibrium gauge theories are different from nongauge ones, because of this fundamental ambiguity.

The description of a gauge-invariant theory within quantum field theory techniques usually involves eliminating the gauge freedom by imposing "gauge fixing" conditions. These conditions are associated with new parameters, whose choice is arbitrary. To ensure the equivalence between the "gauge fixed" and the original theory, new fields (the so-called "ghost" fields) must be included, sometimes even with the wrong spin/statistics connection. It is expected that the predictions of the theory with respect to physical observables are independent of these manipulations: they do not change either if the fields are subject to a gauge transformation (gauge invariance) or if we change the chosen gauge fixing conditions (gauge independence). Nevertheless, oftentimes one is interested in computing objects that are not quite observable, such as a gluon correlation function. Then neither gauge invariance nor independence are guaranteed, and it becomes an important issue to decide which parts of the result really say something about the theory, and which are merely artifacts [Nie75, KobKun89, Bai92, KoKuRe91, GerReb03, ArrSmi02].

Our most important tool to investigate this issue is the observation that the constraints of the theory result in a number of restrictions on the structure of the Green functions such as vertices and propagators. These restrictions take the form of identities linking Green functions of different orders, the so-called Takahashi–Ward and Slavnov–Taylor identities. As it is often the case, there are two possible ways of looking at these identities. On the one hand, they are a check on the quality of a given approach to the problem: if important identities are violated (e.g. a field which ought to be massless is assigned a mass) then the approach is no good. On the other hand, these identities say things about the structure of the theory which may be used to motivate or to improve on a given approach (for example, by using an ansatz for the vertex functions which guarantees that the identities hold to a given order in perturbation theory).

The subject of gauge theory quantization is extremely rich and varied [HenTei92], and the addition of the nonequilibrium dimension only makes it even more so. Within the bounds of a single chapter, only a few of its avenues may be explored. Following the perspective developed in the early chapters, we shall adhere to the approach whereby nonequilibrium dynamics is followed through the evolution of the low-order Green functions. As in Chapter 6, we shall derive the dynamics of these Green functions self-consistently from a suitable closed time path action functional. As a matter of fact, if one is content to make a gauge choice from the start (e.g. to work within the "longitudinal" gauge), then the theory ceases to be "gauge" and the formalism from the earlier chapters may be applied straightforwardly [Gei96, Gei97, Gei99, Son97]. The problem is then whether any given result is valid generally, or limited to the given gauge choice. Our perspective in this chapter shall be the opposite, namely, leaving gauge choices as open and explicit as possible, and trying to learn about the deep structure of the theory from this very same freedom.

More concretely, our goal is to develop the 2PI approach to nonequilibrium gauge theories, as typical of approaches based on the evaluation of Green functions [Mot03, CaKuZa03, KraReb04]. We shall not discuss higher nPI effective actions, for which we refer the reader to the literature [Ber04a].

To set the stage for a discussion of the 2PIEA, we must begin by considering the essentials of the path integral quantization of gauge theories, and in particular how we set the initial conditions for gauge fields in a statistical state (such as a finite temperature one). For reasons of space and clarity we will restrict ourselves to Yang–Mills and to nonlinear abelian theories such as QED and SQED. We shall make no explicit attempt to discuss gravity, form fields or string theories [Wei00].

These self-imposed limitations in our aims here are correlated with some necessary technical choices. We shall discuss only the path integral Fadeev–Popov quantization of gauge theories. Although we shall use Becchi–Rouet–Stora–Tyutin (BRST) invariance at several stages, we shall not apply methods such as BRST or Batalin–Vilkovisky quantization, which really come on their own only in more demanding applications [Wei96]. We are deeply indebted to DeWitt's insights [DeW64, DeW79] and shall use his notation, but we shall not use the gauge-independent formulation of DeWitt and Vilkovisky [Vil84, DeW87] (on this subject, see the discussion in [Reb87]), nor more recent developments by DeWitt and collaborators [DeWMol98].

When gauge symmetries are unbroken, there are no preferred directions in gauge space, and all background fields will vanish identically. Therefore, the only degrees of freedom in the 2PI formalism shall be the propagators or two-point functions. Also, there will be no need to distinguish between the usual and the DeWitt–Abbott gauge invariant EA [DeW81, Abb81, Hart93, Alx99], nor to introduce gauge fixing conditions appropriate to the study of broken gauge theories [Wei96]. We shall only assume that the gauge fixing condition is linear on the quantum fields. On the other hand, we shall be completely general regarding group structure, matter content, (linear) gauge fixing condition and gauge fixing parameter.

As a word of caution, let us observe that symmetries that hold for the exact theory may be broken when the exact 2PI effective action is replaced by an approximated functional. In our case, this will manifest in violations of the Takahashi–Ward or Slavnov–Taylor identities. Usually this problem may be kept under control by working to a high enough order, by going over to a nPI approach with a large enough n, or simply by being careful about the approximations one uses. This problem is not actually exclusive of gauge theories; we will find it again when we attempt to make a consistent field theory of Bose–Einstein condensates, where the symmetry in question is the possibility of adding a phase to the condensate wavefunction. We shall discuss it in more detail in that simpler context, and refer the reader to the literature regarding gauge field theories [ReiSer06].

This chapter contains three sections besides this introduction. Section 7.1 summarizes the main results concerning the path integral quantization of gauge theories to be used in the following, including BRST invariance, the characterization of physical states and how to deal with nonvacuum initial conditions. We shall adopt the Kugo–Hata formalism, where ghost propagators acquire statistical corrections proper of a Bose field. Section 7.2 introduces the 2PIEA for gauge theories. Section 7.3 investigates the two main features of gauge theories which have no equivalent in their "normal" counterparts, namely, the issue of gauge dependence and the possibility of using gauge invariance arguments to investigate the structure of the theory. To develop our arguments, we shall introduce first the powerful tool of the Zinn-Justin identity, and then proceed to discuss these two problems in turn. The results we shall derive are well known in equilibrium settings; our goal is to express them in a way that holds even off-equilibrium.

We assume some familiarity with Grassmann calculus. For more details, we refer the reader to the monographs by Berezin [Ber66], DeWitt [DeW84] and Negele and Orland [NegOrl98].

7.1 Path integral quantization of gauge theories – an overview

7.1.1 Gauge theories

Due to the complexity of the subject, it becomes necessary to adopt a highly compressed notation. For starters, we shall do without explicit spacetime dependence. They may be thought of as so many "continuous" indices to be added to the string of discrete indices identifying each field within the theory.

A gauge theory contains "matter" fields ψ such that there are local (unitary) transformations g which are symmetries of the theory. The g's form a nonabelian (simple) group. Infinitesimal transformations may be written as $g = \exp[i\varepsilon]$, where the Hermitian matrix ε may be expanded as a linear combination of "generators" $\varepsilon = \varepsilon^A T_A$. The generators form a closed algebra under commutation

$$[T_A, T_B] = iC^C_{AB} T_C \tag{7.1}$$

The structure constants C^C_{AB} are antisymmetric on A, B and satisfy the Jacobi identity.

Gauge invariance of kinetic terms within the Lagrangian means that derivatives are written in terms of the gauge covariant derivative operator $D_\mu = \partial_\mu - iA_\mu$. The connection $A_\mu = A_{\mu A} T^A$ transforms upon an infinitesimal gauge transformation as

$$A_\mu \to A_\mu + D_\mu \varepsilon \tag{7.2}$$

where

$$D_\mu \varepsilon = \partial_\mu \varepsilon - i[A_\mu, \varepsilon] \tag{7.3}$$

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Covariant derivatives do not commute, but their commutator contains no derivatives

$$[D_\mu, D_\nu] = -iF_{\mu\nu} \tag{7.4}$$

where F is the field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \tag{7.5}$$

Upon a gauge transformation

$$F_{\mu\nu} \to F_{\mu\nu} + i\,[\varepsilon, F_{\mu\nu}] \tag{7.6}$$

therefore the object

$$\mathcal{L} = \frac{-1}{4g^2}\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu} \tag{7.7}$$

is gauge invariant. This is the classical Lagrangian density for the gauge fields, g being the coupling constant. The total action is $S = S_0 + S_m$, where

$$S_0 = \int d^dx\,\left(\frac{-1}{4g^2}\right)\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu} \tag{7.8}$$

and S_m is the gauge-invariant action for the matter fields.

We may drop the distinction between gauge and matter fields, and consider a theory described by a string of fields ϕ^α invariant under infinitesimal transformations

$$\delta\phi^\alpha = T^\alpha_A\,[\phi]\,\varepsilon^A \tag{7.9}$$

The commutation rules are the statement that the commutator of two gauge transforms is also a gauge transform, namely

$$\frac{\delta T^\alpha_A\,[\phi]}{\delta\phi^\beta}T^\beta_B\,[\phi] - \frac{\delta T^\alpha_B\,[\phi]}{\delta\phi^\beta}T^\beta_A\,[\phi] = T^\alpha_C\,[\phi]\,C^C{}_{AB} \tag{7.10}$$

The classical equations of motion read

$$\frac{\delta S}{\delta\phi^\alpha} = 0 \tag{7.11}$$

and because of gauge invariance we must have the identity

$$\frac{\delta S}{\delta\phi^\alpha}T^\alpha_A\,[\phi] = 0 \tag{7.12}$$

7.1.2 Gauge symmetries and constraints

One important point regarding gauge theories is that a gauge theory is necessarily a constrained theory, and to a large extent vice versa [Dir50, Dir58b, BesKur90, Sun82].

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To understand the reason why a gauge theory must have constraints, we observe that the dynamical information on the theory is carried by the canonical variables ϕ^α and their canonical momenta π^α. The information necessary to evolve these degrees of freedom in time are again the ϕ^α and their time derivatives $\dot{\phi}^\alpha$. Now the existence of gauge freedom means that knowledge of the canonical variables does not determine the evolution (the $\dot{\phi}^\alpha$ are determined only up to a gauge transformation). Therefore the relationship of the $\dot{\phi}^\alpha$ to the π^α is many-to-one. This relationship is usually given through a Lagrangian density $\mathcal{L}$ (for example, as in equation (7.7)). In the simplest case the Lagrangian is quadratic in the velocities, and

$$\pi_\alpha = \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^\alpha \partial \dot{\phi}^\beta} \dot{\phi}^\beta \tag{7.13}$$

The ambiguity in the $\dot{\phi}^\alpha$ means that the operator at the left must have null directions

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^\alpha \partial \dot{\phi}^\beta} T_A^\beta [\phi] = 0 \tag{7.14}$$

We must therefore have a primary constraint

$$T_A^\beta [\phi] \, \pi_\beta = 0 \tag{7.15}$$

and since the primary constraint must hold over time we must also have the secondary constraint

$$\frac{d}{dt} \left[T_A^\beta [\phi] \, \pi_\beta \right] = 0 \tag{7.16}$$

Observe that each gauge freedom engenders two constraints.

Vice versa, assume a theory with fields ϕ and π and Hamiltonian H subject to a constraint $N = 0$. To enforce this constraint introduce a Lagrange multiplier λ and a new Hamiltonian $H + \lambda N$. The momentum Π conjugate to λ vanishes, which is our primary constraint. The secondary constraint is $N = 0$. The canonical equations of motion do not determine the evolution of (ϕ, λ) uniquely; the remaining freedom may be understood as resulting from gauge transformations generated by $\varepsilon N + \dot{\varepsilon} \Pi$, where ε is the gauge parameter.

7.1.3 The measure of integration

The main point in the path integral approach to the quantization of gauge theories is that the measure of integration is highly nontrivial, since it must count only physical histories of the field, each one being represented by many histories within the path integral. To motivate the measure of integration which does the trick let us look into the computation of the vacuum-to-vacuum amplitude.

In the quantum theory, we expect the vacuum-to-vacuum amplitude to be given by the *in–out* path integral

$$Z = \int D\phi \; e^{iS} \tag{7.17}$$

However this integral counts every history, and that means that each *physical* history is counted many times over. Not surprisingly, it is generally ill defined.

To cure this problem, let f^A be functionals in history space which are not gauge invariant. This means that if we begin from a history ϕ^α confined to the surface $f^A[\phi^\alpha]=0$, then any infinitesimal gauge transform will take us out of that surface, unless the gauge transform is the trivial one $\varepsilon^A=0$. In other words,

$$\frac{\delta f^A}{\delta\phi^\alpha}T^\alpha_B[\phi]\,\varepsilon^B=0\Rightarrow\varepsilon^A=0 \tag{7.18}$$

which requires

$$\mathrm{Det}\left[\frac{\delta f^A}{\delta\phi^\alpha}T^\alpha_B[\phi]\right]\neq 0 \tag{7.19}$$

Now let us call $\phi[\varepsilon]$ the result of applying a gauge transform parameterized by ε to the field configuration ϕ. Then we have the identity (which is an elaborate way of saying that a Dirac delta integrates to 1)

$$\int D\varepsilon\ \mathrm{Det}\left[\frac{\delta f^A}{\delta\phi^\alpha}[\phi[\varepsilon]]\,T^\alpha_B[\phi[\varepsilon]]\right]\ \delta\left[f^A[\phi[\varepsilon]]-C^A\right]=1 \tag{7.20}$$

where C^A may be anything, and by inserting this representation of the identity in the vacuum persistence amplitude we can write

$$Z=\int D\varepsilon\int D\phi\ \mathrm{Det}\left[\frac{\delta f^A}{\delta\phi^\alpha}[\phi[\varepsilon]]\,T^\alpha_B[\phi[\varepsilon]]\right]\ \delta\left[f^A[\phi[\varepsilon]]-C^A\right]\ e^{iS[\phi]} \tag{7.21}$$

Of course, $S[\phi]=S[\phi[\varepsilon]]$, and

$$D\phi[\varepsilon]=D\phi\ \left\{1+\varepsilon^A\mathrm{Tr}\frac{\delta T^\alpha_A[\phi]}{\delta\phi^\beta}\right\} \tag{7.22}$$

so, provided

$$\mathrm{Tr}\frac{\delta T^\alpha_A[\phi]}{\delta\phi^\beta}=0 \tag{7.23}$$

we find, up to a constant

$$Z=\int D\phi\ \mathrm{Det}\left[\frac{\delta f^A}{\delta\phi^\alpha}[\phi]\,T^\alpha_B[\phi]\right]\ \delta\left[f^A[\phi]-C^A\right]\ e^{iS[\phi]} \tag{7.24}$$

Since the C^A are arbitrary, any average over different choices will do too. For example, given a suitable metric we may take the Gaussian average

$$\int DC^A\ e^{-(i/2\xi)C^AC_A} \tag{7.25}$$

Integrating over C^A and after a Fourier transform we find

$$Z=\int D\phi Dh_A\ \mathrm{Det}\left[\frac{\delta f^A}{\delta\phi^\alpha}[\phi]\,T^\alpha_B[\phi]\right]\ \exp\left\{i\left[S[\phi]+h_Af^A[\phi]+\frac{\xi}{2}h^Ah_A\right]\right\} \tag{7.26}$$

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where h_A is the Nakanishi–Lautrup (N-L) field [Nak66] and ξ is the gauge fixing parameter.

We may write the determinant as a functional integral

$$Z = \int D\omega^B D\chi_A D\phi Dh_A \exp\left\{i\left[S[\phi] + h_A f^A[\phi] + \frac{\xi}{2}h^A h_A + i\chi_A\Delta^A\right]\right\} \tag{7.27}$$

$$\Delta^A = \frac{\delta f^A}{\delta\phi^\alpha}[\phi]\, T^\alpha_B[\phi]\,\omega^B \tag{7.28}$$

The ω^B, χ_A are *independent* c-number Grassmann variables, namely the ghost and anti-ghost fields, respectively. Following Kugo and Ojima [KugOji79, HatKug80, Oji81], and unlike Weinberg [Wei96], we have included a factor of i in the ghost Lagrangian, which is consistent with taking the ghosts as formally "Hermitian" and demanding the action to be "real." We assign"ghost number" 1 to ω^B, and -1 to χ_A.

7.1.4 BRST invariance

Our goal is to investigate how to formulate a path integral when the initial state is not a vacuum. An important resource in this discussion is the observation that, after breaking the original gauge symmetry by adding gauge fixing conditions and ghosts, the resulting theory has a higher symmetry, the so-called BRST invariance.

We may regard the functional

$$S_{\text{eff}} = S[\phi] + h_A f^A[\phi] + \frac{\xi}{2}h^A h_A + i\chi_A\Delta^A \tag{7.29}$$

as the action of a new theory, built from the original by adding the N-L, ghost and anti-ghost fields. By construction, this action is not gauge invariant in the original sense. However, let us consider a gauge transform parameterized by $\theta\omega^B$, where θ is an anticommuting "constant," namely

$$\delta\phi^\alpha = \theta T^\alpha_A[\phi]\,\omega^A \tag{7.30}$$

Observe that, keeping the other fields invariant for the time being

$$\delta f^A[\phi] = \theta\Delta^A \tag{7.31}$$

$$\delta\Delta^A = \theta f^A_{,\alpha\beta} T^\alpha_B[\phi]\, T^\beta_C[\phi]\,\omega^B\omega^C + \theta f^A_{,\alpha} T^\alpha_B[\phi]_{,\beta}\, T^\beta_C[\phi]\,\omega^C\omega^B \tag{7.32}$$

Since the ghosts are Grassmann, the first term vanishes, and the second may be written in terms of the commutation relations (7.10), whereby this becomes

$$\delta\Delta^A = \frac{-1}{2}\theta f^A_{,\alpha} T^\alpha_D[\phi]\, C^D_{BC}\omega^B\omega^C \tag{7.33}$$

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These results suggest extending the definition of the transformation to

$$\delta h_A = 0 \tag{7.34}$$

$$\delta \chi_A = i\theta h_A \tag{7.35}$$

$$\delta \omega^D = \frac{1}{2}\theta C^D_{BC}\omega^B \omega^C \tag{7.36}$$

Then S_{eff} is invariant under this "BRST" transformation. Let us define the operator Ω

$$\Omega[X] = \frac{d}{d\theta}\delta X \tag{7.37}$$

The operator Ω increases the "ghost number" by one. It is nilpotent ($\Omega^2 = 0$, see [Wei96]). Also, observe that

$$S_{\text{eff}} = S_0 + \Omega[F] \tag{7.38}$$

where

$$S_0 = S[\phi] \tag{7.39}$$

is BRST invariant, and F is the so-called "gauge fixing fermion"

$$F = -i\chi_A \left\{ f^A[\phi] + \frac{1}{2}\xi h^A \right\} \tag{7.40}$$

Recall that

$$\Omega[F] = -i\left(\Omega[\chi_A]\left\{ f^A[\phi] + \frac{1}{2}\xi h^A \right\} - \chi_A \Omega\left[f^A[\phi]\right] \right) \tag{7.41}$$

Also, observe that, *provided* $C^A_{AB} \equiv 0$ the functional volume element is also BRST invariant.

It follows from the above that any gauge fixing dependence (that is, dependence on the choice of the gauge fixing condition f^A, gauge fixing parameter ξ or the metric used to raise indices in the N-L field) may only come from a dependence upon changes in the functional F. Any such change induces a perturbation

$$\delta Z = i\int D\omega^B D\chi_A D\phi Dh_A \,\Omega[\delta F]\exp\{iS_{\text{eff}}\} \tag{7.42}$$

Now, call X^r the different fields in the theory. Then

$$\Omega[\delta F] = (-1)^{g_r+1}\,\delta F_{,r}\Omega[X^r] \tag{7.43}$$

where g_r is the corresponding ghost number. Integrating by parts (see [GoPaSa95]), and *provided the surface term vanishes*, we get

$$\delta Z = i\int D\omega^B D\chi_A D\phi Dh_A \,\delta F\left\{ \frac{\delta}{\delta X^r}\exp\{iS_{\text{eff}}\}\,\Omega[X^r] \right\} \tag{7.44}$$

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But the brackets vanish, because of BRST invariance of S_{eff} and because $\Omega\left[X^r\right]$ is divergence-free. Therefore the *physicality condition* is that the flux of any vector pointing in the direction of $\Omega\left[X^r\right]$ over the boundary of the space of field configurations must vanish.

This shows by the way that F could be any expression of ghost number -1, since S_{eff} must have ghost number zero.

7.1.5 Physical states

BRST invariance allows us to give a simple criterion for physical states. In this section, we shall consider the concrete case where $\phi^\alpha = A^A_\mu$, $f^A_{,\alpha} = \delta^A_B \partial_\mu$ and $T^\alpha_B = \delta^A_B\partial_\mu + C^A_{CB}A^C_\mu$. We can write S_{eff} explicitly:

$$S_{\text{eff}} = \int d^4x \left\{ \frac{-1}{4g^2} F^{A\mu\nu}F_{A\mu\nu} - \partial_\mu h_A A^{\mu A} + \frac{\xi}{2} h^A h_A \right.$$
$$\left. - i\partial_\mu \chi_A \left[\delta^A_B \partial_\mu + iC^A_{CB}A^C_\mu\right]\omega^B \right\} \tag{7.45}$$

If we take A_{Aa} $(a = 1, 2, 3)$, h_A, χ_A and ω^A as canonical variables, then we may identify the corresponding momenta [KugOji79, HatKug80, Oji81]

$$\begin{aligned} p^{Aa}_\phi &= \frac{1}{g^2}F^{Aa0} \\ p^A_h &= -A^{A0} \\ p^A_\chi &= -i\left[\delta^A_B\partial_0 + C^A_{CB}A^C_0\right]\omega^B \\ p_{\omega A} &= i\partial_0\chi_A \end{aligned} \tag{7.46}$$

and impose the ETCCRs

$$\left[p_{X^r}, X^s\right]_\mp = -i\delta^r_s \tag{7.47}$$

where we use anticommutators for ghost fields and momenta, and commutators for all other cases.

The BRST invariance of S_{eff} implies the conservation of the Noether current

$$j^\mu = \Omega\left[X^r\right] \frac{\delta L_{\text{eff}}}{\delta\partial_\mu X^r} \tag{7.48}$$

We define the BRST charge as

$$\Omega = \int d^3x\, \Omega\left[X^r\right] p_{X^r} \tag{7.49}$$

This is the generator of BRST transforms, since

$$\delta X^r = \theta\Omega\left[X^r\right] = i\left[\theta\Omega, X^r\right] \tag{7.50}$$

(Since θ is Grassmann, we use commutators throughout.) Then $\Omega^2 = 0$.

S_{eff} is also invariant upon the scale transformation

$$\omega^B \to e^{\lambda}\omega^B, \qquad \chi^B \to e^{-\lambda}\chi^B \tag{7.51}$$

The corresponding generator

$$Q = \int d^3x \, \left\{\omega^B p_{\omega B} - \chi_A p_\chi^A\right\} \tag{7.52}$$

is the *ghost charge*. Ghost charge is bosonic, so $[Q, Q] = 0$. On the other hand, Ω has ghost charge 1, so

$$i\,[Q, \Omega] = \Omega \tag{7.53}$$

Both Q and $\theta\Omega$ commute with the effective Hamiltonian.

We say that a state $|\alpha\rangle$ is BRST closed if $\Omega\,|\alpha\rangle = 0$ and BRST exact if there is a $|\beta\rangle$ such that $|\alpha\rangle = \Omega\,|\beta\rangle$. Since $\Omega^2 = 0$, an exact state is necessarily closed but there may be closed states that are not exact. Observables are BRST invariant, and so they commute with $\theta\Omega$. Physical states are also BRST invariant, therefore annihilated by Ω. Physical states differing by a BRST transform are physically indistinguishable, in the sense that they lead to the same matrix elements for all observables. We therefore introduce an equivalence relation among states, $|\alpha\rangle \approx |\beta\rangle$ if $|\alpha\rangle - |\beta\rangle$ is BRST exact. A physical state is a representative of an equivalence class of states which are closed but not exact.

7.1.6 Initial conditions for nonvacuum states

We shall now use the above characterization of physical states to introduce a simple way (due to Hata and Kugo) of introducing initial conditions for nonvacuum states in the path integral.

We need one more result from BRST theory, namely, there is an operator R such that

(a) if $|\alpha\rangle$ is exact, $|\alpha\rangle = \Omega\,|\theta\rangle$, then $R\,|\alpha\rangle \approx |\theta\rangle$;
(b) if $|\alpha\rangle$ is not exact, then $R\,|\alpha\rangle \approx 0$.

Given such an operator, then the projector P' orthogonal to the space of physical states has the form $P' = \{\Omega, R\}$. Indeed, if $|\alpha\rangle$ is physical, then it is closed (so $R\Omega\,|\alpha\rangle = 0$) but not exact (so $\Omega R\,|\alpha\rangle = 0$). On the other hand, if $|\alpha\rangle$ is not physical, it is either exact or not closed. If $|\alpha\rangle$ is exact, then $R\Omega\,|\alpha\rangle = 0$ but $\Omega R\,|\alpha\rangle \approx |\alpha\rangle$. If $|\alpha\rangle$ is not closed, then $\Omega R\,|\alpha\rangle = 0$ but $R\Omega\,|\alpha\rangle \approx |\alpha\rangle$.

We may now deal with the construction of statistical operators in gauge theories. In principle, a physical statistical operator should shield nonzero probabilities only for physical states, and so it should satisfy $\rho = P\rho = \rho P$, where P projects over the space of physical states, $P = 1 - P'$. This is a much stronger requirement than BRST invariance $[\Omega, \rho] = 0$. So, given a BRST invariant density

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matrix ρ, we ought to define the physical expectation value of any (BRST invariant) observable C as

$$\langle C \rangle_{\text{phys}} = \text{Tr}\,[P\rho C] \tag{7.54}$$

However, Kugo and Hata [KugOji79, HatKug80, Oji81] (KH) have shown that the same expectation values may be obtained by using the statistical operator $e^{-\pi Q}\rho$. The key to the argument is that the commutation relation $[iQ, \Omega] = \Omega$ implies that, if $|N\rangle$ is an eigenstate of iQ with eigenvalue N, then $\Omega\,|N\rangle$ has eigenvalue $N+1$. It follows that $\left\{e^{-\pi Q}, \Omega\right\} = 0$, since $e^{-\pi Q} = e^{i\pi(iQ)}$. We then find that, for any BRST invariant observable C

$$\langle C \rangle_{\text{phys}} = \text{Tr}\,[P\rho C] = \text{Tr}\left[Pe^{-\pi Q}\rho C\right] = \text{Tr}\left[e^{-\pi Q}\rho C\right] - \text{Tr}\left[\{\Omega, R\}\, e^{-\pi Q}\rho C\right] \tag{7.55}$$

We must show that the second term vanishes, and this follows from $\left\{e^{-\pi Q}, \Omega\right\} = 0$ and $[\Omega, \rho C] = 0$.

This suggests we define the expectation value $\langle C \rangle$ of any observable as $\langle C \rangle = \text{Tr}\left[e^{-\pi Q}\rho C\right]$. Of course, this agrees with the physical expectation value only if C is BRST invariant. For example, the partition function computed from $e^{-\pi Q}\rho$ agrees with the partition function defined by tracing only over physical states, but the generating functionals obtained by adding sources coupled to non-BRST invariant operators will in general be different.

The advantages of the Kugo–Hata ansatz are clearly seen by considering the form of the KMS theorem appropriate to the ghost propagator. Let us define

$$G^{ab}_{AB}\,(x, x') = \left\langle P\left[\chi^a_A\,(x)\,\omega^b_B\,(x')\right]\right\rangle \tag{7.56}$$

where P is the usual (CTP)-ordering operator. Then

$$G^{21}_{AB}\,(x, x') = \langle \chi_A\,(x)\,\omega_B\,(x') \rangle \tag{7.57}$$

$$G^{12}_{AB}\,(x, x') = -\,\langle \omega_B\,(x')\,\chi_A\,(x) \rangle \tag{7.58}$$

(observe the sign change, owing to the anticommuting character of the ghost fields). The Jordan propagator is defined as $G = G^{21} - G^{12}$.

Had we omitted the KH $e^{-\pi Q}$ factor, we would reason, given $\rho = e^{-\beta H}$,

$$\begin{aligned} G^{21}_{AB}\,(x, x') &\approx \text{Tr}\left[e^{-\beta H}\chi_A\,(x)\,\omega_B\,(x')\right] \\ &= \text{Tr}\left[\chi_A\,(x + i\beta)\, e^{-\beta H}\omega_B\,(x')\right] \\ &= -G^{12}_{AB}\,(x + i\beta, x') \end{aligned} \tag{7.59}$$

Therefore $G^{21}_{AB}\,(\omega) = -e^{\beta\omega}G^{12}_{AB}\,(\omega)$, leading to a Fermi–Dirac form of the thermal propagators. This reasoning is incorrect. The proper way is

$$G^{21}_{AB}\,(x, x') = \text{Tr}\left[e^{-\pi Q}\chi_A\,(x + i\beta)\, e^{-\beta H}\omega_B\,(x')\right] = G^{12}_{AB}\,(x + i\beta, x') \tag{7.60}$$

So $G^{21}_{AB}\,(\omega) = e^{\beta\omega}G^{12}_{AB}\,(\omega)$, which leads to the Bose–Einstein form.

The KH factor does not appear explicitly in the path integral representation; it only changes the boundary conditions on ghost fields from anti-periodic to periodic.

We conclude that in this formalism, unphysical degrees of freedom and ghosts get statistical corrections, both being of the Bose–Einstein form, in spite of the ghosts being fermions (for which reason ghost loops do get a minus sign). For an alternative formulation, see [LanReb92, LanReb93].

7.2 The 2PI formalism applied to gauge theories

7.2.1 The 2PI effective action

We can now move towards our real goal, namely, the application of the 2PI CTP formalism to gauge theories. We shall proceed with a fair amount of generality, only assuming that the gauge condition is linear, and that the gauge generators satisfy $T^\alpha_A[\phi] = T^\alpha_{0A} + T^\alpha_{1A\beta}\phi^\beta$. We shall develop the basic formulae in some detail, emphasizing the subtleties associated with having both normal and Grassmann degrees of freedom in the same theory.

The classical action is given by equation (7.29). To this we add sources coupled to the individual degrees of freedom and also to their products

$$X^r J_r + \frac{1}{2}X^r \mathbf{K}_{rs} X^s = j_\alpha x^\alpha + \theta^u \lambda_u + \frac{1}{2}\kappa_{\alpha\beta}x^\alpha x^\beta + \frac{1}{2}\sigma_{uv}\theta^u\theta^v + \theta^u\psi_{u\alpha}x^\alpha \tag{7.61}$$

where x^α represents the bosonic degrees of freedom (ϕ, h) and θ the Grassmann ones (ω, χ), and we introduce the definition $\mathbf{K}_{\alpha u} = -\mathbf{K}_{u\alpha}$. Observe that j, κ and σ are normal, while λ and ψ are Grassmann; σ is antisymmetric.

We therefore define the generating functional

$$e^{iW} = \int DX^r \exp\left\{i\left[S_{\text{eff}} + X^r J_r + \frac{1}{2}X^r \mathbf{K}_{rs} X^s\right]\right\} \tag{7.62}$$

The information about the initial state is implicit in the integration measure and will reappear only as an initial condition on the equations of motion. We find

$$W\frac{\overleftarrow{\delta}}{\delta J_r} = \bar{X}^r \tag{7.63}$$

$$W\frac{\overleftarrow{\delta}}{\delta \mathbf{K}_{rs}} = \frac{\theta^{sr}\theta^s}{2}\left[\bar{X}^r\bar{X}^s + \mathbf{G}^{rs}\right] \tag{7.64}$$

where we introduce the bookkeeping device $\theta^r = (-1)^{q_r}$, where q_r is the ghost charge of the corresponding field, and $\theta^{rs} = (-1)^{q_r q_s}$.

We define the Legendre transform

$$\Gamma = W - \bar{X}^r J_r - \frac{1}{2}\bar{X}^r \mathbf{K}_{rs}\bar{X}^s - \frac{\theta^{sr}\theta^s}{2}\mathbf{G}^{rs}\mathbf{K}_{rs} \tag{7.65}$$

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whereby

$$\frac{\delta}{\delta X^r}\Gamma = -J_r - \frac{1}{2}\mathbf{K}_{rs}\bar{X}^s - \frac{1}{2}\theta^r\bar{X}^s\mathbf{K}_{sr} \tag{7.66}$$

Now observe that $\mathbf{K}_{sr} = \theta^r\theta^s\theta^{rs}\mathbf{K}_{rs}$. In the end

$$\frac{\delta}{\delta X^r}\Gamma = -J_r - \mathbf{K}_{rs}\bar{X}^s \tag{7.67}$$

$$\frac{\delta}{\delta \mathbf{G}^{rs}}\Gamma = -\frac{\theta^{sr}\theta^s}{2}\mathbf{K}_{rs} \tag{7.68}$$

In order to evaluate the 2PIEA, we make the ansatz

$$\Gamma = \bar{S}\left[\bar{X}^r\right] + \frac{1}{2}\theta^{sr}\theta^s\mathbf{G}^{rs}\mathbf{S}_{rs} - \frac{i}{2}\ln\operatorname{sdet}\left[\mathbf{G}^{rs}\right] + \Gamma_2 - \frac{i}{2}\theta^s\mathbf{G}^{rs}\mathbf{G}^{-1}_{Rsr} \tag{7.69}$$

where

$$\mathbf{S}_{rs} = \left[\overrightarrow{\frac{\delta}{\delta\bar{X}^r}}\bar{S}\right]\overleftarrow{\frac{\delta}{\delta\bar{X}^s}} \tag{7.70}$$

and $\bar{S}$ is the classical action (7.29), evaluated at the background fields. The generating functional Γ_2 is the sum of 2PI vacuum bubbles in a theory with free action $i\mathbf{G}^{-1}_{Lrs}$ and interacting terms coming from the cubic and quartic terms in the development of $\bar{S}$ around the mean fields. In spite of appearances, the new term $\theta^s\mathbf{G}^{rs}\mathbf{G}^{-1}_{Rsr}$ is a constant. It may therefore be discarded.

7.2.2 The 2PI Schwinger–Dyson equations

Let us now investigate the 2PI Schwinger–Dyson equations

$$\frac{\delta}{\delta X^r}\Gamma = 0$$

$$\frac{\delta}{\delta \mathbf{G}^{rs}}\Gamma = 0 \tag{7.71}$$

From equation (7.69) we get

$$\frac{\delta}{\delta X^r}\bar{S}\left[\bar{X}^r\right] + \frac{1}{2}\theta^{pq}\theta^q\theta^{rp}\theta^{rq}\mathbf{G}^{pq}\frac{\delta}{\delta X^r}\mathbf{S}_{pq} + \frac{\delta}{\delta X^r}\Gamma_2 = 0$$

$$\theta^{sr}\theta^s\mathbf{S}_{rs} - i\theta^r\theta^{rs}\left(\mathbf{G}_R^{-1}\right)_{rs} + 2\frac{\delta}{\delta\mathbf{G}^{rs}}\Gamma_2 = 0 \tag{7.72}$$

The second set of equations may be rewritten as

$$\theta^{sr}\theta^s\mathbf{S}_{rs} - i\theta^r\left(\mathbf{G}_L^{-1}\right)_{sr} + 2\frac{\delta}{\delta\mathbf{G}^{rs}}\Gamma_2 = 0 \tag{7.73}$$

and finally as

$$\mathbf{S}_{rs} - i\left(\mathbf{G}_L^{-1}\right)_{rs} + 2\theta^{sr}\theta^s\frac{\delta}{\delta\mathbf{G}^{rs}}\Gamma_2 = 0 \tag{7.74}$$

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The classical action is given by equation (7.29). If we expand $X^r = \bar{X}^r + \delta X^r$, then the quadratic terms are

$$\bar{S}^{(2)} = S_0^{(2)}\left[\bar{\phi}, \delta\phi\right] + \delta h_A f_\alpha^A \delta\phi^\alpha + \frac{\xi}{2}\delta h^A \delta h_A + i\delta\chi_A f_\alpha^A T_B^\alpha \left[\bar{\phi}\right]\delta\omega^B$$
$$+ i\bar{\chi}_A f_\alpha^A T_{1B\beta}^\alpha \delta\phi^\beta \delta\omega^B + i\delta\chi_A f_\alpha^A T_{1B\beta}^\alpha \delta\phi^\beta \bar{\omega}^B \tag{7.75}$$

The cubic and quartic terms are

$$\bar{S}^{(3+)} = S_0^{(3+)}\left[\bar{\phi}, \delta\phi\right] + i\delta\chi_A f_\alpha^A T_{1B\beta}^\alpha \delta\phi^\beta \delta\omega^B \tag{7.76}$$

7.2.3 The reduced 2PI effective action

The introduction of the Nakanishi–Lautrup (N-L) field h has been useful to obtain a simple definition of the BRST transformation, but since it only appears quadratically in the action, there are no h field lines in Γ_2. To take advantage of this fact, it is convenient *not* to couple sources to the h field. In this way, Γ_2 is independent of the h field, and the respective variations are exact, namely

$$f_\alpha^A \bar{\phi}^\alpha + \xi \bar{h}^A = 0$$
$$\xi\delta_{AB} - i\left[\mathbf{G}_L^{-1}\right]_{hhAB} = 0$$
$$f_\alpha^A - i\left[\mathbf{G}_L^{-1}\right]_{h\phi\alpha}^A = 0$$
$$\left[\mathbf{G}_L^{-1}\right]_{h\omega B}^A = \left[\mathbf{G}_L^{-1}\right]_{h\chi B}^A = 0 \tag{7.77}$$

where a $L(R)$ superscript denotes a left (right) inverse. Moreover, from the N-L field being Gaussian

$$G_{hX}^{Ar} = \frac{-1}{\xi} f_\beta^A G_{\phi X}^{\beta r}, \qquad X = \phi, \chi, \omega \tag{7.78}$$

and

$$G_{hhA}^C = \frac{1}{\xi}\left[-f_{A\beta}G_{\phi h}^{\beta C} + i\delta_A^C\right] = \frac{1}{\xi}\left[\frac{1}{\xi} f_{A\beta} f_\gamma^C G_{\phi\phi}^{\beta\gamma} + i\delta_A^C\right] \tag{7.79}$$

We could use these formulae to actually eliminate the N-L field from the 2PIEA, thus obtaining a reduced effective action.

Let us explore the solutions to the equations of motion where all fields with nonzero ghost number vanish, i.e.

$$\bar{\omega} = \bar{\chi} = G_{\omega\omega} = G_{\chi\chi} = G_{\omega\phi} = G_{\omega h} = G_{\chi\phi} = G_{\chi h} = 0 \tag{7.80}$$

Since the effective action itself has zero ghost number, it cannot contain terms linear on any of the above, and therefore this condition is consistent with the equations of motion.

Given these conditions, we have, besides the equations determining the h propagators, the further equations

$$\left[\mathbf{G}^{-1}\right]_{\phi\phi\alpha\beta} G^{\beta\gamma}_{\phi\phi} + \left[\mathbf{G}^{-1}\right]_{\phi h\alpha B} G^{B\gamma}_{h\phi} = \delta^{\gamma}_{\alpha}$$

$$\left[\mathbf{G}^{-1}\right]_{\phi\phi\alpha\beta} G^{\beta C}_{\phi h} + \left[\mathbf{G}^{-1}\right]_{\phi h\alpha B} G^{BC}_{hh} = 0 \tag{7.81}$$

The inverse propagators may be read off the variation of the 2PIEA, leading to the equation for the gluon propagator

$$S_{c,\alpha\beta} - \frac{1}{\xi} f_{B\alpha} f^{B}_{\beta} - i\left[G^{-1}_{\phi\phi}\right]_{\alpha\beta} + 2\frac{\delta\Gamma_2}{\delta G^{\alpha\beta}_{\phi\phi}} = 0 \tag{7.82}$$

The other nontrivial equation is

$$-i f^{A'}_{\alpha} T^{\alpha}_{B}\left[\bar{\phi}\right] + i\left[\mathbf{G}^{-1}_{L}\right]^{A'}_{\omega\chi B} + 2\frac{\delta\Gamma_2}{\delta G^{B}_{\chi\omega A}} = 0 \tag{7.83}$$

In deriving this equation we must consider $G^{B}_{\chi\omega A}$ and $G^{B}_{\omega\chi A}$ as independent quantities.

7.3 Gauge dependence and propagator structure

7.3.1 The Zinn-Justin equation

As we have noted in the introduction to this chapter, the most distinctive feature of gauge theories as opposed to "normal" ones is the existence of relationships among propagators of different orders, the so-called Takahashi–Ward or Slavnov–Taylor identities. The powerful BRST formulation allows us to derive them all from a single master identity, the so-called Zinn-Justin (Z-J) equation, which we shall now present.

The key observation is that under a BRST transform within the path integral which defines the generating functional (7.62), only the source terms are really transformed. Therefore

$$\langle\Omega\left[X^r\right]\rangle J_r + \frac{1}{2}\theta^{rs}\theta^{s}\langle\Omega\left[X^r X^s\right]\rangle \mathbf{K}_{rs} = 0 \tag{7.84}$$

The sources are eliminated in terms of derivatives of the 2PIEA (cf. equation (7.68)), leading to

$$0 = \langle\Omega\left[X^r\right]\rangle\frac{\delta\Gamma}{\delta\bar{X}^r} + \left[\langle\Omega\left[X^r X^s\right]\rangle - 2\langle\Omega\left[X^r\right]\rangle\bar{X}^s\right]\left[\frac{\delta}{\delta\mathbf{G}^{rs}}\Gamma\right] \tag{7.85}$$

Taking derivatives of this identity we obtain the desired relationships.

In the remainder of this section we shall give a simple example of how a concrete identity may be derived from equation (7.85).

For simplicity, we shall assume that all background fields vanish. Since the Z-J operator has ghost number 1, it makes no sense to assume that all quantities with nonzero ghost number vanish, as we have done in the previous section.

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However, we may still "turn on" these quantities one by one, and thus obtain partial Z-J identities. For example, we get three identities relating quantities with zero ghost number by requiring that the coefficients of $\bar{\omega}$ and $G_{\omega\phi}$ vanish (we shall not investigate the first, as we are assuming no nonzero backgrounds, and we are working throughout with the reduced 2PIEA). This means that we may still set

$$\bar{\omega} = \bar{\chi} = G_{\omega\omega} = G_{\chi\chi} = G_{\chi\phi} = G_{\chi h} = 0 \tag{7.86}$$

and retain only terms linear in $G_{\omega\phi}$ and $G_{\omega h}$. In this approximation, terms with ghost number neither 0 or 1 must vanish identically, so

$$\left\langle \Omega\left[\omega^{D}\right]\right\rangle = \left\langle \Omega\left[\omega^{D}\omega^{E}\right]\right\rangle = \left\langle \Omega\left[h_{A}\omega^{D}\right]\right\rangle = \left\langle \Omega\left[\phi^{\alpha}\omega^{D}\right]\right\rangle = \left\langle \Omega\left[\chi_{A}\chi_{B}\right]\right\rangle = 0 \tag{7.87}$$

and

$$\frac{\delta\Gamma}{\delta\omega^{D}} = \frac{\delta\Gamma}{\delta G^{\alpha D}_{\phi\omega}} = \frac{\delta\Gamma}{\delta G^{D}_{h\omega A}} = \frac{\delta\Gamma}{\delta G^{DE}_{\omega\omega}} = \frac{\delta\Gamma}{\delta G_{\chi\chi AB}} = 0 \tag{7.88}$$

Also, since there are no preferred directions in gauge space, objects with a single gauge index must vanish out of symmetry, and therefore

$$\left\langle \Omega\left[\phi^{\alpha}\right]\right\rangle = \left\langle \Omega\left[h^{A}\right]\right\rangle = \left\langle \Omega\left[\chi_{A}\right]\right\rangle = \frac{\delta\Gamma}{\delta\bar{\phi}^{\alpha}}\left[0\right] = 0 \tag{7.89}$$

Finally, observe that at zero external sources,

$$\frac{\delta\Gamma}{\delta h^{A}} = \frac{\delta\Gamma}{\delta G^{\alpha}_{\phi hB}} = \frac{\delta\Gamma}{\delta G_{h\chi AB}} = \frac{\delta\Gamma}{\delta G_{hhAB}} \equiv 0 \tag{7.90}$$

In other words, from the terms in equation (7.85) we keep the terms in $\phi\phi$, $\phi\chi$, and $\chi\omega$ only.

Equation (7.85) must vanish at the physical point, since each coefficient vanishes. What is remarkable is that it vanishes identically, even if $G^{\alpha A}_{\phi\omega} \neq 0$. Now $\delta\Gamma/\delta G^{\alpha\beta}_{\phi\phi}$ and $\delta\Gamma/\delta G^{B}_{\omega\chi A}$ have ghost number zero, and therefore contain no terms linear in $G^{\alpha A}_{\phi\omega}$. We conclude that, to linear order in $G^{\alpha A}_{\phi\omega}$, we may write

$$\left\langle \Omega\left[\phi^{\alpha}\chi_{A}\right]\right\rangle \frac{\delta\Gamma}{\delta G^{\alpha A}_{\phi\chi}} \approx 0 \tag{7.91}$$

Here $\approx$ means up to terms proportional to the equations of motion. Now

$$\frac{\delta\Gamma}{\delta G^{\alpha A}_{\phi\chi}} = \frac{-i}{2}\left[\mathbf{G}^{-1}_{L}\right]_{\phi\chi\alpha A} \tag{7.92}$$

Expanding the identity

$$\left[\mathbf{G}^{-1}_{L}\right]_{\phi X\alpha r} G^{r}_{X\omega B} = 0 \tag{7.93}$$

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and using equations (7.77), (7.78), (7.79) and (7.81)

$$\begin{aligned}\left[\mathbf{G}_L^{-1}\right]_{\phi\chi\alpha A} &= -\left[\left[\mathbf{G}_L^{-1}\right]_{\phi\phi\alpha\beta} + \frac{i}{\xi} f_{\alpha C} f_\beta^C\right] G_{\omega\phi}^{B\beta}\left[\mathbf{G}_R^{-1}\right]_{\chi\omega AB} \\ &\approx -\left[G_{\phi\phi}^{-1}\right]_{\alpha\beta} G_{\omega\phi}^{C\beta}\left[\mathbf{G}_R^{-1}\right]_{\chi\omega AC}\end{aligned} \tag{7.94}$$

Since $G_{\omega\phi}^{C\beta}$ can be anything and $\left[G_{\phi\phi}^{-1}\right]_{\alpha\beta}$ and $\left[\mathbf{G}_R^{-1}\right]_{\chi\omega AC}$ are regular, $\langle\Omega\left[\phi^\alpha\chi_A\right]\rangle$ must vanish:

$$\langle\Omega\left[\phi^\alpha\chi_A\right]\rangle = -\langle\chi_A\Omega\left[\phi^\alpha\right]\rangle - \frac{i}{\xi} f_\beta^A G_{\phi\phi}^{\beta\alpha} = 0 \tag{7.95}$$

The point is that this identity links the gluon and ghost propagators to a gluon–ghost–ghost vertex. To see this, observe that

$$\Omega\left[\phi^\alpha\chi_A\right] = T_B^\alpha\left[\phi\right]\omega^B\chi_A + i\phi^\alpha h_A \tag{7.96}$$

involves cubic terms so the missing expectation value may be written as

$$\langle\Omega\left[\phi^\alpha\chi_A\right]\rangle = G_{\omega\chi A}^B T_B^\alpha\left[0\right] - \frac{i}{\xi} G_{\phi\phi}^{\alpha\beta} f_{A\beta} + T_{B,\gamma}^\alpha\left[0\right]\left\langle\phi^\gamma\omega^B\chi_A\right\rangle \tag{7.97}$$

Below we shall use equation (7.95) to investigate the gauge dependence and structure of the propagators.

7.3.2 Gauge dependence of the propagators

There are two issues central to gauge theories with no analog in "normal" theories, namely, to what extent the results of the theory depend on all the machinery associated with the gauge fixing procedure, and second, how the Zinn-Justin identity may be exploited to glean certain facts about the theory over and beyond actual computation. We shall begin by discussing the first issue, taking as case in point how the propagators depend on the gauge fixing conditions.

To investigate the gauge dependence of the 2PIEA, recall equations (7.38), (7.39) and (7.40). Consider a change δF in the gauge fermion F (cf. equation (7.40))

$$\delta F = -i\chi_A\left\{\delta f^A\left[\phi\right] + \frac{1}{2}\delta\xi h^A\right\} \tag{7.98}$$

Holding the background fields constant, we get

$$\delta\Gamma|_{\bar{X}^r,\mathbf{G}^{rs}} = \delta W|_{J_r,\mathbf{K}_{rs}} \tag{7.99}$$

The variation of the generating functional is computed as in equations (7.42), (7.43) and (7.44). However, now the "action" is not BRST invariant, because it includes the source terms, and we get a nontrivial result

$$\delta\Gamma|_{\bar{X}^r,\mathbf{G}^{rs}} = i\left\{\langle\delta F\Omega\left[X^r\right]\rangle J_r + \frac{1}{2}\theta^{rs}\theta^s\langle\delta F\Omega\left[X^rX^s\right]\rangle\mathbf{K}_{rs}\right\} \tag{7.100}$$

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Again we use equation (7.68) to get

$$\delta\Gamma|_{\bar{X}^r,\mathbf{G}^{rs}} = (-i)\left\{\langle\delta F\Omega\left[X^r\right]\rangle\frac{\delta\Gamma}{\delta\bar{X}^r}\right.$$
$$\left.+\left[\langle\delta F\Omega\left[X^rX^s\right]\rangle - 2\langle\delta F\Omega\left[X^r\right]\rangle\bar{X}^s\right]\left[\frac{\delta}{\delta\mathbf{G}^{rs}}\Gamma\right]\right\} \quad (7.101)$$

As before, we shall assume that all background fields vanish and that at such a point $\Gamma_{,r}$ vanishes identically, so the above expression simplifies to

$$\delta\Gamma|_{\bar{X}^r,\mathbf{G}^{rs}} = -Y^{rs}\frac{\delta}{\delta\mathbf{G}^{rs}}\Gamma; \qquad Y^{rs} = i\langle\delta F\Omega\left[X^rX^s\right]\rangle \quad (7.102)$$

At the physical point, the Schwinger–Dyson equations now read

$$\frac{\delta}{\delta\mathbf{G}^{tu}}\Gamma - Y^{rs}\frac{\delta^2}{\delta\mathbf{G}^{tu}\delta\mathbf{G}^{rs}}\Gamma = 0 \quad (7.103)$$

Of course, the solution is now $\mathbf{G}^{tu} + \delta\mathbf{G}^{tu}$, so

$$\left(\frac{\delta^2}{\delta\mathbf{G}^{tu}\delta\mathbf{G}^{rs}}\Gamma\right)\left[\delta\mathbf{G}^{rs} - Y^{rs}\right] = 0 \quad (7.104)$$

Since the Hessian is supposed to be invertible, we must have $\delta\mathbf{G}^{rs} = Y^{rs}$.

Let us also assume that all propagators with nonzero ghost number vanish. Then

$$\delta G^{\alpha\beta}_{\phi\phi} = i\left\langle\delta F\Omega\left[\phi^\alpha\phi^\beta\right]\right\rangle$$
$$= \left\langle\chi_A\left\{\delta f^A\left[\phi\right] + \frac{1}{2}\delta\xi h^A\right\}\left(T^\alpha_C\left[\phi\right]\omega^C\phi^\beta + (\alpha\leftrightarrow\beta)\right)\right\rangle \quad (7.105)$$

Assume δf^A is also linear and use the Gaussianity of h^A to get

$$\delta G^{\alpha\beta}_{\phi\phi} = \left[\delta f^A_\gamma - \frac{\delta\xi}{2\xi}f^A_\gamma\right]\left\langle\chi_A\phi^\gamma\left(T^\alpha_C\left[\phi\right]\omega^C\phi^\beta + (\alpha\leftrightarrow\beta)\right)\right\rangle \quad (7.106)$$

To lowest order, we find

$$\left\langle\chi_A\phi^\gamma\left(T^\alpha_C\left[\phi\right]\omega^C\phi^\beta + (\alpha\leftrightarrow\beta)\right)\right\rangle \sim G^{\beta\gamma}_{\phi\phi}\langle\chi_A\Omega\left[\phi^\alpha\right]\rangle + (\alpha\leftrightarrow\beta) \quad (7.107)$$

Now recall equation (7.95)

$$\delta G^{\alpha\beta}_{\phi\phi} = \frac{(-i)}{\xi}\left[\delta f^A_\gamma - \frac{\delta\xi}{2\xi}f^A_\gamma\right]f^A_\delta G^{\delta\alpha}_{\phi\phi}G^{\beta\gamma}_{\phi\phi} + (\alpha\leftrightarrow\beta) \quad (7.108)$$

or else

$$\delta G^{-1}_{\phi\phi\alpha\beta} = \frac{i}{\xi}f^A_\alpha\left[\delta f^A_\beta - \frac{\delta\xi}{2\xi}f^A_\beta\right] + (\alpha\leftrightarrow\beta) \quad (7.109)$$

This is the result we wanted to show. We will use it below to analyze the structure of the propagators.

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7.3.3 Transverse and longitudinal gluon propagators

We now turn to the second issue outlined above, namely, how we can turn the gauge dependence identities around to investigate the structure of the theory. For simplicity, we shall consider only a pure (nonabelian) Yang–Mills theory to two-loop accuracy.

To this order, variation of the 2PIEA yields the equation for the ghost propagator

$$\left[\mathbf{G}_L^{-1}\right]^A_{\omega\chi B} = f^A_\alpha \left[T^\alpha_B\left[\bar{\phi}\right] - f^C_{\alpha'} T^{\alpha'}_{1B\beta} T^\alpha_{1B'\beta'} G^{\beta\beta'}_{\phi\phi} G^{B'}_{\omega\chi C}\right] \tag{7.110}$$

Multiplying on the right by $G^B_{\omega\chi C}$ we get

$$f^A_\alpha L^\alpha_B = \delta^A_B \tag{7.111}$$

where

$$L^\lambda_C = \left[T^\lambda_B\left[\bar{\phi}\right] - f^A_{\alpha'} T^{\alpha'}_{1B\gamma} T^\lambda_{1B'\beta'} G^{\gamma\beta'}_{\phi\phi} G^{B'}_{\omega\chi A}\right] G^B_{\omega\chi C} \tag{7.112}$$

This suggests defining

$$P^\alpha_{L\beta} = L^\alpha_C f^C_\beta \tag{7.113}$$

which is a projection operator

$$P^\alpha_{L\beta} P^\beta_{L\gamma} = P^\alpha_{L\gamma} \tag{7.114}$$

Now let us return to equation (7.95), which is a consequence of the Z-J identity (7.85). An explicit calculation to two-loop accuracy yields

$$L^\alpha_C = -\left\langle \chi_C \Omega\left[\phi^\alpha\right]\right\rangle = \frac{i}{\xi} f_{C\beta} G^{\beta\alpha}_{\phi\phi} \tag{7.115}$$

Multiply again by L^C_δ to get

$$P_{L\delta\beta} G^{\beta\alpha}_{\phi\phi} = -i\xi L^C_\delta L^\alpha_C \tag{7.116}$$

Therefore we have a decomposition of the gluon propagators into "transverse" and "longitudinal" parts

$$G^{\lambda\beta}_{\phi\phi} = G^{\lambda\beta}_{T\phi\phi} - i\xi L^\lambda_C L^{C\beta}, \qquad P^\gamma_{L\lambda} G^{\lambda\beta}_{T\phi\phi} = 0 \tag{7.117}$$

The corresponding decomposition for the inverse propagators is

$$\left[G^{-1}_{\phi\phi}\right]_{\alpha\lambda} = \left[G^{-1}_{T\phi\phi}\right]_{\alpha\lambda} + \frac{i}{\xi} f^A_\lambda f_{A\alpha} \tag{7.118}$$

Comparing with equation (7.109) we see that the transverse part $\left[G^{-1}_{T\phi\phi}\right]_{\gamma\lambda}$ is gauge-fixing independent to two-loop order. This is the desired result, laying out the gauge dependence of the propagators in its most explicit form. Of course, the projector $P^\alpha_{L\beta}$ is just the generalization of the usual $k^\mu k^\nu/k^2$ to a nonequilibrium setting.

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