UPON A SECOND CONFLUENT FORM OF THE ε-ALGORITHM†

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In two previous papers [**1**], [**2**] the confluent form

$$\{\varepsilon_{s+1}(t)-\varepsilon_{s-1}(t)\}\varepsilon_s'(t) = 1 \tag{1}$$

of the ε-algorithm [**3**]

$$(\varepsilon_{s+1}^{(m)} - \varepsilon_{s-1}^{(m+1)})\,(\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)}) = 1 \tag{2}$$

was established, and various properties which the confluent form of the algorithm possesses were discussed. It was shown, among other things, that if in (1)

$$\varepsilon_1(t) = 0, \qquad \varepsilon_0(t) = f(t) \tag{3}$$

and the notation

$$H_k^{(m)}\{f(t)\} = \begin{vmatrix} f^{(m)}(t) & f^{(m+1)}(t) & \dots & f^{(m+k-1)}(t) \\ f^{(m+1)}(t) & f^{(m+2)}(t) & \dots & f^{(m+k)}(t) \\ \dots & \dots & \dots & \dots \\ f^{(m+k-1)}(t) & f^{(m+k)}(t) & \dots & f^{(m+2k-2)}(t) \end{vmatrix} \tag{4}$$

is used, then (1) is satisfied by

$$\varepsilon_{2s}(t) = \frac{H_{s+1}^{(0)}\{f(t)\}}{H_s^{(1)}\{f(t)\}}, \quad \varepsilon_{2s+1}(t) = \frac{H_s^{(3)}\{f(t)\}}{H_s^{(1)}\{f(t)\}}, \tag{5}$$

and further that under certain conditions, and for a certain *n*,

$$\varepsilon_{2n}(t) = \lim_{t\to\infty} f(t) \tag{6}$$

identically. It is the purpose of this note to derive another confluent form of the ε-algorithm and to discuss its properties.

The ε-algorithm has as its main application the transformation of the slowly convergent or divergent series

$$S \sim \sum_{s=0}^{\infty} u_s, \tag{7}$$

and if in (2) the initial conditions

$$\varepsilon_{-1}^{(m)} = 0, \quad \varepsilon_0^{(m)} = S_m = \sum_{s=0}^{m-1} u_s \quad (m = 1, 2, \dots), \quad \varepsilon_0^{(0)} = 0 \tag{8}$$

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are used, then, under favourable conditions, the sequence $\varepsilon_{2n}^{(0)}(n = 1, 2, \ldots)$ provides increasingly good estimates of S. This principle will be applied to the transformation of the sum

$$h\sum_{s=0}^{\infty} f(a+sh). \tag{9}$$

Under the assumption that $f(t)$ is infinitely differentiable for $a \leqq t \leqq \infty$, limiting forms for the expressions for $\varepsilon_s^{(0)}$ as h tends to zero will be derived.

It may be shown that if the initial conditions (8) are used in (2), then [3, p. 91]

$$\varepsilon_{2n}^{(0)} = \begin{vmatrix} S_0 & S_1 & \ldots & S_n \\ \Delta S_0 & \Delta S_1 & \ldots & \Delta S_n \\ \ldots & \ldots & \ldots & \ldots \\ \Delta S_{n-1} & \Delta S_n & \ldots & \Delta S_{2n-1} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \Delta S_0 & \Delta S_1 & \ldots & \Delta S_n \\ \ldots & \ldots & \ldots & \ldots \\ \Delta S_{n-1} & \Delta S_n & \ldots & \Delta S_{2n-1} \end{vmatrix}, \tag{10}$$

$$\varepsilon_{2n+1}^{(0)} = \{\varepsilon_{2n}(\Delta S_0)\}^{-1}. \tag{11}$$

On substituting

$$u_s = f(a+sh) \tag{12}$$

in (8), making the changes of notation

$$h^{-1}\varepsilon_{2s}^{(m)} = \varepsilon_{2s}(t), \qquad h\varepsilon_{2s+1}^{(m)} = \varepsilon_{2s+1}(t) \quad (m, s = 0, 1, \ldots), \tag{13}$$

where

$$t = a+mh, \tag{14}$$

and letting h tend to zero, there follow, after appropriate operations upon rows and columns of the determinantal expressions (10) and (11),

$$\varepsilon_{2s}(t) = \frac{H_{s+1}^{(-1)}\{f(t)\}}{H_s^{(1)}\{f(t)\}}, \qquad \varepsilon_{2s+1}(t) = \frac{H_s^{(2)}\{f(t)\}}{H_{s+1}^{(0)}\{f(t)\}}, \tag{15}$$

where, in (4)

$$f^{(-1)}(t) = 0. \tag{16}$$

These may be shown to satisfy the difference-differential relations

$$\{\varepsilon_{2s+2}(t) - \varepsilon_{2s}(t)\}\varepsilon'_{2s+1}(t) = 1, \tag{17}$$

$$\{\varepsilon_{2s+1}(t) - \varepsilon_{2s-1}(t)\}\{\varepsilon'_{2s}(t) + f(t)\} = 1, \tag{18}$$

with the initial conditions $\varepsilon_{-1}(t) = \varepsilon_0(t) = 0$.

The latter will be proved in detail; it is slightly the more difficult of the two cases. Using an expansion of Schweins [4, p. 108] there follows firstly

$$\{\varepsilon_{2s+1}(t)\}^{-1} - \{\varepsilon_{2s-1}(t)\}^{-1} = \frac{-\{H_s^{(1)}\{f(t)\}\}^2}{H_s^{(2)}\{f(t)\}H_{s-1}^{(2)}\{f(t)\}} \tag{19}$$

and, upon multiplying this result by the product $\varepsilon_{2s+1}(t)\ \varepsilon_{2s-1}(t)$, the result

$$\varepsilon_{2s+1}(t) - \varepsilon_{2s-1}(t) = \frac{\{H_s^{(1)}\{f(t)\}\}^2}{H_{s+1}^{(0)}\{f(t)\}H_s^{(0)}\{f(t)\}}. \tag{20}$$

Further

$$\{H_s^{(1)}\{f(t)\}\}^2\varepsilon'_{2s}(t)$$

$$=H_s^{(1)}\{f(t)\}\begin{vmatrix} 0 & d_1 & d_2 & \cdots d_s \\ d_0 & d_2 & d_3 & \cdots d_{s+1} \\ d_1 & d_3 & d_4 & \cdots d_{s+2} \\ \hdotsfor{4} \\ \hdotsfor{4} \\ d_{s-1} & d_s & d_{s+1} & \cdots d_{2s-1} \end{vmatrix} + \begin{vmatrix} 0 & d_0 & d_1 & \cdots d_s \\ d_0 & d_1 & d_2 & \cdots d_{s+1} \\ d_1 & d_2 & d_3 & \cdots d_{s+2} \\ \hdotsfor{4} \\ d_{s-2} & d_{s-1} & d_s & \cdots d_{2s-2} \\ d_s & d_{s+1} & d_{s+2} & \cdots d_{2s} \end{vmatrix}$$

$$-H_{s+1}^{(1)}\{f(t)\}\begin{vmatrix} d_1 & d_2 & d_3 & \cdots d_s \\ d_2 & d_3 & d_4 & \cdots d_{s+1} \\ d_3 & d_4 & d_5 & \cdots d_{s+2} \\ \hdotsfor{4} \\ d_{s-1} & d_s & d_{s+1} & \cdots d_{2s} \end{vmatrix}, \qquad (21)$$

where

$$d_s = f^{(s)}(t) \quad (s = 0, 1, \ldots); \qquad (22)$$

this may be transformed into

$$\{H_s^{(1)}\{f(t)\}\}^2\{\varepsilon'_{2s}(t)+f(t)\}$$

$$=\begin{vmatrix} d_0 & d_1 & \cdots d_{s-1} & 1 \\ d_1 & d_2 & \cdots d_s & 0 \\ d_2 & d_3 & \cdots d_{s+1} & 0 \\ \hdotsfor{4} \\ d_{s-1} & d_s & \cdots d_{2s-2} & 0 \\ d_s & d_{s+1} & \cdots d_{2s-1} & 0 \end{vmatrix} \begin{vmatrix} d_0 & d_1 & \cdots d_{s-1} & 0 \\ d_1 & d_2 & \cdots d_s & d_0 \\ d_2 & d_3 & \cdots d_{s+1} & d_1 \\ \hdotsfor{4} \\ d_{s-1} & d_s & \cdots d_{2s-2} & d_{s-2} \\ d_{s+1} & d_{s+2} & \cdots d_{2s} & d_s \end{vmatrix}$$

$$-\begin{vmatrix} d_0 & d_1 & \cdots d_{s-1} & 0 \\ d_1 & d_2 & \cdots d_s & d_0 \\ d_2 & d_3 & \cdots d_{s+1} & d_1 \\ \hdotsfor{4} \\ d_{s-1} & d_s & \cdots d_{2s-2} & d_{s-2} \\ d_s & d_{s+1} & \cdots d_{2s-1} & d_{s-1} \end{vmatrix} \begin{vmatrix} d_0 & d_1 & \cdots d_{s-1} & 1 \\ d_1 & d_2 & \cdots d_s & 0 \\ d_2 & d_3 & \cdots d_{s+1} & 0 \\ \hdotsfor{4} \\ d_{s-1} & d_s & \cdots d_{2s-2} & 0 \\ d_{s+1} & d_{s+2} & \cdots d_{2s} & 0 \end{vmatrix}, \qquad (23)$$

which reduces, by using a theorem on compound determinants [**4**, p. 49] to

$$H_s^{(0)}\{f(t)\}H_{s+1}^{(0)}\{f(t)\}. \qquad (24)$$

Thus (18) has been established and (17) follows in a similar manner.

These results may be generalised by letting

$$f(a+t) = e^{-zt}\phi(a+t). \qquad (25)$$

The determinantal expressions (15) then become, as h tends to zero,

$$\varepsilon_{2s}(z\,;a) = X_s/Y_s, \qquad (26)$$

where

$$X_s = \begin{vmatrix} 0 & c_0 & zc_0+c_1 & \dots z^{s-1}c_0+z^{s-2}c_1+\dots+c_{s-1} \\ c_0 & c_1 & c_2 & \dots \qquad c_s \\ \cdots & \cdots & \cdots & \cdots \\ c_{s-1} & c_s & c_{s+1} & \dots \qquad c_{2s-1} \end{vmatrix}, \tag{27}$$

$$Y_s = \begin{vmatrix} 1 & z & z^2 & \dots z^s \\ c_0 & c_1 & c_2 & \dots c_s \\ \cdots & \cdots & \cdots & \cdots \\ c_{s-1} & c_s & c_{s+1} & \dots c_{2s-1} \end{vmatrix} \tag{28}$$

and

$$\varepsilon_{2s+1}(z, a) = \begin{vmatrix} 0 & 1 & z & \dots z^s \\ 1 & c_0 & c_1 & \dots c_s \\ z & c_1 & c_2 & \dots c_{s+1} \\ \cdots & \cdots & \cdots & \cdots \\ z^s & c_s & c_{s+1} & \dots c_{2s} \end{vmatrix} \Bigg/ \begin{vmatrix} c_0 & c_1 & \dots c_s \\ c_1 & c_2 & \dots c_{s+1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ c_s & c_{s+1} & \dots c_{2s} \end{vmatrix}, \tag{29}$$

where

$$c_s = \phi^{(s)}(a), \tag{30}$$

and these expressions satisfy the relationships

$$\{\varepsilon_{2s+1}(z\,;a)-\varepsilon_{2s-1}(z\,;a)\}\left\{\phi(a)-z\varepsilon_{2s}(z\,;a)+\frac{\partial}{\partial a}\varepsilon_{2s}(z\,;a)\right\}=z, \tag{31}$$

$$\{\varepsilon_{2s+2}(z\,;a)-\varepsilon_{2s}(z\,;a)\}\left\{z\varepsilon_{2s+1}(z, a)+\frac{\partial}{\partial a}\varepsilon_{2s+1}(z\,;a)\right\}=z. \tag{32}$$

In a subsequent paper a convergence theory for the process (17), (18), (31) and (32) will be discussed; in order to prepare the assault, a number of results will be established.

Expression (26) may be recognised [6] as the sth convergent of the Stieltjes J-fraction [7]

$$\frac{\phi(a)}{z-Q_1(a)-}\;\frac{E_1(a)}{z-Q_2(a)-}\;\cdots\;\frac{E_r(a)}{z-Q_{r+1}(a)-}\;\cdots \tag{33}$$

equivalent to the formal power series

$$F(z)=\int_0^\infty e^{-zt}\phi(a+t)\,dt \sim \sum_{s=0}^{\infty}\phi^{(s)}(a)z^{-s-1}. \tag{34}$$

The sequence of functions $\varepsilon_{2s}(z\,;a)$ may therefore be constructed in a number of ways. For example, the discrete ε-algorithm relationships (2) may be applied to the initial conditions (8) in which

$$u_s=\phi^{(s)}(a)z^{-s-1} \quad (s=0, 1, \dots), \tag{35}$$

when

$$\varepsilon_{2s}^{(0)}=\varepsilon_{2s}(z\,;a). \tag{36}$$

In this context it may be remarked that

$$\varepsilon_{2s+1}^{(0)} = z\varepsilon_{2s+1}(z\,;a). \tag{37}$$

(This follows by comparing formula (3.8.4) of [**6**, p. 160] with (29).)

Alternatively the coefficients in (30) may be constructed by application of the $q-d$ algorithm [**8**] relationships

$$q_r^{(m)}+e_r^{(m)} = q_r^{(m+1)}+e_{r-1}^{(m+1)}\ , \quad q_{r+1}^{(m)}e_r^{(m)} = q_r^{(m+1)}e_r^{(m+1)}\,, \tag{38}$$

$$Q_r(a) = q_r^{(0)}+e_{r-1}^{(0)}\ , \quad E_r(a) = q_r^{(0)}e_r^{(0)}\,, \tag{39}$$

to the initial conditions

$$q_1^{(m)} = \frac{c_{m+1}}{c_m}\ , \quad e_0^{(m)} = 0. \tag{40}$$

The coefficients in (30) may also be constructed by use of the confluent form [**9**]

$$E_r(t)-E_{r-1}(t) = Q_r'(t), \quad Q_{r+1}(t)-Q_r(t) = E_r'(t)/E_r(t), \tag{41}$$

$$Q_r(t) = \phi'(t)/\phi(t), \quad E_1(t) = 0, \tag{42}$$

of the quotient-difference algorithm.

Relationships (41) may be used to show [**10**] that

$$Q_r(t) = \frac{H_{r+1}^{(1)}\{\phi(t)\}H_r^{(0)}\{\phi(t)\}}{H_{r+1}^{(0)}\{\phi(t)\}H_r^{(1)}\{\phi(t)\}} + \frac{H_{r+1}^{(0)}\{\phi(t)\}H_{r-1}^{(1)}\{\phi(t)\}}{H_r^{(0)}\{\phi(t)\}H_r^{(1)}\{\phi(t)\}} \tag{43}$$

$$E_r(t) = \frac{H_{r+1}^{(0)}\{\phi(t)\}H_{r-1}^{(0)}\{\phi(t)\}}{[H_r^{(0)}\{\phi(t)\}]^2} \tag{44}$$

The successive numerators A_s, and denominators B_s ($s = 0, 1, \ldots$) of (30) are given by

$$A_s = X_s/H_s^{(0)}\{\phi(a)\}, \quad B_s = Y_s/H_s^{(0)}\{\phi(a)\}, \tag{45}$$

and satisfy the recursions

$$A_s = \{z-Q_s(a)\}A_{s-1}+E_{s-1}(a)A_{s-2}, \quad B_s = \{z-Q_s(a)\}B_{s-1}+E_{s-1}(a)B_{s-2}, \tag{46}$$

where

$$E_2(a) = \phi(a), \tag{47}$$

based on

$$A_{-1} = 1, \quad A_0 = 0, \quad B_{-1} = 0, \quad B_0 = 1. \tag{48}$$

Finally

$$\varepsilon_{2s}(z\,;a) = A_s/B_s \quad (s = 0, 1, \ldots). \tag{49}$$

The quantities $q_r^{(0)}, e_r^{(0)}$ may easily be recovered from the quantities $E_r(a)$, $Q_r(a)$ by application of equations (39). This remark, in conjunction with equations (36) and (37) implies that the theoretical possibility exists of varying the mode of application of both the $q-d$ and ε-algorithms at any desired stage, changing from the discrete forms (38) and (2) to the differential forms (41) or (31) and (32), or back again at will.

The functions produced by the confluent form of the ε-algorithm may also be derived from those produced by application of the confluent form of the $q-d$ algorithm, simply by using formulae (46), (49), (32) and (37) in that order. The reverse is also made possible by observing that A_s and B_s may be extracted from $\varepsilon_{2s}(z, a)$ from the condition that the coefficient of z^s in B_s is unity; the recursions (46) are then solved for $Q_s(a)$ and $E_{s-1}(a)$.

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