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REMARKS ON A MEASURE OF WEAK NONCOMPACTNESS IN THE LEBESGUE SPACE

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Using the concept of equi–integrability we introduce a measure of weak noncompactness in the Lebesgue space $L^1(0,1)$. We show that this measure is equal to the classical De Blasi measure of weak noncompactness.

1. INTRODUCTION

The theory of measures of weak noncompactness was initiated by De Blasi in the paper [4] where he introduced the first measure of weak noncompactness. This measure was applied successfully to nonlinear functional analysis, to operator theory and to the theory of differential and integral equations (see [**1**, **2**, **3**, **5**, **7**, **8**], for example).

On the other hand it is rather difficult to express the De Blasi measure of weak noncompactness by a convenient formula. Up to now the only formula of this type was obtained by Appell and De Pascale in the Lebesgue space $L^1(0,1)$ [**1**].

The aim of this paper is to construct another formula for De Blasi measure in the space $L^1(0,1)$. This formula is strongly connected with the concept of equi–integrability.

2. NOTATION AND DEFINITIONS

Let E be an infinite dimensional real Banach space with norm $\|\cdot\|$ and zero element θ. Denote by B_E the closed unit ball of the space E. The usual algebraic operations on sets will be denoted in the standard way.

Next, let $\mathfrak{M}_E$ be the family of all nonempty and bounded subsets of E and let $\mathfrak{N}_E^w$ be its subfamily consisting of all relatively weakly compact sets.

The main concept used in this paper is the so–called De Blasi measure of weak noncompactness $\beta : \mathfrak{M}_E \longrightarrow [0,\infty)$ defined in the following way:

$$\beta(x) = \inf\{\varepsilon > 0 : \text{ there exists } Y \in \mathfrak{N}_E^w \text{ such that } X \subset Y + \varepsilon B_E\}.$$

This function was introduced by De Blasi in the paper [4].

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Let us observe that in the case when E is a reflexive Banach space then $\beta(X) = 0$ for every $X \in \mathfrak{M}_E$. Hence it is only of interest to consider the function β in nonreflexive Banach spaces. Recall that $\beta(B_E) = 1$ when E is nonreflexive. For other properties of the De Blasi measure of noncompactness β we refer to [**4**, **8**], for example.

In what follows let $L^1 = L^1(I)$ denote the space of Lebesgue integrable real functions defined on the interval $I = (0,1)$, furnished with the standard norm

$$\|x\| = \int_I |x(t)|\, dt.$$

Obviously the restriction to the interval I has no significance in our considerations.

Further assume that $a > 0$ is a fixed number. For an arbitrary function $x \in L^1$ denote by $I(x,a)$ the set defined by

$$I(x,a) = \{t \in I : |x(t)| > a\}.$$

For our further purposes we quote the definition of equi-integrability [**6**].

Definition: We say that the set $X \in \mathfrak{M}_{L^1}$ is equi-integrable if

$$\lim_{a\to\infty} \int_{I(x,a)} |x(t)|\, dt = 0$$

uniformly with respect to $x \in X$.

The following result will be fundamental in the sequel (see [**6**]).

Theorem 1. *Let $X \in \mathfrak{M}_{L^1}$. Then X is relatively weakly compact if and only if X is equi-integrable.*

Now, based on the above theorem we introduce the function H defined on the family $\mathfrak{M}_{L^1}$ by the formula

$$H(X) = \lim_{a\to\infty} H_a(X),$$

where

$$H_a(X) = \sup\left\{ \int_{I(x,a)} |x(t)|\, dt : x \in X \right\}.$$

The properties of the function H will be investigated in the next section. In particular, we show that H provides a lower estimate for β.

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3. Results

We start with the following result describing the simplest properties of the function H.

Theorem 2. *The function H satisfies the following conditions:*

(i) $H(X) = 0 \iff X \in \mathfrak{N}^w_E$,

(ii) $X \subset Y \Longleftarrow H(X) \leqslant H(Y)$,

(iii) $H(cX) = |c|\, H(X)$ *for* $c \in \mathbb{R}$.

Proof: Observe that (i) is an easy consequence of Theorem 1 while (ii) follows immediately from the definition of the function H.

In order to prove (iii) let us take a set $X \in \mathfrak{M}_{L^1}$ and choose arbitrarily $x \in X$ and $a > 0$. Further assume that c is a fixed number, $c \neq 0$. Then we have

$$\int_{I(cx,a)} |cx(t)|\, dt = |c| \int_{I(cx,a)} |x(t)|\, dt = |c| \int_{I(x,a/|c|)} |x(t)|\, dt.$$

Hence

$$H_a(cX) = |c|\, H_{a/|c|}(X)$$

and consequently

$$H(cX) = |c|\, H(X).$$

In the case when $c = 0$ we have that $cX = \{\theta\}$ and $I(cx, a) = \emptyset$. Thus in this case the equality (iii) is also satisfied.

This completes the proof. □

Our next result is connected with estimates for the values of the function H.

Theorem 3. $H(B_{L^1}) = 1$.

Proof: Choose arbitrarily $x \in B_{L^1}$ and fix $a > 0$. Then

$$\int_{I(x,a)} |x(t)|\, dt \leqslant \int_I |x(t)|\, dt \leqslant 1$$

which gives

$$H(B_{L^1}) \leqslant 1.$$

In order to prove the converse inequality let us take the set $X \subset B_{L^1}$ consisting of all functions x_δ $(0 < \delta < 1)$ defined by the formula

$$x_\delta(t) = \begin{cases} 1/\delta & \text{for } t \in (0, \delta) \\ 0 & \text{otherwise.} \end{cases}$$

Then, for δ such that $1/\delta > a$ we get

$$\int_{I(x_\delta,a)} |x_\delta(t)|\,dt = \int_{(0,\delta)} |x_\delta(t)|\,dt = 1.$$

This allows us to infer that $H_a(X) = 1$. Hence

$$H_a(B_{L^1}) \geqslant H_a(X) = 1$$

and consequently

$$H(B_{L^1}) \geqslant 1.$$

This ends the proof. ∎

Theorem 4. *The function H is subadditive that is,*

$$H(X+Y) \leqslant H(X) + H(Y)$$

for all $X, Y \in \mathfrak{M}_{L^1}$.

Proof: Fix arbitrarily $a > 0$. For an arbitrary function $z \in L^1$ let us denote:

$$I_+(z,a) = \{t \in I : z(t) > a\},$$
$$I_-(z,a) = \{t \in I : z(t) < -a\}.$$

Obviously $I(z,a) = I_+(z,a) \cup I_-(z,a)$.

Next, let us take $X, Y \in \mathfrak{M}_{L^1}$ and $x \in X$, $y \in Y$.

Denote:

$$I_y^+(x,a) = \{t \in I : x(t) > a \text{ and } y(t) \leqslant a\},$$
$$I_x^+(y,a) = \{t \in I : y(t) > a \text{ and } x(t) \leqslant a\}.$$

Observe that

$$I_+(x+y,2a) \subset I_y^+(x,a) \cup I_x^+(y,a) \cup I_+(x,a) \cap I_+(y,a).$$

Further, we get

$$\begin{aligned}
\int\limits_{I_+(x+y,2a)} |x(t)+y(t)|\,dt &= \int\limits_{I_+(x+y,2a)} (x(t)+y(t))dt \\
&\leqslant \int\limits_{I_y^+(x,a)} (x(t)+y(t))dt + \int\limits_{I_x^+(y,a)} (x(t)+y(t))dt + \int\limits_{I_+(x,a)\cap I_+(y,a)} (x(t)+y(t))dt \\
&= \int\limits_{I_x^+(y,a)} x(t)dt + \left\{ \int\limits_{I_y^+(x,a)} x(t)dt + \int\limits_{I_+(x,a)\cap I_+(y,a)} x(t)dt \right\} \\
&\quad + \int\limits_{I_y^+(x,a)} y(t)dt + \left\{ \int\limits_{I_x^+(y,a)} y(t)dt + \int\limits_{I_+(x,a)\cap I_+(y,a)} y(t)dt \right\} \\
&= \int\limits_{I_x^+(y,a)} x(t)dt + \int\limits_{I_y^+(x,a)} y(t)dt + \int\limits_{I_+(x,a)} x(t)dt + \int\limits_{I_+(y,a)} y(t)dt \\
&\leqslant a\left\{\operatorname{meas}\left(I_x^+(y,a)\right) + \operatorname{meas}\left(I_y^+(x,a)\right)\right\} + \int\limits_{I_+(x,a)} x(t)dt + \int\limits_{I_+(y,a)} y(t)dt.
\end{aligned}$$

Now, let us take $A > a$. Then, arguing similarly th that above, it is easily seen that

$$\int\limits_{I_x^+(y,A)} x(t)dt \leqslant a\operatorname{meas}\left(I_x^+(y,A)\right),$$

$$\int\limits_{I_y^+(x,A)} y(t)dt \leqslant a\operatorname{meas}\left(I_y^+(x,A)\right).$$

Hence, taking into account the above estimates we derive

$$(1) \quad \int\limits_{I_+(x+y,2A)} (x(t)+y(t))dt \leqslant a\left\{\operatorname{meas}\left(I_x^+(y,A)\right) + \operatorname{meas}\left(I_y^+(x,A)\right)\right\} + \int\limits_{I_+(x,A)} x(t)dt + \int\limits_{I_+(y,A)} y(t)dt.$$

Next, let us take an arbitrary number $\varepsilon > 0$. Observe that

$$\lim_{A\to\infty} \left\{\operatorname{meas}\left(I_x^+(y,A)\right) + \operatorname{meas}\left(I_y^+(x,A)\right)\right\} = 0.$$

Consequently, we can find $A_0 > a$ such that for $A \geqslant A_0$ we have

$$\operatorname{meas}\left(I_x^+(y,A)\right) + \operatorname{meas}\left(I_y^+(x,A)\right) \leqslant \varepsilon/a.$$

Combining the above inequality and (1) we obtain

$$\int\limits_{I_+(x+y,2A)} (x(t)+y(t))dt \leqslant \varepsilon + \int\limits_{I_+(x,A)} x(t)dt + \int\limits_{I_+(y,A)} y(t)dt. \tag{2}$$

By reasoning similar to that above we can deduce that for $A \geqslant A_0$ the following inequality holds

$$\int\limits_{I_-(x+y,2A)} (x(t)+y(t))dt \leqslant -\varepsilon + \int\limits_{I_-(x,A)} x(t)dt + \int\limits_{I_-(y,A)} y(t)dt. \tag{3}$$

Finally, by linking the inequalities (2) and (3) we arrive at the following inequality:

$$\begin{aligned}\int\limits_{I(x+y,2A)} |x(t)+y(t)|\,dt \leqslant 2\varepsilon + \int\limits_{I(x,A)} |x(t)|\,dt + \int\limits_{I(y,A)} |y(t)|\,dt \\ \leqslant 2\varepsilon + H_A(X) + H_A(Y).\end{aligned}$$

Hence

$$H_{2A}(X+Y) \leqslant 2\varepsilon + H_A(X) + H_A(Y).$$

Consequently, keeping in mind the arbitrariness of ε, we get

$$H(X+Y) \leqslant H(X) + H(Y).$$

Thus the proof is complete. ∎

Our last result will show that the function H coincides with the De Blasi measure of weak noncompactness.

Theorem 5. *$H(X) = \beta(X)$ for any $X \in \mathfrak{M}_{L^1}$.*

Proof: Fix arbitrarily $a > 0$ and observe that every function $x \in L^1$ may be written in the form

$$x = xK_{I\setminus I(x,a)} + xK_{I(x,a)}, \tag{4}$$

where by K_Ω we denote the characteristic function of the set Ω.

Next, for a given set $X \in \mathfrak{M}_{L^1}$ let us consider the set X_a defined by

$$X_a = \left\{xK_{I\setminus I(x,a)} : x \in X\right\}.$$

Observe that the set X_a is relatively weakly compact in the space L^1.

Now, using the representation (4) it may be easily shown that

$$X \subset X_a + H_a(X)B_{L^1}.$$

Hence, in view of the properties of the De Blasi measure of weak noncompactness β [4] we get

$$\beta(X) \leqslant \beta(X_a) + H_a(X)\beta(B_{L^1}) = H_a(X).$$

Since a was chosen arbitrarily, the above estimate yields

$$\beta(X) \leqslant H(X). \tag{5}$$

Conversely, suppose that $\beta(X) = r$. Then for any $\varepsilon > 0$ we can find a set $Y \in \mathfrak{N}^w_{L^1}$ such that

$$X \subset Y + (r + \varepsilon)B_{L^1}$$

(see Section 2). Hence, using Theorems 2, 3 and 4 we obtain

$$H(X) \leqslant H(Y) + (r + \varepsilon)H(B_{L^1}) = r + \varepsilon.$$

In virtue of the arbitrariness of ε the above inequality implies

$$H(X) \leqslant \beta(X).$$

This inequality in conjunction with (5) finishes the proof. ∎

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