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A NOTE ON ANALYTIC CAPACITY

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Let K be a 2-dimensional Cantor set. In this note we prove, in two cases, the analytic capacity and the continuous analytic capacity of K are equal.

1. Introduction

Let K be a compact set of the complex plane $\mathbf{C}$ and let $U = \mathbf{C}\backslash K$. Let $H^\infty(U)$ denote the algebra of all bounded analytic functions on U and let $A(U)$ denote the algebra of those functions in $H^\infty(U)$ that have continuous extentions throughout the entire plane. The analytic capacity and the continuous analytic capacity of K are defined respectively by

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H^\infty(U), |f|_U \leqslant 1\},$$
$$\alpha(K) = \sup\{|f'(\infty)| : f \in A(U), |f|_{\mathbf{C}} \leqslant 1\}.$$

If E is an arbitrary subset of $\mathbf{C}$, then $\gamma(E)$ (respectively $\alpha(E)$) is defined as the supremum over $\gamma(K)$ (respectively $\alpha(K)$) for all compact $K \subseteq E$. It is clear that both α and γ are monotonic set functions with $\alpha(E) \leqslant \gamma(E)$ for arbitrary set E and $\alpha(E) = \gamma(E)$ if E is open. It is also true that $\alpha(K) = \gamma(K)$ if K is bounded by a finite number of disjoint analytic Jordan curves. (See [4, Chapter 1, Section 4]). In general, however, these two quantities are not commensurate. For example, if K is a circle or a line segment, then $\gamma(K) > 0$ by the Riemann mapping theorem but by Morera's theorem $\alpha(K) = 0$. We refer to [2, 4, 7], and [9] for more results concerning analytic capacity. For our purpose, we shall mention the following results.

In the sequel, $\Delta(z, r)$ will denote an open disk of radius r and center z, and $A(U)$ is said pointwise boundedly dense in $H^\infty(U)$ if for each $f \in H^\infty(U)$, there exists a sequence $\{f_n\}$ in $A(U)$ such that $\sup_n \|f\| < \infty$ and $f_n(z)$ converges to $f(z)$ for each $z \in U$.

Theorem 1. (Gamelin and Garnett [3].) *Let K be a compact set of the complex plane and let $U = \mathbf{C}\backslash K$. Then the following are equivalent.*

(i) *$A(U)$ is pointwise boundedly dense in $H^\infty(U)$.*

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(ii) *There exist constants $r > 1$ and $A > 0$ such that for all $\delta > 0$ sufficiently small,*

$$\gamma(\Delta(z,\delta) \cap K) \leqslant A\alpha(\Delta(z,r\delta) \cap K), \qquad z \in \partial U.$$

Theorem 2. (Davie [1]) *If $A(U)$ is pointwise boundedly dense in $H^\infty(U)$, then for every $f \in H^\infty(U)$, there exists a sequence $\{f_n\}$ in $A(U)$ with $\sup_n \|f_n\| \leqslant \|f\|$ and $f_n(z) \to f(z)$ pointwise.*

Remark: Suppose $A(U)$ is pointwise boundedly dense in $H^\infty(U)$. Then the Ahlfors function g for K will be a pointwise limit of a sequence $\{g_n\}$ in $A(U)$ with norms bounded by 1. This implies $g_n'(\infty) \to g'(\infty)$ and $\alpha(K) = \gamma(K)$.

2. Analytic Capacity and Cantor Sets

It is known that if K is any compact subset of a rectifiable curve, then $\alpha(K) = 0$ (see [4, p.66]) but $\gamma(K) = 0$ if and only if K has a zero arc length (see [6]). In view of these results, we consider analytic capacity of 2-dimentional Cantor sets K defined as follows and examine the relationship between $\alpha(K)$ and $\gamma(K)$.

Let $\{\lambda_n\}$ $(n = 0,1,2,\dots)$ be a decreasing sequence of positive numbers with $\lambda_0 = 1$ and $\lambda_n < \lambda_{n-1}/2$ for $n = 1,2,\dots$ Let $L_0 = [0,1]$, and at each stage $n \geqslant 1$, construct L_n from L_{n-1} by replacing each component of L_{n-1} by its two endmost intervals of length λ_n. Let $Q_n = L_n \times L_n$. Then Q_n is a union of 4^n squares $Q_{n,j}$ $(j = 1,2,\dots,4^n)$. Let $K = \cap Q_n$ and $K_{n,j} = K \cap Q_{n,j}$. We will prove $\alpha(K) = \gamma(K)$ in two cases. First, we need the following theorem.

Theorem 3. *Let K be a Cantor set and let $U = \mathbf{C}\backslash K$. Then $A(U)$ is pointwise boundedly dense in $H^\infty(U)$ if and only if there exists a constant $B > 0$ such that for all n sufficiently large,*

$$(1) \qquad \gamma(K_{n,j}) \leqslant B\alpha(K_{n,j}), \qquad 1 \leqslant j \leqslant 4^n.$$

Proof: First, observe that if two compact sets K and K' are separated by a straight line L that is, if K is contained in one open half-plane defined by L and K' is contained in the other open half-plane, then there exists an absolute constant $C > 0$ such that

$$(2) \qquad \alpha(K \cup K') \leqslant C(\alpha(K) + \alpha(K'))$$

and

$$(3) \qquad \gamma(K \cup K') \leqslant C(\gamma(K) + \gamma(K')).$$

Relation (2) follows from [2, Chapter 8, Corollary 12.8]. To obtain (3), let $\{K_n\}$ and $\{K'_n\}$ be two sequences of compact sets that are also separated by L and each has a boundary consisting of a finite number of pairwise disjoint analytic Jordan curves. We will also assume that K_n and K'_n decreases to K and K' respectively. Under this assumption $\alpha(K_n) = \gamma(K_n)$, $\alpha(K'_n) = \gamma(K'_n)$, and $\alpha(K_n \cup K'_n) = \gamma(K_n \cup K'_n)$. Therefore, by (2) $\gamma(K_n \cup K'_n) \leqslant C(\gamma(K_n) + \gamma(K'_n))$ and (3) is obtained by letting $n \to \infty$.

Now suppose $A(U)$ is pointwise boundedly dense in $H^\infty(U)$. Take n sufficiently large so that $\delta = 2\lambda_n$ is sufficiently small. Let $z \in K_{n,j}$. Then $K_{n,j} \subseteq \Delta(z,\delta)$ and by Theorem 1,

$$\begin{aligned}\gamma(K_{n,j}) &\leqslant \gamma(\Delta(z,\delta) \cap K)\\ &\leqslant A\alpha(\Delta(z,r\delta) \cap K).\end{aligned}$$

Furthermore, since $\Delta(z,r\delta)$ intersects at most N $K_{n,j}$'s, where N depends only on r, we may apply (2) repeatedly to obtain $\alpha(\Delta(z,r\delta)) \leqslant NC^N\alpha(K_{n,j})$. Thus (1) holds with $B = NAC^N$.

Conversely, suppose (1) holds for n sufficiently large and take $\delta > 0$ sufficiently small so that $\lambda_{n+1} \leqslant \delta < \lambda_n$. Then $\Delta(z,\delta)$, $z \in K$, can intersect atmost 9 $K_{n,j}$'s. Apply (3) repeatedly to obtain

$$\begin{aligned}\gamma(\Delta(z,\delta) \cap K) &\leqslant 9C^9\gamma(K_{n,j})\\ &\leqslant 144C^{25}\gamma(K_{n+2,j})\\ &\leqslant 144AC^{25}\alpha(K_{n+2,j})\\ &\leqslant 144AC^{25}\alpha(\Delta(z,\delta) \cap K)\end{aligned}$$

because $\Delta(z,\delta) \cap K$ contains some $K_{n+2,j}$. Then, it follows from Theorem 1 that $A(U)$ is pointwise boundedly dense in $H^\infty(U)$. □

Theorem 4. *If K is a Cantor set, then $\alpha(K) = \gamma(K)$ in each of the following cases.*

(i) *K has a positive plane Lebesgue measure.*

(ii) *There exists a constant $\rho > 0$ such that $\lambda_{n+1}/\lambda_n = \rho$ for all n sufficiently large.*

Proof: (i) Let $|E|$ denote the Lebesgue measure of a measurable set E. Define

$$f_{n,j}(\zeta) = \iint_{K_{n,j}} \frac{dx\,dy}{z - \zeta} \qquad z = x + iy.$$

Then $f_{n,j}$ is continuous on $\mathbf{C}$, $f_{n,j}(\infty) = 0$, $f'_{n,j}(\infty) = -|K_{n,j}|$ and $\|f_{n,j}\| \leqslant 2\pi^{1/2}|dK_{n,j}|^{1/2}$. (See [4, pp.1–3]). Thus $\alpha(K_{n,j}) \geqslant 2^{-1}\pi^{-1/2}|K_{n,j}|^{1/2}$. On the other hand, since $|K| = 4^n|K_{n,j}|$ and $K_{n,j}$ is contained in a circle of radius 2^{-n}, we obtain

$$\begin{aligned}\gamma(K_{n,j}) &\leqslant 2^{-n}\\ &= |K|^{-1/2}|K_{n,j}|^{1/2}\\ &\leqslant 2\pi^{1/2}|K|^{-1/2}\alpha(K_{n,j}).\end{aligned}$$

By Theorem 3 and the remark that follows Theorem 2, we have $\alpha(K) = \gamma(K)$.

(ii) If $0 < \rho \leqslant 1/4$, then $\alpha(K) = \gamma(K) = 0$. For the case $0 < \rho < 1/4$, this result follows from the fact that K can be enclosed by a finite number of rectifiable curves of total length arbitrary small. The case $\rho = 1/4$ was proved by Garnett [5]. Thus we may assume $\rho > 1/4$. First, we shall show that in this case $\alpha(K_{n,j}) > 0$ for all n and j.

Let μ be a positive Borel measure of compact support. The Newtonian potential of μ is defined by

$$U_\mu(\zeta) = \int \frac{d\mu(z)}{z-\zeta} \qquad \zeta \in \mathbf{C},$$

and the Newtonian capacity of a set E is defined by

$$\nu(E) = \sup\{\mu(E),\ \operatorname{supp}(\mu) \subseteq E, U_\mu \text{ is continuous and } U_\mu \leqslant 1\}.$$

Ohtsuka [8] proved $\nu(E) > 0$ if and only if $\sum 4^{-n}/\lambda_n < \infty$. Thus, if $\rho > 1/4$ and $\lambda_{n+1}/\lambda_n = \rho$ for n sufficiently large, say $n \geqslant p$, then $\nu(K) > 0$. Then, it is also true that $\alpha(K) > 0$ because $\nu(K) \leqslant \alpha(K)$ (see [4, p.71]). Since α and γ are translation invariant and homogeneous of degree 1, this implies $\alpha(K_{n,j}) > 0$ for all n and j, and furthermore, for $n \geqslant p$,

$$\alpha(K_{n,j}) = \rho^{n-p}\alpha(K_{p,1})$$
$$\gamma(K_{n,j}) = \rho^{n-p}\gamma(K_{p,1})$$

which implies

$$\gamma(K_{n,j}) = \frac{\gamma(K_{p,1})}{\alpha(K_{p,1})}\alpha(K_{n,j}).$$

This completes the proof of (ii). ∎

REMARK: Cases (i) and (ii) represent the two extreme structures of Cantor sets. In case (i), each group of 4 $K_{n,j}$'s cluster rapidly together while in case (ii) they are scattered from each other. For this reason, it seems likely that $\alpha(K) = \gamma(K)$ for any Cantor set K.

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