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Abstract

Let X be a Shimura curve of genus zero. In this paper, we first characterize the spaces of automorphic forms on X in terms of Schwarzian differential equations. We then devise a method to compute Hecke operators on these spaces. An interesting by-product of our analysis is the evaluation

$$ {}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; -\frac{2^{10}\cdot 3^3\cdot 5}{11^4}\right) = \sqrt{6}\sqrt[6]{\frac{11}{5^5}} $$

and other similar identities.

1. Introduction

Let K be a totally real number field and R be its ring of integers. Let B be a quaternion algebra over K that splits exactly at one infinite place, i.e.,

$$ B\otimes_{\mathbb{Q}}\mathbb{R}\simeq M(2,\mathbb{R})\times\mathbb{H}^{[K:\mathbb{Q}]-1}, $$

where $M(2,\mathbb{R})$ is the algebra of 2×2 matrices over $\mathbb{R}$ and $\mathbb{H}$ is Hamilton's quaternion algebra. Thus, up to conjugation, there is a unique embedding $\iota : B\hookrightarrow M(2,\mathbb{R})$ from B into $M(2,\mathbb{R})$. Given an order $\mathcal{O}$ of B, let $\mathcal{O}_1^*$ denote the group of elements of reduced norm 1 inside $\mathcal{O}$. Then the image $\Gamma(\mathcal{O}) := \iota(\mathcal{O}_1^*)$ of $\mathcal{O}_1^*$ under ι is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and the quotient space $\Gamma(\mathcal{O})\backslash\mathfrak{H}$ is called the *Shimura curve* associated to $\mathcal{O}$, where $\mathfrak{H}$ denotes the upper half-plane. For example, when $K=\mathbb{Q}$, $B=M(2,\mathbb{Q})$, and $\mathcal{O}=M(2,\mathbb{Z})$, we have $\mathcal{O}_1^*=\mathrm{SL}(2,\mathbb{Z})$ and the Shimura curve associated to $M(2,\mathbb{Z})$ is simply the classical modular curve $Y_0(1)$. Thus, Shimura curves are generalizations of classical modular curves. From now on, the term *Shimura curve* will be reserved strictly for the case $B\neq M(2,\mathbb{Q})$. The compact Riemann surface associated to a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ commensurable to some $\mathcal{O}_1^*$ will also be called a Shimura curve. (See [AB04, ch. 2] and [Elk98, ch. 2] for a more detailed introduction to Shimura curves.)

When B is an indefinite quaternion algebra over $\mathbb{Q}$, Shimura curves have moduli-space interpretation similar to their classical counterpart. Namely, they are moduli spaces of principally polarized abelian surfaces with quaternionic multiplication. (See [Shi67].) Also, many theories and properties about classical modular curves can be extended to the case of Shimura curves. For example, if, for a positive even integer k, we let $S_k(\Gamma(\mathcal{O}))$ or simply $S_k(\mathcal{O})$ denote the space

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of automorphic forms of weight k, i.e., the space consisting of holomorphic functions $f:\mathfrak{H}\to\mathbb{C}$ satisfying

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau)$$

for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma(\mathcal{O})$ and all $\tau\in\mathfrak{H}$, then, similar to the case of classical modular curves, we can define a family of mutually commuting self-adjoint linear operators on $S_k(\mathcal{O})$, called *Hecke operators*. (Here self-adjointness is with respect to the Petersson inner product defined by

$$\langle f,g\rangle=\int_{\Gamma(\mathcal{O})\backslash\mathfrak{H}} f(\tau)\overline{g(\tau)}y^k\frac{dx\,dy}{y^2},\quad \tau=x+iy,$$

where the integration is taken over any fundamental domain of $\Gamma(\mathcal{O})$.) Then the space $S_k(\mathcal{O})$ contains a basis consisting of simultaneous eigenforms, called *Hecke eigenforms*, for all Hecke operators. In fact, according to the Jacquet–Langlands correspondence [JL70, ch. 16], each Hecke eigenform corresponds to an irreducible automorphic representation of $\mathrm{GL}(2,\mathbb{Q})$ that is an inner twist of a certain irreducible cuspidal representation. In other words, for each Hecke eigenform on $\Gamma(\mathcal{O})$, there corresponds a Hecke eigenform on a certain modular group with the same eigenvalues.

On the other hand, even though it is true that many theoretical aspects of classical modular curves can be extended to the case of Shimura curves, it is not true for explicit methods. The main obstacle lies at the lack of cusps on Shimura curves. Namely, in the case of classical modular curves, many problems about modular curves can be answered using q-expansions (i.e., expansions with respect to a local parameter at a *cusp*) of modular forms or modular functions involved, and there are many explicit methods for constructing modular functions and modular forms and computing their q-expansions. In fact, because the Fourier coefficients of a normalized Hecke eigenform on congruence subgroups are identical with the eigenvalues of Hecke operators, one can compute the q-expansions of Hecke eigenforms without actually constructing them. However, because there are no cusps on Shimura curves, any method for classical modular curves that uses q-expansions cannot possibly be extended to the case of Shimura curves. Moreover, as far as we know, eigenvalues for Hecke operators on automorphic forms on Shimura curves do not say anything about Taylor coefficients of automorphic forms. Thus, it is both interesting and challenging to find explicit methods for Shimura curves.

In this paper, for a Shimura curve of genus zero, we will first characterize the spaces of automorphic forms in terms of Schwarzian differential equations. (See Remark 2 for the definition of Schwarzian differential equations.) In other words, spaces of automorphic forms will be represented using solutions of certain differential equations. This makes explicit computation on automorphic forms possible. For example, in the second half of the paper, we will devise a method to compute Hecke operators and hence determine Hecke eigenforms. Two examples will be worked out. As by-products of our analysis, we find the following intriguing evaluations

$$_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};-\frac{2^{10}\cdot3^3\cdot5}{11^4}\right)=\sqrt{6}\sqrt[6]{\frac{11}{5^5}},$$

$$_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};-\frac{2^{10}\cdot3^3\cdot5}{11^4}\right)=\frac{\sqrt[3]{2}(1+\sqrt{2})\sqrt[6]{11^5}}{20\sqrt{3}}\frac{\Gamma(7/6)\Gamma(13/24)\Gamma(19/24)}{\Gamma(5/6)\Gamma(17/24)\Gamma(23/24)},$$

and other similar identities, where $_2F_1(a,b;c;x)$ denotes the $_2F_1$-hypergeometric function. (See Corollary 18 and Remark 19 below.)

Since Schwarzian differential equations play a crucial role in our approach, it is an important problem to determine the differential equation associated to each Shimura curve of genus zero.

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Some work has already been done in this direction. In particular, Bayer and Travesa [BT07] considered Shimura curves associated to a maximal order in the quaternion algebra of discriminant 6 over $\mathbb{Q}$. In [Tu11], Tu determined the Schwarzian differential equations associated to certain Eichler orders in quaternion algebras over $\mathbb{Q}$.

As of now, our method only works for Shimura curves of genus zero. We hope to extend our method to Shimura curves of higher genus in the future. Note that Sijsling [Sij12] has already considered the cases of Shimura curves of genus one with exactly one elliptic point and obtained differential equations associated to these curves. It will be an interesting problem to combine his equations with our approach to study Shimura curves of genus one.

Finally, we remark that the results in this paper can be used to compute modular equations for Shimura curves, which in turn can be used to determine the coordinates of CM-points on Shimura curves. This is currently a work in progress.

2. Schwarzian differential equations and automorphic forms on Shimura curves

Let $X(\mathcal{O})$ be a Shimura curve with the associated norm-one group $\Gamma(\mathcal{O})$. It is known since the nineteenth century that if $F(\tau)$ is a meromorphic automorphic form of weight k (with some multiplier system) and $t(\tau)$ is a nonconstant automorphic function on $\Gamma(\mathcal{O})$, then $F, \tau F, \ldots, \tau^k F$, as functions of t, span the solution space of a $(k+1)$st order linear ordinary differential equation

$$\theta^{k+1}F + r_k(t)\theta^k F + \cdots r_0(t)F = 0, \quad \theta = t\frac{d}{dt},$$

with algebraic functions as coefficients $r_j(t)$. (See [Sti84, Theorem 5.1] and also [Yan04, Theorem 1].) In this paper, we refer to this kind of differential equations as *automorphic differential equations*. If the compact Riemann surface $\Gamma\backslash\mathfrak{H}$ has genus zero, which we assume from now on, and $t(\tau)$ is a generator of the function field on $\Gamma(\mathcal{O})$, which we call a *Hauptmodul* of $\Gamma(\mathcal{O})$, then the coefficients $r_j(t)$ are actually rational functions and all the singularities of the differential equations are regular. In fact, if $F(\tau)$ is a holomorphic automorphic form, then the singularities of the differential equation are precisely the points where the function $\tau \to t(\tau)$ fails to be locally one-to-one, that is, where the values of t correspond to elliptic points. If the number of elliptic points is 3, i.e., if $\Gamma(\mathcal{O})$ is a triangle group, then it is a classical fact that a second order ordinary differential equation with exactly three regular singular points is completely determined by the local exponents. In this way, one can write down an automorphic differential equation associated to a triangle group without actually finding an automorphic form and a Hauptmodul first. (In fact, such a differential equation corresponds to the symmetric power of an algebraic transformation of a ${}_2F_1$-hypergeometric function.) Then one can study properties of Shimura curves using this differential equation. This method has been used by [Elk98, Voi06] to study CM (complex multiplication) points on Shimura curves.

When a Shimura curve $X(\mathcal{O})$ of genus zero has more than three elliptic points, the determination of automorphic differential equations is more complicated. In [Elk98], Elkies determined a differential equation associated to the normalizer of a maximal order in a quaternion algebra of discriminant 10 over $\mathbb{Q}$ and then used this differential equation to numerically compute the coordinates of CM points on a Hauptmodul. In [BT07], Bayer and Travesa obtained automorphic differential equations for the Shimura curve associated to the maximal order in the indefinite quaternion algebra of discriminant 6 and its various Atkin–Lehner quotients. In both cases, the determination uses geometry of Shimura curves. For instance, in [Elk98], Elkies used the covering between two Shimura curves. Other than these two isolated results, as far as we know, there is no systematic attempt in literature to determine such differential equations. (Note that in [Elk98], such an automorphic differential equation is called a Schwarz equation. However, in this paper, we will reserve the term *Schwarz equation* for a certain normalized automorphic differential equation because of its connection to the Schwarzian derivative. See Proposition 1 below.)

To fully realize the potential of the method of automorphic differential equations, we shall first normalize such differential equations. The idea is that if $t(\tau)$ is an automorphic function on $\Gamma(\mathcal{O})$, then $t'(\tau)$ is a meromorphic automorphic form of weight 2 on $\Gamma(\mathcal{O})$. Thus, $t'(\tau)^{1/2}$, as a function of t, satisfies a second-order ordinary differential equation. This differential equation can be regarded as a normal form for all automorphic differential equations associated to $\Gamma(\mathcal{O})$ because it depends only on the chosen automorphic function $t(\tau)$. In fact, there is a simple formula to convert a general automorphic differential equation to such a differential equation satisfied by $t'(\tau)$ and $t(\tau)$.

Proposition 1 [For72, pp. 99–102, 290]. *Let $X(\mathcal{O})$ be a Shimura curve. Let $F(\tau)$ be an automorphic form of weight* 1 *(with some multiplier system) and $t(\tau)$ be a nonconstant automorphic function on $\Gamma(\mathcal{O})$. If the second-order differential equation satisfied by $F(\tau(t))$ by expressing τ as a t series is*

$$\theta^2 F + r_1(t)\theta F + r_0(t)F = 0, \quad \theta = t\frac{d}{dt},$$

then the differential equation satisfied by $t'(\tau)^{1/2}$ and $t(\tau)$ is

$$\frac{d^2}{dt^2}G + Q(t)G = 0, \tag{1}$$

where

$$Q(t) = \frac{1 + 4r_0 - 2t(dr_1/dt) - r_1^2}{4t^2}$$

and satisfies

$$2Q(t)t'(\tau)^2 + \{t, \tau\} = 0, \quad \{t, \tau\} = \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\left(\frac{t''(\tau)}{t'(\tau)}\right)^2.$$

The reader who is unfamiliar with the relation between automorphic forms and differential equations may find the proof of this proposition given in [Yan04] easier to comprehend.

Remark 2. The function $\{t, \tau\}$ is classically known as the *Schwarzian derivative*. (See [Hil97, ch. 10].) In our setting, it is a meromorphic automorphic form of weight 4 on $\Gamma(\mathcal{O})$. In view of its connection to the Schwarzian derivative, we call the differential equation in (1) satisfied by $t'(\tau)^{1/2}$ and $t(\tau)$ the *Schwarzian differential equation* associated to t. Note that the function $-\{t, \tau\}/2t'(\tau)^2$, up to a factor of -4, is called the *automorphic derivative* in [BT07]. Here we will follow the same terminology.

Notation 3. For a thrice-differentiable function f of z, we let $D(f, z)$ denote the automorphic derivative

$$D(f, z) = -\frac{\{f, z\}}{2f'(z)^2}, \quad \{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Now the upshot is that if $X(\mathcal{O})$ has genus zero and $t(\tau)$ is a Hauptmodul, then the analytic behavior of $t'(\tau)$ is simple to describe and it is easy to express all (holomorphic) automorphic

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forms with trivial character in terms of $t'(\tau)$. Here the dimension formula in the following theorem is taken from [Shi94].

Theorem 4. *Assume that a Shimura curve X has genus zero with elliptic points $\tau_1, \ldots, \tau_r$ of order $e_1, \ldots, e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of X and set $a_i = t(\tau_i)$, $i = 1, \ldots, r$. For a positive even integer $k \geqslant 4$, let*

$$d_k = \dim S_k(\mathcal{O}) = 1 - k + \sum_{j=1}^{r} \left\lfloor \frac{k}{2}\left(1 - \frac{1}{e_j}\right) \right\rfloor .$$

Then a basis for the space of automorphic forms of weight k on X is

$$t'(\tau)^{k/2} t(\tau)^j \prod_{i=1, a_i \neq \infty}^{r} (t(\tau) - a_i)^{-\lfloor k(1-1/e_i)/2 \rfloor}, \quad j = 0, \ldots, d_k - 1.$$

Proof. If $a_j \neq \infty$, then we have

$$t(\tau) - a_j = C_j (\tau - \tau_j)^{e_j} + O((\tau - \tau_j)^{e_j + 1}), \quad C_j \in \mathbb{C},$$

near τ_j since τ_j is assumed to be an elliptic point of order e_j. The constant C_j cannot be zero because $t - a_j$ is also a Hauptmodul and cannot have a zero of order greater than 1 (as a function on X) at a_j. Then

$$t'(\tau) = C_j e_j (\tau - \tau_j)^{e_j - 1} + O((\tau - \tau_j)^{e_j}) \tag{2}$$

near τ_j. Thus, the function

$$t'(\tau)^{k/2} \prod_{i=1, a_i \neq \infty}^{r} (t(\tau) - a_i)^{-\lfloor k(1-1/e_i)/2 \rfloor} \tag{3}$$

is a (possibly meromorphic) automorphic form of weight k that is holomorphic throughout $\mathfrak{H}$ except for possibly the point where $t(\tau) = \infty$.

Consider first the case $a_j \neq \infty$ for all j. Let τ_0 be the point where $t = \infty$. Since t is a Hauptmodul, t must have a simple pole at τ_0, that is

$$t(\tau) = \frac{C_0}{\tau - \tau_0} + O(1), \quad C_0 \in \mathbb{C}.$$

Then $t'(\tau) = -C_0/(\tau - \tau_0)^2 + O(1)$ and the order of the function in (3) at τ_0 is

$$A := -k + \sum_{j=1}^{r} \left\lfloor \frac{k}{2}\left(1 - \frac{1}{e_j}\right) \right\rfloor .$$

If $\dim S_k(\mathcal{O}) \neq 0$, then $A \geqslant 0$ and thus the function in (3) is holomorphic throughout $\mathfrak{H}$. In fact, we can multiply the function in (3) by a polynomial of degree $\leqslant A$ in t and still get a function holomorphic throughout $\mathfrak{H}$. Since the dimension of the space of polynomials of degree $\leqslant A$ is the same as d_k, we conclude that the functions in the statement of the theorem form a basis for $S_k(\mathcal{O})$.

Now assume that $a_j = \infty$ for some j, say, $a_1 = \infty$. Again, because t is assumed to be a Hauptmodul, we must have

$$t(\tau) = \frac{C_1}{(\tau - \tau_1)^{e_1}} + O((\tau - \tau_1)^{1 - e_1}), \quad C_1 \in \mathbb{C},$$

near τ_1. Then the order of the function (3), as a function of τ, at τ_1 is

$$B := -\frac{k}{2}(e_1+1) + e_1 \sum_{i=2}^{r} \left\lfloor \frac{k}{2}\left(1-\frac{1}{e_j}\right)\right\rfloor .$$

We can multiply the function in (3) by a polynomial in t of degree not exceeding

$$\left\lfloor \frac{B}{e_1} \right\rfloor = \left\lfloor -\frac{k}{2}\left(1+\frac{1}{e_1}\right)\right\rfloor + \sum_{i=2}^{r} \left\lfloor \frac{k}{2}\left(1-\frac{1}{e_j}\right)\right\rfloor = -k + \sum_{j=1}^{r} \left\lfloor \frac{k}{2}\left(1-\frac{1}{e_j}\right)\right\rfloor$$

and still get a function holomorphic throughout $\mathfrak{H}$. Again, we conclude that the functions in the statement form a basis for $S_k(\mathcal{O})$. This completes the proof. □

The combination of Proposition 1 and Theorem 4 gives us a concrete space of functions that can be used to study properties of automorphic forms, provided that a Schwarzian differential equation has been determined. In the second half of the paper, we will provide a method to compute Hecke operators using these functions.

In view of the importance of Schwarzian differential equations, here we shall review some analytic properties of Schwarzian differential equations. This information will be helpful in determining the differential equations.

Proposition 5. *Automorphic derivatives have the following properties:*

(i) $D((az+b)/(cz+d), z) = 0$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})$;

(ii) $D(g\circ f, z) = D(g,f) + D(f,z)/(dg/df)^2$.

Proof. Both properties are well-known and can be verified directly. □

Proposition 6. *Assume that $X(\mathcal{O})$ has genus zero with elliptic points $\tau_1,\ldots,\tau_r$ of order $e_1,\ldots,e_r$, respectively. Let $t(\tau)$ be a Hauptmodul of $X(\mathcal{O})$ and set $a_i = t(\tau_i)$, $i=1,\ldots,r$. Then the automorphic derivative $Q(t)$ in Proposition 1 is equal to*

$$Q(t) = \frac{1}{4} \sum_{j=1, a_j\neq\infty}^{r} \frac{1-1/e_j^2}{(t-a_j)^2} + \sum_{j=1, a_j\neq\infty}^{r} \frac{B_j}{t-a_j}$$

for some constants B_j. Moreover, if $a_j \neq \infty$ for all j, then the constants B_j satisfy

$$\sum_{j=1}^{r} B_j = \sum_{j=1}^{r} \left(a_j B_j + \frac{1}{4}(1-1/e_j^2)\right) = \sum_{j=1}^{r} \left(a_j^2 B_j + \frac{1}{2} a_j (1-1/e_j^2)\right) = 0.$$

Also, if $a_r = \infty$, then B_j satisfy

$$\sum_{j=1}^{r-1} B_j = 0, \quad \sum_{j=1}^{r-1} \left(a_j B_j + \frac{1}{4}(1-1/e_j^2)\right) = \frac{1}{4}(1-1/e_r^2).$$

In principle, the properties stated above were known for a long time. However, in literature, usually the proposition is stated under the assumption that a_i are all real. The reason for this assumption is that in the standard books on ordinary differential equations in the complex domain, one usually starts from a second-order linear differential equation and builds an automorphic function from it. At some point, one would need to employ the Schwarz reflection principle in order to extend the domain of the definition of the automorphic function to the whole upper half-plane. This is where the assumption that a_i are all real comes in.

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(See [Hil97, Theorem 10.2.1].) Here, because we actually start from an automorphic function first, we do not need this assumption. For convenience of the reader, we provide a complete proof of the proposition here.

Proof. We first consider the analytic behavior of

$$Q(t) = -\frac{1}{2t'(\tau)^2}\left(\frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\left(\frac{t''(\tau)}{t'(\tau)}\right)^2\right)$$

at a point $t_0 = t(\tau_0) \neq \infty$ that does not correspond to any elliptic point. We have

$$t(\tau) = t_0 + c(\tau - \tau_0) + \cdots$$

for some constant c. Since $t(\tau) - t_0$ is also a Hauptmodul, the constant c cannot be 0. From this we easily see that $Q(t)$ is holomorphic at t_0.

We next consider the case $t_0 = a_j \neq \infty$ corresponding to an elliptic point τ_j of order e_j.

$$t(\tau) = a_j + c_1(\tau - \tau_0)^{e_j} + c_2(\tau - \tau_0)^{e_j+1} + \cdots$$

for some constants $c_1, c_2 \in \mathbb{C}$.

Again, because $t(\tau) - a_j$ is also a Hauptmodul, c_1 cannot be equal to 0. When $e_j \geqslant 3$, we have

$$\begin{aligned}\frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\frac{t''(\tau)^2}{t'(\tau)^2} &= \frac{c_1e_j(e_j-1)(e_j-2)(\tau-\tau_0)^{e_j-3} + \cdots}{c_1e_j(\tau-\tau_0)^{e_j-1} + \cdots} - \frac{3}{2}\frac{c_1^2e_j^2(e_j-1)^2(\tau-\tau_0)^{2e_j-4} + \cdots}{c_1^2e_j^2(\tau-\tau_0)^{2e_j-2} + \cdots}\\ &= \frac{1}{2}\frac{1-e_j^2}{(\tau-\tau_0)^2} + \cdots .\end{aligned}$$

When $e_j = 2$, we have

$$\begin{aligned}\frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\frac{t''(\tau)^2}{t'(\tau)^2} &= \frac{6c_2 + \cdots}{2c_1(\tau-\tau_0) + \cdots} - \frac{3}{2}\frac{4c_1^2 + \cdots}{4c_1^2(\tau-\tau_0)^2 + \cdots}\\ &= \frac{1}{2}\frac{1-e_j^2}{(\tau-\tau_0)^2} + \cdots .\end{aligned}$$

Either way, we have

$$Q(t) = -\frac{1}{2c_1^2e_j^2(\tau-\tau_0)^{2e_j-2} + \cdots}\left(\frac{1}{2}\frac{1-e_j^2}{(\tau-\tau_0)^2} + \cdots\right) = \frac{1-1/e_j^2}{4c_1^2(\tau-\tau_0)^{2e_j}} + \cdots ,$$

which implies that

$$Q(t) - \frac{1}{4}\frac{1-1/e_j^2}{(t-a_j)^2},$$

as a function of t, has at most a simple pole at a_j. Thus, if we let B_j denote the residue of $Q(t)$ at a_j, then

$$P(t) := Q(t) - \frac{1}{4}\sum_{j=1, a_j\neq\infty}^{r} \frac{1-1/e_j^2}{(t-a_j)^2} - \sum_{j=1, a_j\neq\infty}^{r} \frac{B_j}{t-a_j} \tag{4}$$

will be a polynomial function in t. We now consider the behavior of $Q(t)$ at ∞.

If $a_j \neq \infty$ for all elliptic points τ_j, letting $\tau_0 \in \mathfrak{H}$ be a point with $t(\tau_0) = \infty$, we have

$$t(\tau) = \frac{c}{\tau - \tau_0} + \cdots$$

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for some nonzero constant c. Then

$$\begin{aligned} Q(t) &= -\frac{1}{2(-c(\tau-\tau_0)^{-2}+O(1))^2}\left(\frac{-6c(\tau-\tau_0)^{-4}+O(1)}{-c(\tau-\tau_0)^{-2}+O(1)} - \frac{3}{2}\frac{4c^2(\tau-\tau_0)^{-6}+O((\tau-\tau_0)^{-3})}{c^2(\tau-\tau_0)^{-4}+O((\tau-\tau_0)^{-2})}\right) \\ &= O((\tau-\tau_0)^4) = O(t^{-4}) \end{aligned} \tag{5}$$

as $\tau \to \tau_0$. In particular, we have $Q(t) \to 0$ as $t \to \infty$, which implies that $P(t) = 0$. Moreover, the coefficients of t^{-1}, t^{-2}, t^{-3} in the expansion of

$$\frac{1}{4}\sum_{j=1, a_j\neq\infty}^{r} \frac{1-1/e_j^2}{(t-a_j)^2} + \sum_{j=1, a_j\neq\infty} \frac{B_j}{t-a_j} \tag{6}$$

at $t = \infty$ are

$$\sum_{j=1}^{r} B_j, \quad \sum_{j=1}^{r}\left(a_jB_j + \frac{1}{4}(1-1/e_j^2)\right), \quad \sum_{j=1}^{r}\left(a_j^2B_j + \frac{1}{2}a_j(1-1/e_j^2)\right),$$

respectively. In view of (5), all the three sums must be equal to 0. This proves the proposition in the case $a_j \neq \infty$ for all j.

Likewise, if $a_j = \infty$ for some elliptic point τ_j, say, $a_r = \infty$, we have

$$t(\tau) = \frac{c}{(\tau-\tau_r)^{e_r}} + \cdots$$

for some nonzero constant c. Then

$$\begin{aligned} Q(t) &= -\frac{1}{2(-ce_r(\tau-\tau_r)^{-e_r-1}+\cdots)^2}\left(\frac{-ce_r(e_r+1)(e_r+2)(\tau-\tau_0)^{-e_r-3}+\cdots}{-ce_r(\tau-\tau_r)^{-e_r-1}+\cdots}\right. \\ &\qquad \left. - \frac{3}{2}\frac{c^2e_r^2(e_r+1)^2(\tau-\tau_r)^{-2e_r-4}+\cdots}{c^2e_r^2(\tau-\tau_r)^{-2e_r-2}+\cdots}\right) \\ &= \frac{1-1/e_r^2}{4c^2}(\tau-\tau_r)^{2e_r} + \cdots = \frac{1-1/e_r^2}{4t^2} + O(t^{-3}) \end{aligned}$$

as $t \to \infty$. This also shows that the polynomial $P(t)$ in (4) is actually 0. Also, comparing the coefficients of t^{-1} and t^{-2} in the expansion of (6) with the asymptotic behavior of $Q(t)$ at ∞ given above, we find

$$\sum_{j=1}^{r-1} B_j = 0, \quad \sum_{j=1}^{r-1}\left(a_jB_j + \frac{1}{4}(1-1/e_j^2)\right) = \frac{1}{4}(1-1/e_r^2).$$

This proves the proposition for the case $a_j = \infty$ for some j. □

Here we will give two examples of automorphic derivatives. Let us first fix some notations.

Notation 7. For an Eichler order $\mathcal{O}$ of level N in an indefinite quaternion algebra of discriminant D over $\mathbb{Q}$, we let $\Gamma_D(N)$ denote the group $\Gamma(\mathcal{O})$ and $X_D(N)$ denote the Shimura curve associated to $\mathcal{O}$. If we let W_D be the group of Atkin–Lehner involutions $\{w_e : e \mid D\}$, then $\Gamma_D^*(N)$ and $X_D^*(N)$ will denote the group $\bigcup_{e|D} w_e\Gamma_D(N)$ and the quotient curve $X_D(N)/W_D$, respectively.

Also, as is customary, if a Shimura curve has genus g with m_i elliptic points of order e_i, we use the notation $(g; e_1^{m_1}, \ldots, e_r^{m_r})$ to encode the signature of the curve.

Example 8. Consider the Shimura curve $X_6^*(1)$. The signature of $X_6^*(1)$ is $(0; 2, 4, 6)$. Choose a Hauptmodul t of $X_6^*(1)$ by requiring that t takes values 0, 1, and ∞ at the elliptic points of

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orders 6, 2, and 4, respectively. By Proposition 6,

$$Q(t) = \frac{35/144}{t^2} + \frac{3/16}{(t-1)^2} + \frac{B_1}{t} + \frac{B_2}{t-1},$$

where B_1 and B_2 satisfy

$$B_1 + B_2 = 0, \quad B_2 + \frac{35}{144} + \frac{3}{16} = \frac{15}{64}.$$

Thus,

$$Q(t) = \frac{35/144}{t^2} + \frac{3/16}{(t-1)^2} + \frac{113/576}{t} + \frac{113/576}{1-t}.$$

The local exponents provided by $(1 \pm 1/e_i)/2$ of the differential equation $d^2G/dt^2 + Q(t)G = 0$ at 0, 1, and ∞, are $\{5/12, 7/12\}$, $\{1/4, 3/4\}$, and $\{-5/8, -3/8\}$, respectively. Thus, the differential equation satisfied by $t^{-5/12}(1-t)^{-1/4}t'(\tau)^{1/2}$ (as a function of t) will have local exponents $\{0, 1/6\}$, $\{0, 1/2\}$, and $\{1/24, 7/24\}$ at 0, 1, and ∞, respectively. In other words, $t^{-5/12}(1-t)^{-1/4}t'(\tau)^{1/2}$, as a function of t, is a solution of the hypergeometric differential equation

$$\theta\left(\theta - \frac{1}{6}\right)F - t\left(\theta + \frac{1}{24}\right)\left(\theta + \frac{7}{24}\right)F = 0, \quad \theta = t\frac{d}{dt}.$$

Therefore, if we fix a representative $\tau_1 \in \mathfrak{H}$ of the elliptic point of 6, then in a neighborhood of τ_1, we have

$$t'(\tau) = t^{5/6}(1-t)^{1/2}\left(C_1 \cdot {}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; t\right) + C_2 t^{1/6} {}_2F_1\left(\frac{5}{24}, \frac{11}{24}; \frac{7}{6}; t\right)\right)^2, \tag{7}$$

where C_1 and C_2 are complex numbers depending on the choice of the embedding of the quaternion algebra into $M(2, \mathbb{R})$ and the choice of τ_1. Note that from (2), we know that C_1 is nonzero. Then by Theorem 4, for even positive integers $k \geqslant 4$, the automorphic forms whose t-expansions near the point τ_1 are

$$t^{\{5k/12\}}(1-t)^{\{k/4\}}t^j\left({}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; t\right) + C t^{1/6} {}_2F_1\left(\frac{5}{24}, \frac{11}{24}; \frac{7}{6}; t\right)\right)^k, \tag{8}$$

$j = 0, \ldots, \lfloor 5k/12 \rfloor + \lfloor k/4 \rfloor + \lfloor 3k/8 \rfloor - k$, form a basis for the space of automorphic forms of weight k on $\Gamma_6^*(1)$, where $C = C_2/C_1$ and for a rational number x we let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. In § 4, we will compute the constant C and Hecke operators relative to this basis.

More generally, automorphic forms on triangle groups can all be expressed in terms of hypergeometric functions.

Theorem 9. *Assume that a Shimura curve X has signature $(0; e_1, e_2, e_3)$. Let $t(\tau)$ be the Hauptmodul of X with values 0, 1, and ∞ at the elliptic points of order e_1, e_2, and e_3, respectively. Let $k \geqslant 4$ be an even integer. Then a basis for the space of automorphic forms of weight k on X is given by*

$$t^{\{k(1-1/e_1)/2\}}(1-t)^{\{k(1-1/e_2)/2\}}t^j({}_2F_1(a, b; c; t) + C t^{1/e_1} {}_2F_1(a', b', c'; t))^k,$$

$j = 0, \ldots, \lfloor k(1-1/e_1)/2 \rfloor + \lfloor k(1-1/e_2)/2 \rfloor + \lfloor k(1-1/e_3)/2 \rfloor - k$, for some constant C, where

$$a = \frac{1}{2}\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right), \quad b = a + \frac{1}{e_3}, \quad c = 1 - \frac{1}{e_1}$$

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and

$$a' = a + \frac{1}{e_1}, \quad b' = b + \frac{1}{e_1}, \quad c' = c + \frac{2}{e_1}.$$

Proof. The proof follows the same argument as in the preceding example. Here we just remind the reader that the hypergeometric function ${}_2F_1(a, b, c; t)$ is a solution of

$$\theta(\theta + c - 1)F - t(\theta + a)(\theta + b)F = 0, \quad \theta = \frac{d}{dt},$$

whose local exponents at 0, 1, and ∞ are $\{0, 1 - c\}$, $\{0, c - a - b\}$, and $\{a, b\}$, respectively. From this, it is easy to figure out the parameters in the hypergeometric functions. We omit the details. □

Example 10. Consider the Shimura curve $X_{10}^*(1)$. The signature of $X_{10}^*(1)$ is $(0; 3, 2^3)$. In [Elk98], Elkies showed that there is a Hauptmodul on $X_{10}^*(1)$ with values 0 at the elliptic point of order 3 and values 2, 27, and ∞ at the three elliptic points of order 2. Moreover, using the covering $X_{10}^*(3) \to X_{10}^*(1)$, he also showed that an automorphic differential equation for $X_{10}^*(1)$ is

$$t(t - 2)(t - 27)F'' + \frac{10t^2 - 203t + 216}{6}F' + \frac{7t - 56}{144}F = 0,$$

which is the same as

$$\theta^2 F + \frac{4t^2 - 29t - 108}{6(t - 2)(t - 27)}\theta F + \frac{7t(t - 8)}{144(t - 2)(t - 27)}F = 0, \quad \theta = t\frac{d}{dt}.$$

Using Proposition 1, we find that the automorphic derivative $D(t, \tau)$ associated to t is

$$D(t, \tau) = \frac{3t^4 - 119t^3 + 3157t^2 - 7296t + 10368}{16t^2(t - 2)^2(t - 27)^2}. \tag{9}$$

Here we will use Propositions 5 and 6 to obtain the same result.

According to Proposition 6, we have

$$D(t, \tau) = \frac{2/9}{t^2} + \frac{3/16}{(t - 2)^2} + \frac{3/16}{(t - 27)^2} + \frac{B_1}{t} + \frac{B_2}{t - 2} + \frac{B_3}{t - 27},$$

where B_i satisfy

$$B_1 + B_2 + B_3 = 0, \quad 2B_2 + 27B_3 + \frac{2}{9} + \frac{3}{16} + \frac{3}{16} = \frac{3}{16},$$

which yield

$$B_1 = \frac{25}{2}B_3 + \frac{59}{288}, \quad B_2 = -\frac{27}{2}B_3 - \frac{59}{288}.$$

On the other hand, the Shimura curve $X_{10}^*(3)$ has signature $(0; 2^4, 3)$. According to [Elk98], there is a Hauptmodul u on $X_{10}^*(3)$ whose relation with t is

$$t = \frac{216(u - 1)^3}{(u + 1)^2(9u^2 - 10u + 17)} \tag{10}$$

and whose values at the four elliptic points of order 2 and the elliptic point of order 3 are $(5 \pm 2\sqrt{-5})/9$, $(5 \pm 8\sqrt{-2})/9$, and ∞, respectively. (Note that in (57) of [Elk98], the factor $9x^2 - 10x + 17$ in the denominator was misprinted as $9x^2 - 10x + 7$.) Also, the action of Atkin–Lehner involution w_3 is

$$w_3 : u \longmapsto \frac{10}{9} - u. \tag{11}$$

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By Proposition 6,

$$\begin{aligned} D(u,\tau) &= \frac{3(162u^2-180u+10)/16}{(9u^2-10u+5)^2} + \frac{3(162u^2-180u-206)/16}{(9u^2-10u+17)^2} \\ &\quad + \frac{C_1+C_2u}{9u^2-10u+5} + \frac{C_3+C_4u}{9u^2-10u+17} \end{aligned} \tag{12}$$

for some constants $C_1,\ldots,C_4$. According to the proof of Proposition 6 the constants $C_1,\ldots,C_4$ satisfy $Q(u)=2/9t^2+O(t^{-3})$ as $t\to\infty$. Thus,

$$C_4=-C_2,\quad C_3=-C_1-\frac{19}{4}.$$

Now observe that the Atkin–Lehner involution $w_3: u\mapsto 10/9-u$ switches the two roots of $9u^2-10u+5$ and the two roots of $9u^2-10u+17$ and fixes ∞. From this we deduce that $D(u,\tau)=D(10/9-u,\tau)$, which implies that the right-hand side of (12) is invariant under the substitution $u\mapsto 10/9-u$. From this, we infer that $C_2=C_4=0$. That is,

$$\begin{aligned} D(u,\tau) &= \frac{3(162u^2-180u+10)/16}{(9u^2-10u+5)^2} + \frac{3(162u^2-180u-206)/16}{(9u^2-10u+17)^2} \\ &\quad + \frac{C_1}{9u^2-10u+5} + \frac{-C_1-19/4}{9u^2-10u+17}. \end{aligned}$$

Now let $R(x)=216(x-1)^3/(x+1)^2/(9x^2-10x+17)$. We have $t=R\circ u$. Thus, by part (ii) of Proposition 5, we have

$$D(t,\tau)=D(R(u),u)+\frac{D(u,\tau)}{(dR(u)/du)^2}.$$

Expressing the left-hand side in terms of u and comparing it with the right-hand side, we find that the constants B_3 and C_1 are

$$B_3=-\frac{953}{97\,200},\quad C_1=-\frac{1}{24}.$$

This gives us (9).

Remark 11. In [Tu11], Tu determined the Schwarzian differential equations associated to $X_D^*(N)$ for the following pairs of (D,N):

$$\begin{gathered}(6,1),\ (6,5),\ (6,7),\ (6,13),\ (10,1),\ (10,3),\ (10,7),\\ (14,1),\ (14,3),\ (14,5),\ (15,1),\ (15,2),\ (15,4),\ (21,1),\\ (21,2),\ (26,1),\ (26,3),\ (35,1),\ (35,2),\ (39,1),\ (39,2).\end{gathered}$$

For example, for $(D,N)=(39,1)$, she showed that there exists a Hauptmodul $t(\tau)$ for $X_{39}^*(1)$ that takes values $\pm 2i$, $(-1\pm\sqrt{-3})/2$, and $(-23+\sqrt{-3})/14$ at the CM-points of discriminants -52, -39, and -156, respectively, and the function $Q(t)$ in the Schwarzian differential equation associated to $t(\tau)$ is

$$Q(t)=-\frac{P_1(t)}{P_2(t)}$$

with

$$P_1(t)=3(2596+7104t+9692t^2+12\,348t^3+13\,149t^4+9522t^5+4367t^6+1086t^7+97t^8)$$

and

$$P_2(t)=4(4+t^2)^2(1+t+t^2)^2(19+23t+7t^2)^2.$$

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The rest of the paper will be devoted to the computation of Hecke operators on automorphic forms. But before we work on the case of Shimura curves, let us first work out a familiar example from classical modular curves to give the reader a clearer idea about our approach.

3. Computing Hecke operators – an example

Let $\Delta(\tau)=\eta(\tau)^{24}$ be the unique normalized Hecke eigenform on $\mathrm{SL}(2,\mathbb{Z})$. Of course, from the Fourier expansion $\Delta(\tau)=q-24q^2+\cdots$, we immediately see that the eigenvalue for the Hecke operator T_2 is -24. Our goal in this section is to obtain the same result without resorting to Fourier expansions.

It is a classical identity that

$$\Delta(\tau)=t\cdot {}_2F_1\left(\frac{1}{12},\frac{5}{12};1;1728t\right)^{12},\quad t(\tau)=\frac{1}{j(\tau)},$$

where $j(\tau)$ is the elliptic j-function. (In fact, this can also be verified using a slightly modified version of our Theorem 9.) Thus, as long as the imaginary part of τ is large, we may expand $\Delta(\tau)$ with respect to t. Assuming that $\operatorname{Im}\tau$ is large, by the definition of T_2, we have

$$\begin{aligned}T_2\Delta(\tau)&=2^{11}t(2\tau){}_2F_1\left(\frac{1}{12},\frac{5}{12};1;t(2\tau)\right)^{12}+\frac{1}{2}t(\tau/2){}_2F_1\left(\frac{1}{12},\frac{5}{12};1;t(\tau/2)\right)^{12}\\&\quad+\frac{1}{2}t((\tau+1)/2){}_2F_1\left(\frac{1}{12},\frac{5}{12};1;t((\tau+1)/2)\right)^{12}.\end{aligned}\tag{13}$$

Now suppose that we are allowed to use Fourier expansions for the moment. We have $t(\tau)=1/j(\tau)=q-744q^2+\cdots$ and

$$\begin{aligned}t(2\tau)&=q^2-744q^4+\cdots=t(\tau)^2+1488t(\tau)^3+\cdots,\\t(\tau/2)&=q^{1/2}-744q+\cdots=t(\tau)^{1/2}-744t(\tau)+\cdots,\\t((\tau+1)/2)&=-q^{1/2}-744q+\cdots=-t(\tau)^{1/2}-744t(\tau)+\cdots.\end{aligned}\tag{14}$$

Substituting these expressions into (13), we get

$$\begin{aligned}T_2\Delta(\tau)&=2^{11}(t^2+\cdots)(1+60t^2+\cdots)^{12}\\&\quad+\tfrac{1}{2}(t^{1/2}-744t+\cdots)(1+60t^{1/2}+\cdots)^{12}\\&\quad+\tfrac{1}{2}(-t^{1/2}-744t+\cdots)(1-60t^{1/2}+\cdots)^{12}\\&=-24t(\tau)+\cdots=-24\Delta(\tau).\end{aligned}$$

From this, we see that the eigenvalue for T_2 is indeed -24. Note that there is an ambiguity in the choice of the square root of t in (14), but it does not affect the final result.

Of course, we have cheated a little bit in the above computation by using Fourier expansion in (14). We now discuss how to obtain the same t-expansions without using q-expansions. The idea is to use the so-called *modular equation*, which is the polynomial relation satisfied by $j(\tau)$ and $j(2\tau)$.

Observe that $t(\tau)$ and $t(2\tau)$ are both modular functions on $\Gamma_0(2)$. Let $u(\tau)$ be a Hauptmodul of $\Gamma_0(2)$. Since $X_0(2)\to X_0(1)$ is a covering of degree 3, we have $t(\tau)=R(u(\tau))$ for some rational function R of exactly degree 3. Now $t(2\tau)=t(-1/2\tau)=R(u(-1/2\tau))$. Since $\left(\begin{smallmatrix}0&-1\\2&0\end{smallmatrix}\right)$

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normalizes $\Gamma_0(2)$, $u(-1/2\tau)$ is also a Hauptmodul and therefore

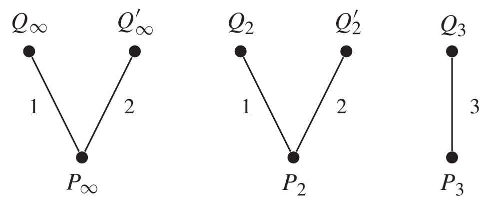
$$u(-1/2\tau) = \frac{au(\tau)+b}{cu(\tau)+d}$$

for some $a, b, c, d \in \mathrm{GL}(2, \mathbb{C})$. Hence,

$$t(2\tau) = R\left(\frac{au(\tau)+b}{cu(\tau)+d}\right).$$

In other words, the polynomial relation between $t(\tau)$ and $t(2\tau)$ is just the relation between $t = R(u)$ and $s = R((au+b)/(cu+d))$.

Now t has values 0, 1/1728, and ∞ at the cusp P_∞, the elliptic point P_2 of order 2 and the elliptic point P_3 of order 3, respectively. Above these three points, we have the following ramification data.



Here the numbers next to the lines are the ramification indices.

Choose the Hauptmodul u of $\Gamma_0(2)$ with values $u(Q_\infty) = 0$, $u(Q_2) = 1$, and $u(Q_3) = \infty$. From the ramification data at P_∞ and P_3, we have $R(u) = Au(u-\alpha)^2$ for some $\alpha \in \mathbb{C}$. Also, the ramification data P_2 implies $Au(u-\alpha)^2 - 1/1728 = A(u-1)(u-\beta)^2$ for some $\beta \in \mathbb{C}$. Comparing the coefficients we find $A = 1/108$, $\alpha = 3/4$, and $\beta = 1/4$. Furthermore, the Atkin–Lehner involution w_2 switches the two cusps Q_∞ and Q'_∞ and fixes the elliptic point Q_2 of order 2. Thus,

$$w_2 : u \longmapsto \frac{4u-3}{5u-4}.$$

Eliminating u, we find that the relation between $t = R(u) = u(u-3/4)^2/108$ and $s = R((4u-3)/(5u-4))$ is

$$\begin{aligned}\Phi_2(s,t) = s^3 + t^3 - st + 1488s^2t - 162000s^3t + 1488st^2 \\ + 40773375s^2t^2 + 8748000000s^3t^2 - 162000st^3 \\ + 8748000000s^2t^3 - 157464000000000s^3t^3.\end{aligned}$$

Solving $\Phi_2(s,t) = 0$ for s, we find the three roots are

$$\begin{aligned} s &= t^2 + 1488t^3 + 2053632t^4 + \cdots, \\ s &= t^{1/2} - 744t + 357024t^{3/2} + \cdots, \\ s &= -t^{1/2} - 744t - 357024t^{3/2} + \cdots, \end{aligned}$$

which agree with the t-expansions of $t(2\tau)$, $t(\tau/2)$, and $t((\tau+1)/2)$ given in (14).

Indeed, using differential equations and modular equations, we can compute Hecke operators on the spaces of modular forms on $\mathrm{SL}(2, \mathbb{Z})$ without resorting to Fourier expansions. In the next two sections, we will use the same idea to compute Hecke operators in the case of Shimura curves.

4. Hecke operators on $X_6^*(1)$

Hecke operators on the space of automorphic forms on Shimura curves associated to Eichler orders are defined in the same way as in the case of classical modular curves. For simplicity, we assume that the quaternion algebra B is over $\mathbb{Q}$ and has discriminant D. Fix an embedding $\iota : B \to M(2,\mathbb{R})$. Let $\mathcal{O}$ be an Eichler order of level N. For a prime p not dividing DN, we pick an element of reduced norm p in $\mathcal{O}$. Then one can show that $\Gamma(\mathcal{O})$ and $\iota(\alpha)^{-1}\Gamma(\mathcal{O})\iota(\alpha)$ are commensurable so that $\Gamma(\mathcal{O})\backslash\Gamma(\mathcal{O})\iota(\alpha)\Gamma(\mathcal{O})$ has finitely many right cosets. (In fact, the number of right cosets is $p+1$.) Then for an automorphic form $f(\tau)$ of even weight k on $\Gamma(\mathcal{O})$, the action of Hecke operator T_p on $f(\tau)$ is defined by

$$T_p : f \longmapsto p^{k/2-1} \sum_{\gamma\in\Gamma(\mathcal{O})\backslash\Gamma(\mathcal{O})\iota(\alpha)\Gamma(\mathcal{O})} \frac{(\det\gamma)^{k/2}}{(c\tau+d)^k} f(\gamma\tau), \tag{15}$$

where for a coset representative γ, we write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Hecke operators T_n for general n with $(n, DN) = 1$ are slightly more complicated.

In this section, we will compute Hecke operators on automorphic forms on $X_6^*(1)$. The computation follows that in the previous section in principle, but several issues arise.

(i) Proposition 1 only says that $t'(\tau)^{1/2}$ satisfies the Schwarzian differential equation, but it does not say which solution corresponds to $t'(\tau)^{1/2}$. This is not a problem in the example in the previous section because the hypergeometric differential equation $\theta^2F - 1728t(\theta+1/12)(\theta+5/12)F = 0$ has a unique solution (up to scalars) that is holomorphic at $t=0$ and $t^{-1/2}t'(\tau)^{1/2}$ must be a multiple of this solution. (The other solutions have a logarithmic singularity at $t=0$.) Here we need to find two linearly independent solutions of the differential equation and then find an appropriate linear combination that corresponds to $t'(\tau)^{1/2}$.

(ii) Unlike the example in the previous section, here we also need to find the t-expansion for τ. Nonetheless, this problem is relatively simple to settle once the first problem is answered.

(iii) In the example in the previous section, since the t-expansions converge only for τ with large imaginary parts, there is an obvious choice of coset representatives $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$, but here it is not immediately clear how we should choose coset representatives.

(iv) Even if we are able to find the polynomial relation $\Phi(s,t) = 0$ between $t(\tau)$ and $s(\tau) = t(\gamma\tau)$, $\gamma \in \Gamma(\mathcal{O})\iota(\alpha)\Gamma(\mathcal{O})$, and solve the equation for s as t-series, we still need to determine which solution of the equation is matched with which coset representatives. In the example in the previous section, this is relatively simple. The solution starting with $t^2 + \cdots$ must correspond to $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, while it does not really matter how the other two solutions are matched with coset representatives.

(v) Unlike the example in the previous section where the choice of coset representatives makes $t(\gamma\tau) \to 0$ as $t(\tau) \to 0$ so that to find the t-expansion of $f(\gamma\tau)$, we only need to substitute t by the t-expansion of $t(\gamma\tau)$ in f, here we also need to find a method to determine the t-expansion of $f(\gamma\tau)$ for each coset representative. This is perhaps the most complicated part of the computation. The Jacquet–Langlands correspondence will be very useful in this part. We now recall an explicit version of the correspondence in the case of quaternion algebras over $\mathbb{Q}$.

Proposition 12 [JL70, Shi72]. *Let D be the discriminant of an indefinite quaternion algebra over $\mathbb{Q}$. Let N be a positive integer relatively prime to D. For an Eichler order $\mathcal{O}(D,N)$ of level (D,N) and a positive even integer, let $S_k(\mathcal{O}(D,N))$ denote the space of automorphic forms* *on $\Gamma(\mathcal{O}(D,N))$. Then*

$$S_k(\mathcal{O}(D,N)) \simeq S_k^{D\text{-new}}(DN) := \bigoplus_{d|N} \bigoplus_{m|N/d} S_k^{\mathrm{new}}(dD)^{[m]}$$

as Hecke modules. Here

$$S_k^{\mathrm{new}}(dD)^{[m]} = \{f(m\tau) : f(\tau) \in S_k^{\mathrm{new}}(dD)\}$$

and $S_k^{\mathrm{new}}(dD)$ denotes the newform subspace of cusp forms of weight k on $\Gamma_0(dD)$. In other words, for each Hecke eigenform $f(\tau)$ in $S_k^{D\text{-new}}(DN)$, there corresponds a Hecke eigenform $\widetilde{f}(\tau)$ in $S_k(\mathcal{O}(D,N))$ that shares the same Hecke eigenvalues. Moreover, for a prime divisor p of D, if the Atkin–Lehner involution W_p acts on f by $W_p f = \epsilon_p f$, then

$$W_p \widetilde{f} = -\epsilon_p \widetilde{f}.$$

We now consider the case $X_6^*(1)$. According to Example 8, if we choose the Hauptmodul t with values 0, 1, and ∞ at the elliptic points of order 6, 2, and 4, respectively, then the space of automorphic forms of weight k has a basis given by (8). Here we rescale the Hauptmodul t such that it has values 0, -540, and ∞ at the elliptic points of order 6, 2, and 4, respectively. (The purpose of this scaling is to make the coefficients of t-series in the future computation simpler. Without this scaling, the coefficients will become algebraic numbers of high degrees.) Then the basis for the space of automorphic forms becomes

$$g_\ell = t^{\{5k/12\}}(1+t/540)^{\{k/4\}} t^\ell \left({}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; -\frac{t}{540}\right) - Ct^{1/6} {}_2F_1\left(\frac{5}{24}, \frac{1}{24}; \frac{7}{6}; -\frac{t}{540}\right)\right)^k, \tag{16}$$

$\ell = 0, \ldots, \lfloor 5k/12 \rfloor + \lfloor k/4 \rfloor + \lfloor 3k/8 \rfloor - k$, where C is a nonzero constant. We will compute Hecke operators relative to this basis.

Let us first fix the quaternion algebra B of discriminant 6 to be $((-1,3)/\mathbb{Q})$, i.e., the algebra generated by I and J over $\mathbb{Q}$ with the relations

$$I^2 = -1, \quad J^2 = 3, \quad IJ = -JI,$$

and choose the embedding $\iota : B \to M(2,\mathbb{R})$ to be

$$I \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J \longmapsto \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$$

as in [AB04, § 5.5.2]. Fix the maximal order $\mathcal{O}$ to be $\mathbb{Z} + \mathbb{Z}I + \mathbb{Z}J + \mathbb{Z}(1+I+J+IJ)/2$. Then

$$\Gamma(\mathcal{O}) = \left\{ \frac{1}{2} \begin{pmatrix} \alpha & \beta \\ -\beta' & \alpha' \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) : \alpha, \beta \in \mathbb{Z}[\sqrt{3}], \alpha \equiv \beta \bmod 2 \right\},$$

where α' and β' denote the Galois conjugates of α and β, respectively.

As in [AB04, § 5.5.2], we choose the representatives of elliptic points of order 2, 4, 6 by

$$P_2 = (\sqrt{6} - \sqrt{2})i/2, \quad P_4 = i, \quad P_6 = (-1+i)/(1+\sqrt{3})$$

with the isotropy subgroups generated by

$$M_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -3+\sqrt{3} \\ 3+\sqrt{3} & 0 \end{pmatrix}, \quad M_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$M_6 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 3+\sqrt{3} & 3-\sqrt{3} \\ -3-\sqrt{3} & 3-\sqrt{3} \end{pmatrix}, \tag{17}$$

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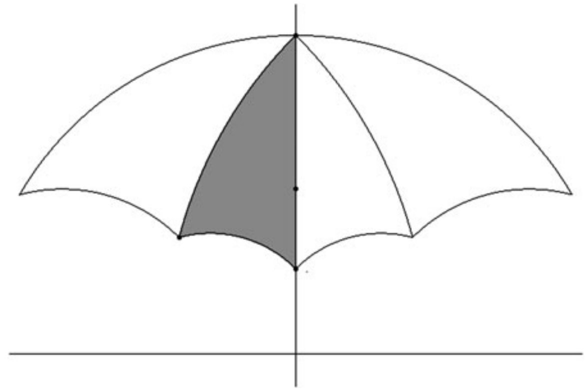
respectively. A fundamental domain for $X_6^*(1)$ is given in the following diagram.



Here the grey area represents a fundamental domain for $X_6^*(1)$. The four marked points on the boundary are

$$P_4 = i, \quad P_6 = \frac{-1+i}{1+\sqrt{3}}, \quad (2-\sqrt{3})i, \quad P_2 = \frac{(\sqrt{6}-\sqrt{2})i}{2},$$

respectively. (Note that the action of M_2 maps P_4 to $(2-\sqrt{3})i$.) The grey area and three other white areas form a fundamental domain for $X_6(1)$. (See [AB04, Figure 5.1] and [Voi09].)

To compute T_5 on the functions in (8), we need to choose appropriate coset representatives γ_j, $j = 1, \ldots, 6$, for $\Gamma^*(\mathcal{O})\backslash\Gamma^*(\mathcal{O})\left(\begin{smallmatrix}1 & -2\\ 2 & 1\end{smallmatrix}\right)\Gamma^*(\mathcal{O})$, where $\left(\begin{smallmatrix}1 & -2\\ 2 & 1\end{smallmatrix}\right)$ is the image of the element $1+2I$ of reduced norm 5 in $\mathcal{O}$ under ι. With hindsight, if the goal is just to compute Hecke operators, it does not really matter how we choose γ_j, as long as $\gamma_j P_6$ are the same point for $j = 1, \ldots, 6$. Here we take the somehow natural choice having the property that $\gamma_j P_6$ is in the fundamental domain given above.

LEMMA 13. *Let the notations be given as above. A complete set of right coset representatives of* $\Gamma^*(\mathcal{O})\backslash\Gamma^*(\mathcal{O})\left(\begin{smallmatrix}1 & -2\\ 2 & 1\end{smallmatrix}\right)\Gamma^*(\mathcal{O})$ *is given by*

$$\gamma_0 = \frac{1}{\sqrt{6}}\begin{pmatrix}6+\sqrt{3} & \sqrt{3}\\ \sqrt{3} & 6-\sqrt{3}\end{pmatrix}, \quad \gamma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}3+\sqrt{3} & 2\\ -2 & 3-\sqrt{3}\end{pmatrix},$$
$$\gamma_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}3+2\sqrt{3} & 6-\sqrt{3}\\ -6-\sqrt{3} & 3-2\sqrt{3}\end{pmatrix}, \quad \gamma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{3} & 4-\sqrt{3}\\ -4-\sqrt{3} & -\sqrt{3}\end{pmatrix},$$
$$\gamma_4 = \frac{1}{\sqrt{6}}\begin{pmatrix}-3+\sqrt{3} & 6-2\sqrt{3}\\ -6-2\sqrt{3} & -3-\sqrt{3}\end{pmatrix}, \quad \gamma_5 = \frac{1}{\sqrt{2}}\begin{pmatrix}-3 & 2-\sqrt{3}\\ -2-\sqrt{3} & -3\end{pmatrix}.$$

These coset representatives have the property that

$$\gamma_j = \gamma_0 M_6^j \tag{18}$$

for all j *and*

$$\gamma_j P_6 = \frac{-1+5i}{1+3\sqrt{3}}$$

is in the fundamental domain given above for all j*, where* M_6 *is given in (17). (The indices are arranged such that* $\gamma_0 P_2, \ldots, \gamma_5 P_2$ *are located counterclockwise around* $\gamma_j P_6$*.) Moreover, letting* $\gamma_j = \left(\begin{smallmatrix}a_j & b_j\\ c_j & d_j\end{smallmatrix}\right)$ *and* $z = e^{2\pi i/24}$*, we have*

$$c_j P_6 + d_j = z^{-2j}(z + z^5 - z^7). \tag{19}$$

Proof. Everything can be verified by a direct computation. We omit the details. □

We next determine the expansion of τ as a t-series in a neighborhood of P_6. In the lemma below, the sixth root $t(\tau)^{1/6}$ of $t(\tau)$ is defined in a neighborhood of P_6 such that it becomes a holomorphic function of τ near P_6 and takes positive real values along the boundary of the fundamental domain from P_6 to P_4. Note that in view of $t(M_6\tau)=t(\tau)$, we have $t(M_6\tau)^{1/6}=\epsilon t(\tau)^{1/6}$ for some sixth root of unity ϵ. Since the function $\tau\to t(\tau)$ preserves orientation and is locally 6-to-1 at P_6, this root of unity is actually $e^{2\pi i/6}$. In other words, we have

$$t(M_6^j\tau)^{1/6}=e^{2\pi ij/6}t(\tau)^{1/6}. \tag{20}$$

Similarly, the function $(1+t/540)^{1/2}$ is defined in a way such that it becomes a holomorphic function near P_2 and takes positive values along the boundary from P_2 to P_6 and from P_6 to P_4. Note that we have

$$(1+t(M_2\tau)/540)^{1/2}=-(1+t(\tau)/540)^{1/2},$$

even though this fact is not needed in the following.

Lemma 14. *Let*

$$F_1={}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};-\frac{t}{540}\right),\quad F_2=t^{1/6}{}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};-\frac{t}{540}\right)$$

be two linearly independent solutions of

$$\theta\left(\theta-\frac{1}{6}\right)+\frac{t}{540}\left(\theta+\frac{1}{24}\right)\left(\theta+\frac{7}{24}\right)=0,\quad \theta=t\frac{d}{dt}. \tag{21}$$

We have

$$\frac{\tau-P_6}{\tau-\overline{P}_6}=C\frac{F_2}{F_1},\quad C=\left(\frac{P_2-P_6}{P_2-\overline{P}_6}\right)\left(\frac{e^{\pi i/6}}{\sqrt[6]{540}}\right)\left(\frac{\Gamma(5/6)\Gamma(17/24)\Gamma(23/24)}{\Gamma(7/6)\Gamma(13/24)\Gamma(19/24)}\right). \tag{22}$$

Moreover, we have

$$\begin{aligned} t'(\tau)&=\frac{6t^{5/6}(1+t/540)^{1/2}}{C(P_6-\overline{P}_6)}\\ &\quad\times\left({}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};-\frac{t}{540}\right)-Ct^{1/6}{}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};-\frac{t}{540}\right)\right)^2. \end{aligned}\tag{23}$$

That is, the constants C in (16) and (22) are the same.

Proof. The existence of a constant C such that (22) holds is well-known in the classical theory of automorphic functions. (Cf. [Elk98, Equations (48) and (49)].) Here we sketch a proof.

From Example 8, we know that

$$t^{-5/12}(1+t/540)^{-1/4}t'(\tau)^{1/2},\quad \tau t^{-5/12}(1+t/540)^{-1/4}t'(\tau)^{1/2}$$

are both solutions of the same differential equation (21). Thus,

$$\tau=\frac{aF_1+bF_2}{cF_1+dF_2} \tag{24}$$

for some complex numbers a,b,c,d. Now let γ be a generator of the isotropy subgroup for P_6. It is an elementary computation to show that

$$\frac{\gamma\tau-P_6}{\gamma\tau-\overline{P}_6}=\epsilon\frac{\tau-P_6}{\tau-\overline{P}_6} \tag{25}$$

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for some primitive 6th root of unity. The two facts (24) and (25) together imply that $(\tau - P_6)/(\tau - \overline{P}_6) = CF_1/F_2$ or $(\tau - P_6)/(\tau - \overline{P}_6) = CF_2/F_1$ for some nonzero complex number C. Since the left-hand side approaches 0 as $\tau \to P_6$, it must be the second possibility that occurs. We then let $\tau \to P_2$ and use Gauss' formula

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

to get the value of C. We now prove (23).

From (22), we have

$$ \tau = \frac{P_6F_1 - C\overline{P}_6F_2}{F_1 - CF_2}. $$

Differentiating the two sides with respect to t, we get

$$ \frac{d\tau}{dt} = C(P_6 - \overline{P}_6)\frac{F_1\,dF_2/dt - F_2\,dF_1/dt}{(F_1 - CF_2)^2}. $$

Recall the formula that if f_1 and f_2 are two linearly independent solutions of a second-order Fuchsian differential equation $d^2f/dt^2 + p_1(t)\,df/dt + p_2(t)f = 0$, then

$$ f_1\frac{df_2}{dt} - f_2\frac{df_1}{dt} = c\exp\left(-\int_{t_0}^{t} p_1(t)\,dt\right) \tag{26} $$

for some constant c depending on the starting point t_0. Here we express the differential equation in (21) in the form $d^2f/dt^2 + p_1(t)\,df/ft + p_2(t)f = 0$ and find that the coefficients are

$$ p_1(t) = \frac{2(2t+675)}{3t(t+540)}, \quad p_2(t) = \frac{7}{576t(t+540)}. $$

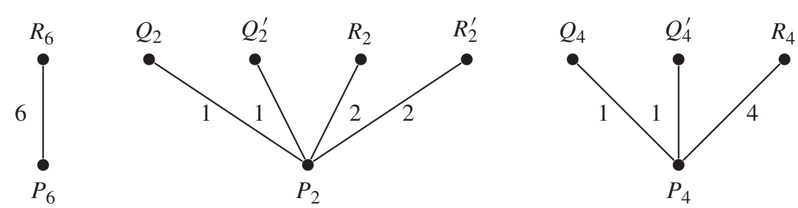
Thus, by (26), we have

$$ F_1\frac{dF_2}{dt} - F_2\frac{dF_1}{dt} = ct^{-5/6}(1+t/540)^{-1/2} $$

for some c. Considering the leading coefficients, we find $c = 1/6$. From this, we get the formula (23) for $t'(\tau)$. □

In the next lemma we determine the 'modular equation' of level 5, i.e., the polynomial relation between $t(\tau)$ and $t(\gamma\tau)$ for $\gamma \in \Gamma^*(\mathcal{O})\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\Gamma^*(\mathcal{O})$.

Lemma 15. *The Shimura curve $X_6^*(5)$ has signature $(0; 2^2, 4^2)$. The ramification data of the covering $X_6^*(5) \to X_6^*(1)$ are as follows.*



If we let t be the Hauptmodul on $X_6^*(1)$ with values $0, -540, \infty$ at P_6, P_2, and P_4, respectively, and let u be a Hauptmodul on $X_6^*(5)$ with values 0 and ∞ at R_6 and R_4, respectively, then with a suitable scaling of u, we have

$$ t = \frac{(30u)^6}{1 + 18u + 225u^2}. $$

Moreover, the Atkin–Lehner involution w_5 switches the two elliptic points Q_2 and Q_2' of order 2 and switches the two elliptic points Q_4 and Q_4' of order 4, so that

$$w_5 : u \longmapsto \frac{11u+2}{252u-11}.$$

Finally, the polynomial relation between $t(\tau)$ and $s(\tau)=t(w_5\tau)$ is given by the polynomial in Appendix A.

Proof. The Shimura curve $X_6(5)$ has totally

$$\left(1-\left(\frac{-4}{2}\right)\right)\left(1-\left(\frac{-4}{3}\right)\right)\left(1+\left(\frac{-4}{5}\right)\right)=4$$

CM points of discriminant -4. These are elliptic points of order 2 on $X_6(5)$. The Atkin–Lehner involution w_2 fixes these points and the Atkin–Lehner involution w_3 switches them pairwise. Thus, $X_6^*(5)$ has 2 elliptic points of order 4. The curve $X_6(5)$ has no elliptic points of order 3 since $(\frac{-3}{5})=-1$. Thus, all the other elliptic points on $X_6^*(5)$ are the fixed points of the Atkin–Lehner involutions w_2, w_3, and w_6, which, if they exist, are CM points of discriminant -8, -3 or -12, and -24, respectively. Since $(\frac{-8}{5})=(\frac{-3}{5})=(\frac{-12}{5})=-1$, CM points of discriminant -8, -3, or -12 do not exist on $X_6(5)$. The number of CM points of discriminant -24 on $X_6(5)$ is

$$2\left(1-\left(\frac{-24}{2}\right)\right)\left(1-\left(\frac{-24}{3}\right)\right)\left(1+\left(\frac{-24}{5}\right)\right)=4.$$

(The integer 2 stands for the class number of imaginary quadratic order of discriminant -24.) The Atkin–Lehner involution w_6 fixes these points, while the Atkin–Lehner involution w_2 switches them pairwise. Therefore, $X_6^*(5)$ has only 2 elliptic points of order 2 coming from CM points of discriminant -24. Then the genus formula shows that $X_6^*(5)$ has genus zero. We conclude that $X_6^*(5)$ has signature $(0;2^2,4^2)$.

The ramification data of $X_6^*(5)$ follow immediately from the above information.

Now suppose that the Hauptmodul t of $X_6^*(1)$ is chosen in a way that $t(P_6)=0$, $t(P_2)=-540$, and $t(P_4)=\infty$. If u is a Hauptmodul on $X_6^*(5)$ with $u(R_6)=0$ and $u(R_4)=\infty$, then

$$t=\frac{Au^6}{1+au+bu^2}$$

for some complex numbers A, a, and b. Then the ramification data at P_2 imply that

$$Au^6+540(1+au+bu^2)=540(1+cu+du^2)(1+eu+fu^2)^2$$

for some complex numbers c, d, e, and f. To have nicer coefficients, we scale u such that $a=18$. (The case $a=0$ yields $t=-270u^6/(1-3u^2/2)$, but then $w_5:u\to -u$, which implies that t is a rational function of a Hauptmodul on $X_6^*(5)/w_5$. This is absurd.) Comparing the coefficients of the two sides above, we get $t=(30u)^6/(1+18u+225u^2)$. □

Lemma 16. *For $k=8,12,16,22,30,38$, let f_k denote the automorphic form of weight k on $\Gamma^*(\mathcal{O})$ that spans the one-dimensional space $S_k(\Gamma^*(\mathcal{O}))$. Then the eigenvalues λ of T_5 for f_k are as follows.*

k	8	12	16	22	30	38
λ	−114	3630	77646	−23245050	−21003872250	4477461318150

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Proof. By the Jacquet–Langlands correspondence (Proposition 12),

$$S_k(\Gamma^*(\mathcal{O})) \simeq S_k^{\text{new}}(\Gamma_0(6), -1, -1),$$

where $S_k^{\text{new}}(\Gamma_0(6), -1, -1)$ denotes the Atkin–Lehner subspace of $S_k^{\text{new}}(\Gamma_0(6))$ with eigenvalues -1 for both W_2 and W_3. We then look up the eigenvalues of T_5 in William Stein's modular form database [Ste04]. Alternatively, one can use the trace formulas of Eichler and Yamauchi [Eic73, Yam73] to find the eigenvalues. (See [Yan11, Proposition 52] for a simplified trace formula, specifically for the group $\Gamma_0(6)$.) □

Using the information above, we can determine which root of the modular equation corresponds to $t(\gamma_j\tau)$.

Corollary 17. *Let $\gamma_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$, $j = 0, \ldots, 5$, be the coset representatives given in Lemma 13. There are rational numbers A_n, $n = 0, 1, 2, \ldots$ with*

$$A_0 = \frac{74649600}{14641}, A_1 = \frac{2918799360}{161051}, A_2 = \frac{69264688896}{1771561}, \ldots,$$

such that in a small neighborhood of P_6, the t-expansion of $t(\gamma_j\tau)$ is given by

$$t(\gamma_j\tau) = \sum_{n=0}^{\infty} A_n(\zeta^j t(\tau)^{1/6})^n,$$

where $\zeta = e^{2\pi i/6}$ and $t^{1/6}$ is defined as in the paragraph preceding Lemma 14. In particular, at $\tau = \gamma_j P_6$, we have $t(\gamma_j P_6) = A_0 = 74649600/14641 = 2^{12} \cdot 3^6 \cdot 5^2/11^4$.

In addition, at $t = A_0$, we have

$$F_1(A_0) - CF_2(A_0) = \sqrt[6]{\frac{11}{5^5}}(z + z^5 - z^7), \quad z = e^{2\pi i/24}. \tag{27}$$

Proof. Let $\Phi(s,t)$ be the modular equation given in Appendix A. We have

$$\Phi(s, 0) = -625(14641s - 74649600)^6,$$

which implies that $t(\gamma_j P_6) = 74649600/14641 = A_0$ for all j. Setting $s = \widetilde{s} + A_0$, the modular equation becomes

$$\begin{aligned} -625(11^6 + O(t))^4\widetilde{s}^6 + O(t)\widetilde{s}^5 + O(t)\widetilde{s}^4 + O(t)\widetilde{s}^3 + O(t)\widetilde{s}^2 + O(t)\widetilde{s} \\ + 625t(2^{66} \cdot 3^{36} \cdot 5^6 \cdot 17^6 \cdot 23^6/11^6 + O(t)) = 0. \end{aligned} \tag{28}$$

Using Newton's polygon and Hensel's lemma, we see that each root of the equation $\widetilde{s}^6 - A_1^6 t = 0$ lifts uniquely to a solution of (28), where $A_1 = 2^{11} \cdot 3^6 \cdot 5 \cdot 17 \cdot 23/11^5 = 2918799360/161051$. Since all coefficients in $\Phi(s,t)$ are rational numbers, the solution of (28) with the initial term $A_1 t^{1/6} + \cdots$ has rational numbers as coefficients. We now show that the series $A_0 + A_1 t^{1/6} + A_2 t^{2/6} + \cdots$ is the t-expansion of $t(\gamma_0\tau)$.

By Theorem 9, the space $S_{12}(\Gamma^*(\mathcal{O}))$ is spanned by $F = (F_1 - CF_2)^{12}$, where F_1, F_2, and C are given as in Lemma 14. Now by Lemma 16, we have

$$5^{11} \sum_{j=1}^{6} \frac{1}{(c_j\tau + d_j)^{12}} F(t(\gamma_j\tau)) = 3630F(t(\tau)),$$

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valid in a neighborhood of P_6, where γ_j are the coset representatives given in Lemma 13. Specializing τ to P_6 and using (19), we get

$$5^{11}(z+z^5-z^7)^{-12}\sum_{j=1}^{6}F(t(\gamma_jP_6))=3630,\quad z=e^{2\pi i/24}.$$

Now we have $t(\gamma_jP_6)=A_0$ for all j. Thus,

$$F(A_0)=\frac{11^2}{5^{10}}(z+z^5-z^7)^{12},$$

i.e., $F_1(A_0)-CF_2(A_0)=z^{2j}(z+z^5-z^7)\sqrt[6]{11/5^5}$ for some j. Approximating numerically, we find this integer j is equal to 0. This proves (27).

Now assume that the t-expansion of $t(\gamma_j\tau)$ is $B_0+B_1t^{1/6}+\cdots$ with $B_0=A_0$. We have, by (22),

$$B_1=\lim_{\tau\to P_6}\frac{t(\gamma_0\tau)-B_0}{C^{-1}(\tau-P_6)/(\tau-\overline{P}_6)}.$$

By L'Hôpital's rule and (19), it is equal to

$$B_1=C(P_6-\overline{P}_6)\frac{5}{(c_0P_6+d_0)^2}t'(\gamma_0P_6). \tag{29}$$

Combining (23) and (27), we find

$$t'(\gamma_0P_6)=\frac{6(z+z^5-z^7)^2}{C(P_6-\overline{P}_6)}\sqrt[3]{\frac{11}{5^5}}B_0^{5/6}(1+B_0/540)^{1/2}=\frac{2^{11}\cdot 3^6\cdot 17\cdot 23(z+z^5-z^7)^2}{11^5C(P_6-\overline{P}_6)}.$$

Substituting this and (19) into (29), we arrive at

$$B_1=\frac{2^{11}\cdot 3^6\cdot 5\cdot 17\cdot 23}{11^5}=A_1.$$

This shows that the solution $A_0+A_1t^{1/6}+\cdots$ of the modular equation corresponds to $t(\gamma_0\tau)$. By (18) and (20), it follows that

$$t(\gamma_j\tau)=t(\gamma_0M_6^j\tau)=\sum_{n=0}^{\infty}A_nt(M_6^j\tau)^{n/6}=\sum_{n=0}^{\infty}A_n(\zeta^jt(\tau)^{1/6})^n.$$

This completes the proof. □

An interesting consequence of the above calculation is the following evaluation of hypergeometric functions.

Corollary 18. *We have the evaluations*

$${}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};-\frac{2^{10}\cdot 3^3\cdot 5}{11^4}\right)=\sqrt{6}\sqrt[6]{\frac{11}{5^5}},$$

$${}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};-\frac{2^{10}\cdot 3^3\cdot 5}{11^4}\right)=\frac{\sqrt[3]{2}(1+\sqrt{2})\sqrt[6]{11^5}}{20\sqrt{3}}\frac{\Gamma(7/6)\Gamma(13/24)\Gamma(19/24)}{\Gamma(5/6)\Gamma(17/24)\Gamma(23/24)}.$$

Proof. Let the notations z, t_0, F_1, and F_2 be the same as in Corollary 17. By (22), we have

$$\frac{CF_2(t_0)}{F_1(t_0)}=\frac{\gamma_jP_6-P_6}{\gamma_jP_6-\overline{P}_6}=\frac{1}{6}(1+z^2+z^4-z^6).$$

Combining this with (27) and simplifying, we get the claimed formulas. □

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Remark 19. If we consider the Hecke operator T_7 instead, we will obtain analogous formulas

$$
{}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};\frac{2^{10}\cdot 3^3\cdot 5^6\cdot 7}{11^4\cdot 23^4}\right)=\frac{2\sqrt{2}}{7}\sqrt[6]{11\cdot 23},
$$
$$
{}_2F_1\left(\frac{5}{24},\frac{11}{24};\frac{7}{6};\frac{2^{10}\cdot 3^3\cdot 5^6\cdot 7}{11^4\cdot 23^4}\right)=\frac{\sqrt[3]{2}(1+\sqrt{2})}{140\sqrt{3}}\sqrt[6]{\frac{11^5\cdot 23^5}{7}}\frac{\Gamma(7/6)\Gamma(13/24)\Gamma(19/24)}{\Gamma(5/6)\Gamma(17/24)\Gamma(23/24)}.
$$

We will not give a proof here.

Note that the numbers $-2^{10}\cdot 3^3\cdot 5/11^4$ and $2^{10}\cdot 3^3\cdot 5^6\cdot 7/(11^4\cdot 23^4)$ correspond to the CM-points of discriminants -75 and -147 on the Shimura curve $X_6^*(1)$, respectively. In fact, with a little extra work, one can show that at CM-points τ of discriminants $-3n^2$, $(n,6)=1$, the values of ${}_2F_1(1/24,7/24;5/6;-t(\tau)/540)$ are all algebraic numbers. It will be an interesting problem to determine when s and ${}_2F_1(1/24,7/24;5/6;s)$ are both algebraic over $\mathbb{Q}$.

The last information we need in order to compute Hecke operators is the t-expansion of $F(\tau)=F_1(t(\gamma_j\tau))-CF_2(t(\gamma_j\tau))$ near P_6. We will use the Jacquet–Langlands correspondence for this purpose.

Set

$$
f_0=F^{12},\quad f_1=t^{1/6}(1+t/540)^{1/2}F^{22},\quad f_2=t^{1/3}F^8,
$$
$$
f_3=t^{1/2}(1+t/540)^{1/2}F^{30},\quad f_4=t^{2/3}F^{16},\quad f_5=t^{5/6}(1+t/540)^{1/2}F^{38}.
$$

By (16), these span the one-dimensional spaces of automorphic forms of weights 12, 22, 8, 30, 16, and 38, respectively. Let k_ℓ and λ_ℓ, $\ell=0,\dots,5$, be the weights of f_j and the eigenvalues for T_5 given in Lemma 16. In other words, if we let $\gamma_j=\left(\begin{smallmatrix}a_j & b_j\\ c_j & d_j\end{smallmatrix}\right)$, $j=0,\dots,5$ be the coset representatives given in Lemma 13, we have

$$
5^{k_\ell/2-1}\sum_{j=0}^{5}\frac{1}{(c_j\tau+d_j)^{k_\ell}}f_\ell(\gamma_j\tau)=\lambda_\ell f_\ell(\tau) \tag{30}
$$

for $\ell=0,\dots,5$. Note that the indices are arranged such that

$$
k_\ell/2+\ell\equiv 0 \mod 6. \tag{31}
$$

Now the t-expansion of τ is known by (22). Since γ_jP_6 lies on the boundary from P_6 to P_4 of the fundamental domain, according to the agreement on $t^{1/6}$ and $(1+t/540)^{1/2}$ made in the paragraph preceding Lemma 14, $t(\gamma_jP_6)^{1/6}$ and $(1+t(\gamma_jP_6)/540)^{1/2}$ are both positive. Then the t-expansions of $t(\gamma_j\tau)^{1/6}$ and $(1+t(\gamma_j\tau)/540)^{1/2}$ can be determined from that of $t(\gamma_j\tau)$ given in Corollary 17. They are

$$
t(\gamma_j\tau)^{1/6}=12\sqrt[3]{\frac{5}{11^2}}\left(1+\frac{391}{660}\zeta^jt^{1/6}+\frac{14543}{36300}(\zeta^jt^{1/6})^2+\cdots\right)
$$

and

$$
(1+t(\gamma_j\tau)/540)^{1/2}=\frac{391}{121}\left(1+\frac{6912}{4301}\zeta^jt^{1/6}+\frac{514656}{236555}(\zeta^jt^{1/6})^2+\cdots\right),
$$

respectively, where $\zeta=e^{2\pi i/6}$. Now assume that the t-expansion of $F(\gamma_0\tau)$ near P_6 is $B_0+B_1t^{1/6}+\cdots$. By (18) and (20), we have

$$
F(\gamma_j\tau)=\sum_{n=0}^{\infty}B_n(\zeta^jt^{1/6})^n.
$$

We now determine B_n inductively.

The value of B_0 is already determined in Corollary 17. It is equal to $(z+z^5-z^7)\sqrt[6]{11/5^5}$, where $z=e^{2\pi i/24}$. Now assume that the values of B_m are known up to $m=n-1$. To determine B_n, we let $\ell\in\{0,\dots,5\}$ be the integer satisfying $\ell\equiv n \bmod 6$ and consider (30). The coefficient of $t^{n/6}$ on the left-hand side of (30) is equal to

$$(\text{a known number})+\sum_{j=0}^{5}\frac{5^{k_\ell/2-1}k_\ell}{(c_jP_6+d_j)^{k_\ell}}C_\ell B_0^{k_\ell-1}B_n\zeta^{jn}, \tag{32}$$

where

$$C_\ell=12^\ell(5/121)^{\ell/3}\times\begin{cases}1 & \text{if } \ell\equiv 0 \bmod 2,\\ 391/121 & \text{if } \ell\equiv 1 \bmod 2.\end{cases}$$

By (19), we have

$$(c_jP_6+d_j)^{-k_\ell}=\zeta^{jk_\ell/2}(c_0P_6+d_0)^{-k_\ell}.$$

In view of (31), (32) is equal to

$$(\text{a known number})+\frac{6\cdot 5^{k_\ell/2-1}k_\ell}{(c_0P_6+d_0)^{k_\ell}}C_\ell B_0^{k_\ell-1}B_n.$$

This number must be equal to the coefficient of $t^{n/6}$ on the right-hand side of (30). This determines the value of B_n inductively. The first few B_n are given in Appendix B.

In general, if we wish to compute the Hecke operator T_5 on the space of automorphic forms of weight k on $X_6^*(1)$ with dimension d_k, we just have to determine the t-expansions of τ, $t(\gamma_j\tau)$ and $F(\gamma_j\tau)$ up to the term $t^{d_k-1+\{5k/12\}}$ and then express

$$5^{k/2-1}\sum_{j=0}^{5}\frac{1}{(c_j\tau+d)^k}g_\ell$$

as a linear combination of g_m by comparing the coefficients up to the term $t^{d_k-1+\{5k/12\}}$ for each g_ℓ in (16). In Appendix C, we give the matrices for T_5 up to weight 48.

Remark 20. (i) It may look miraculous that the constant C never appears in the matrices for T_5. The explanation is that the constant C depends on the choice of the embedding of the quaternion algebra into $M(2,\mathbb{R})$ and also the choice of representatives for elliptic points, but the matrices for Hecke operators do not.

(ii) Using the Jacquet–Langlands correspondence, it is easy to deduce the matrices for other Hecke operators from that of T_5. For example, for the case of weight 24, according to William Stein's modular form database [Ste04], the pair of Galois-conjugate normalized Hecke eigenforms in $S_{24}^{\text{new}}(6,-1,-1)$ have Fourier expansions

$$q+2048q^2+177147q^3+\cdots+aq^5+\cdots+(-25a+3197833334)q^7+\cdots,$$

where a is a root of the characteristic polynomial of T_5, which is irreducible over $\mathbb{Q}$. Now, the matrix for T_5 relative to our basis of automorphic forms on $X_6^*(1)$ is

$$A=\begin{pmatrix}10980750 & 3111696/5\\ 55987200000 & 14267406\end{pmatrix}.$$

Thus, the matrix for T_7 relative to the same basis is

$$-25A+3197833334=\begin{pmatrix}2923314584 & -15558480\\ -1399680000000 & 2841148184\end{pmatrix}.$$

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5. Hecke operators on $X_{10}^*(1)$

In this section, we will consider the Shimura curve $X_{10}^*(1)$. The argument runs completely parallel to the case of $X_6^*(1)$, so we will just sketch our computation.

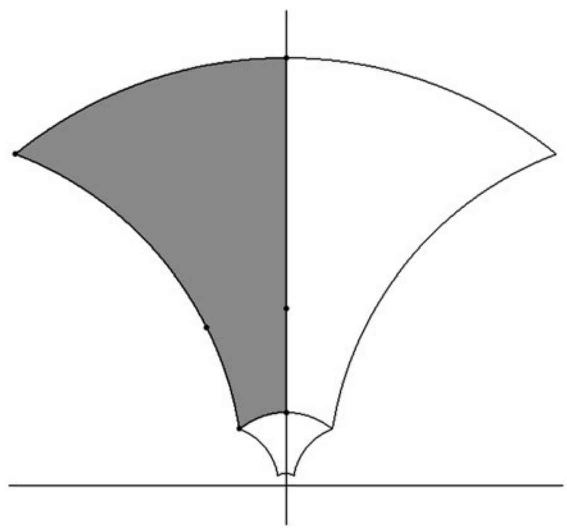
As in [AB04, § 5.5.3], we let B be the algebra generated by I and J over $\mathbb{Q}$ with the relations

$$I^2 = 2, \quad J^2 = 5, \quad IJ = -JI.$$

Then B is a quaternion algebra of discriminant 10 over $\mathbb{Q}$. Fix the maximal order $\mathcal{O}$ to be $\mathbb{Z} + \mathbb{Z}I + \mathbb{Z}(1+J)/2 + \mathbb{Z}(I+IJ)/2$ and choose the embedding $\iota : B \to M(2,\mathbb{R})$ to be

$$I \longmapsto \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \quad J \longmapsto \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}.$$

A fundamental domain for $\Gamma(\mathcal{O})$ is given in [AB04, § 5.5.3], from which we deduce that a fundamental domain for $\Gamma^*(\mathcal{O})$ is



Here the grey area represents a fundamental domain for $X_{10}^*(1)$. The six marked points on the boundary, in clockwise order from the top one on the imaginary axis, are

$$\frac{(\sqrt{2}+1)i}{\sqrt{5}}, \quad \frac{i}{\sqrt{5}}, \quad \frac{(\sqrt{2}-1)i}{\sqrt{5}}, \quad \frac{-\sqrt{2}+\sqrt{3}i}{5(\sqrt{2}+1)}, \quad \frac{-1+2i}{5}, \quad \frac{-\sqrt{2}+\sqrt{3}i}{5(\sqrt{2}-1)},$$

respectively. The grey area together with the three other white areas form a fundamental domain for $X_{10}(1)$.

The representatives of the elliptic point of order 3 and the three elliptic points of order 2 are

$$P_3 = \frac{-\sqrt{2}+\sqrt{3}i}{5(\sqrt{2}-1)}, \quad P_2 = \frac{-1+2i}{5}, \quad P_2' = \frac{i}{\sqrt{5}(\sqrt{2}-1)}, \quad P_2'' = \frac{i}{\sqrt{5}}$$

with the isotropy subgroups generated by

$$\begin{aligned} M_3 &= \frac{1}{2}\begin{pmatrix} -1-\sqrt{2} & -1-\sqrt{2} \\ 5(-1+\sqrt{2}) & -1+\sqrt{2} \end{pmatrix}, \quad M_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -5\sqrt{2} & -\sqrt{2} \end{pmatrix}, \\ M_2' &= \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 1-\sqrt{2} \\ 5(1+\sqrt{2}) & 0 \end{pmatrix}, \quad M_2'' = \frac{1}{\sqrt{10}}\begin{pmatrix} 0 & \sqrt{2} \\ -5\sqrt{2} & 0 \end{pmatrix}, \end{aligned} \tag{33}$$

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respectively. Note that the points P_2, P_2', and P_2'' are the fixed points of the Atkin–Lehner involutions w_2, w_5, and w_{10}, respectively. That is, they are CM-points of discriminant -8, -20, and -40, respectively. According to [Elk98], there is a Hauptmodul $t(\tau)$ on $\Gamma^*(\mathcal{O})$ that takes values $0, \infty, 2$, and 27 at P_3, P_2, P_2', and P_2'', respectively. Also, by (9), the Schwarzian differential equation associated to t is

$$\frac{d^2}{dt^2}f + \frac{3t^4 - 119t^3 + 3157t^2 - 7296t + 10368}{16t^2(t-2)^2(t-27)^2}f = 0.$$

In other words, near the point P_3, the t-expansion of $t'(\tau)$ is the square of a linear combination of two solutions

$$F_1(t) = t^{1/3}\left(1 - \frac{10}{81}t - \frac{18539}{839808}t^2 - \frac{168605}{25509168}t^3 - \frac{107269219465}{46548313473024}t^4 + \cdots\right),$$
$$F_2(t) = t^{2/3}\left(1 - \frac{5}{81}t - \frac{99095}{5878656}t^2 - \frac{8353325}{1428513408}t^3 - \frac{851170821485}{385081502367744}t^4 + \cdots\right)$$

of the differential equation above. To determine the linear combination, we follow the computation in Lemma 14.

Similar to (22), we have

$$\frac{\tau - P_3}{\tau - \overline{P}_3} = C\frac{F_2}{F_1},$$

where C is a nonzero constant. (The constant C will not appear in the matrices for Hecke operators, so its exact value is not important for our purpose. Cf. Remark 20.) Thus,

$$\frac{d\tau}{dt} = C(P_3 - \overline{P}_3)\frac{F_1\,dF_2/dt - F_2\,dF_1/dt}{(F_1 - CF_2)^2}.$$

Here because the differential equation is normalized, the numerator $F_1\,dF_2/dt - F_2\,dF_1/dt$ is just a constant. In fact, by computing the leading coefficients, we find that it is $1/3$. Thus,

$$t'(\tau) = \frac{3(F_1 - CF_2)^2}{C(P_3 - \overline{P}_3)}. \tag{34}$$

Note that the function $t(\tau)^{1/3}$ is defined in a way such that it takes negative values along the arc from P_3 to P_2 and becomes a holomorphic function near the point P_3.

Now, by Theorem 4, for an even integer $k \geqslant 4$, a basis for $S_k(\Gamma^*(\mathcal{O}))$ is

$$\frac{t^j(F_1(t) - CF_2(t))^{k/2}}{t^{\lfloor k/3\rfloor}(1-t/2)^{\lfloor k/4\rfloor}(1-t/27)^{\lfloor k/4\rfloor}}, \quad j = 0, \ldots, d_k - 1, \tag{35}$$

for some constant C, where $d_k = 1 - k + \lfloor k/3\rfloor + 3\lfloor k/4\rfloor$ is the dimension of $S_k(\Gamma^*(\mathcal{O}))$. We now compute the Hecke operator T_3 with respect to this basis.

Let $\gamma = 1 + 2I + IJ$. An Eichler order of level 3 is given by $\mathcal{O}\bigcap\gamma^{-1}\mathcal{O}\gamma$. Choose the coset representatives of $\Gamma^*(\mathcal{O})\backslash\Gamma^*(\mathcal{O})\begin{pmatrix}1+2\sqrt{2} & \sqrt{2}\\ -5\sqrt{2} & 1-2\sqrt{2}\end{pmatrix}\Gamma^*(\mathcal{O})$ to be

$$\gamma_0 = \frac{1}{2}\begin{pmatrix}3+\sqrt{2} & 1+\sqrt{2}\\ 5-5\sqrt{2} & 3-\sqrt{2}\end{pmatrix}, \quad \gamma_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}-5 & -1-\sqrt{2}\\ -5+5\sqrt{2} & -5\end{pmatrix},$$
$$\gamma_2 = \frac{1}{2\sqrt{10}}\begin{pmatrix}5\sqrt{2} & 4+5\sqrt{2}\\ 20-25\sqrt{2} & -5\sqrt{2}\end{pmatrix}, \quad \gamma_3 = \frac{1}{2\sqrt{10}}\begin{pmatrix}10-5\sqrt{2} & -2-3\sqrt{2}\\ -10+15\sqrt{2} & 10+5\sqrt{2}\end{pmatrix}.$$

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These coset representatives have the properties

$$\gamma_0 P_3 = P_3, \quad \gamma_1 P_3 = \gamma_2 P_3 = \gamma_3 P_3 = \frac{-2\sqrt{2}+3\sqrt{3}i}{5(2\sqrt{2}-1)},$$

and

$$\gamma_2 = \gamma_1 M_3, \quad \gamma_3 = \gamma_1 M_3^2,$$

where M_3 is the generator of the isotropy subgroup of P_3 given in (33). Also, for $j = 0, \ldots, 3$, if we write $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, then

$$c_0 P_3 + d_0 = \frac{3-\sqrt{3}i}{2}, \quad c_j P_3 + d_j = \zeta^{j-1}\frac{-5-\sqrt{2}+\sqrt{3}i}{\sqrt{10}}, \quad \zeta = e^{2\pi i/3}. \tag{36}$$

By (10) and (11), the modular equation of level 3 is the relation between

$$t = \frac{216(u-1)^3}{(u+1)^2(9u^2-10u+17)}, \quad s = \frac{216(1/9-u)^3}{(19/9-u)^2(9u^2-10u+17)}.$$

We find that it is equal to

$$\begin{aligned} &t(25t+192)^3 + (7077888 + 2908160t - 12612480t^2 + 1674720t^3 - 36750t^4)s \\ &\quad + (2764800 - 12612480t + 8025390t^2 - 798210t^3 + 21609t^4)s^2 \\ &\quad + (360000 + 1674720t - 798210t^2 + 33614t^3)s^3 + (147t-125)^2 s^4 = 0. \end{aligned}$$

As an equation in s, the 4 roots of the above equation are

$$s_0 = -t - \frac{10}{27}t^2 - \frac{100}{729}t^3 - \frac{16675}{314928}t^4 - \frac{90125}{4251528}t^5 - \cdots \tag{37}$$

and

$$s_j = -\frac{192}{25} - \frac{2992}{125}\zeta^{j-1}t^{1/3} - \frac{25044}{625}(\zeta^{j-1}t^{1/3})^2 - \frac{501163}{9375}(\zeta^{j-1}t^{1/3})^3 - \cdots \tag{38}$$

for $j = 1, 2, 3$. It is clear that s_0 is the t-expansion of $t(\gamma_0\tau)$ near P_3. To determine how s_j are matched with $t(\gamma_m\tau)$ for $j, m = 1, 2, 3$, we consider the space of automorphic forms on $X_{10}^*(1)$ of weight 18. The space $S_{18}(\Gamma(\mathcal{O}))$ has dimension 1 and is spanned by

$$f_{18} = \frac{(F_1(t) - CF_2(t))^{18}}{t^6(1-t/2)^4(1-t/27)^4}.$$

By the Jacquet–Langlands correspondence and the modular form database of William Stein's [Ste04], we have $T_3 f_{18} = -14976 f_{18}$. In other words,

$$3^{17}\sum_{j=0}^{3}\frac{1}{(c_j\tau+d_j)^{18}}f_{18}(\gamma_j\tau) = -14976 f_{18}(\tau).$$

Evaluating the two sides at $\tau = P_3$ and using (36), we get

$$3^{17}\left(-\frac{1}{3^9} + \frac{3\cdot 10^9(F_1(-192/25) - CF_2(-192/25))^{18}}{(5+\sqrt{2}-\sqrt{3}i)^{18}(192/25)^6(121/25)^4(289/225)^4}\right) = -14976.$$

Therefore,

$$(F_1(-192/25) - CF_2(-192/25))^2 = -\epsilon\frac{2^3\cdot 11\cdot 17\cdot(5+\sqrt{2}-\sqrt{3}i)^2}{3^2\cdot 5^4} \tag{39}$$

for some 9th root of unity ϵ. Approximating numerically, we find this root of unity is equal to 1. If we assume that the t-expansion of $t(\gamma_j\tau)$, $j=1,2,3$, is $B_0+B_1t^{1/3}+\cdots$, then, similar to (29),

$$B_1 = C(P_3-\overline{P}_3)\frac{3}{(c_jP_3+d_j)^2}t'(\gamma_jP_3).$$

From (36), (34) and (39), it follows that

$$B_1 = -\zeta^{j-1}\frac{2992}{125}, \quad \zeta = e^{2\pi i/3}.$$

Therefore, we have $t(\gamma_j\tau)=s_j$, where s_j is the power series in (38).

The last piece of information needed for computing the Hecke operator T_3 is the t-expansion of $F(t(\gamma_j\tau))$ near P_3, where $F(t)=F_1(t)-CF_2(t)$. For γ_0, since $t(\gamma_0P_3)=0$, the expansion is just $F_1(s_0)-CF_2(s_0)$, where s_0 is given by (37). For $\gamma_1,\gamma_2,\gamma_3$, we use the Jacquet–Langlands correspondence. Set

$$f_0=\frac{F^{18}}{t^6(1-t/2)^4(1-t/27)^4}, \quad f_1=\frac{F^4}{t(1-t/2)(1-t/27)}, \quad f_2=f_1^2.$$

These functions span the one-dimensional spaces of automorphic forms of weights 18, 4, and 8, respectively. By the Jacquet–Langlands correspondence and the modular form database, we have

$$T_3f_0=-14976f_0, \quad T_3f_1=-8f_1, \quad T_3f_2=28f_2.$$

Using the same idea in the case of $X_6^*(1)$, we can inductively determine the t-expansion of $F(t(\gamma_j\tau))$. This is enough to compute T_3 for automorphic forms of general weights. The matrices for weights up to 32 are given in Appendix D.

Remark 21. Similar to the case of $X_6^*(1)$, the Hecke operator T_3 gives rise to special values of $F_1(t)$ and $F_2(t)$ at $t=-192/25$. However, because there does not seem to be a simple description of F_1 and F_2, we do not work out the values here.

Appendix A. Modular equation of level 5 for $X_6^*(1)$

The relation between $t(\tau)$ and $s(\tau)=t(w_5\tau)$ in Lemma 15 is given by $\Phi(s,t)=a_0(t)+a_1(t)s+\cdots+a_6(t)s^6=0$ with

$$\begin{aligned}
a_0(t) &= 625(14641t-74649600)^6,\\
a_1(t) &= -1272739237185948389085264117497856000000000000000\\
&\quad -80334423751172973765172796218289002905600000000t\\
&\quad -4299645575007916355456871833989545984000000000t^2\\
&\quad -584929357876511069442449306458521600000000t^3\\
&\quad -15730132485905280227703650941440000000t^4\\
&\quad -6539287187545403426159665668843750t^5\\
&\quad -1081298279045282670656361 0000t^6,\\
a_2(t) &= 62405475620899075027184847421440000000000000000\\
&\quad -4299645575007916355456871833989545984000000000t\\
&\quad +1785753116574594713648207726896742400000000t^2\\
&\quad +1456815138486987230121098076557760000000t^3
\end{aligned}$$

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$$
\begin{aligned}
&+ 1662794876557402809482114589943710937 5t^4\\
&- 4345461774128783231852293307250000t^5\\
&+ 7122260106437394560116860000t^6,\\
a_3(t) = &-1631941887727165061787360952320000000000\\
&- 58492935787651106944244930645852160000000t\\
&+ 14568151384869872301210980765577600000000t^2\\
&+ 67656628873325479898031088625171875 00t^3\\
&- 12631131715161053121163548347250 0000t^4\\
&- 1512641834504765277067940724 00000t^5\\
&- 208500754270394065865616 0000t^6,\\
a_4(t) = &\ 2400541447463893678227554304000000000\\
&- 157301324859052802277036509414400000000t\\
&+ 1662794876557402809482114589943710937 5t^2\\
&- 12631131715161053121163548347250 0000t^3\\
&- 224960987576019075545123651160000t^4\\
&+ 1682182876508576171986568256 00t^5\\
&+ 2288909904387176525313 60000t^6,\\
a_5(t) = &-1883269425814411187356911360 00000\\
&- 65392871875454034261596656688437 50t\\
&- 4345461774128783231852293307250000t^2\\
&- 1512641834504765277067940724 00000t^3\\
&+ 1682182876508576171986568256 00t^4\\
&+ 386001766275853449228885504t^5,
\end{aligned}
$$

and

$$a_6(t) = 625(777924t - 1771561)^4.$$

Appendix B. The t-expansion of automorphic forms on $X_6^*(1)$

Let $z = e^{2\pi i/24}$. The first few terms in

$$F(\tau) := F_1(t(\gamma_0\tau)) - CF_2(\gamma_0\tau) = \sqrt[6]{\frac{11}{5^5}}(z + z^5 - z^7)\left(1 + \sum_{n=1}^{\infty} a_n t^{n/6}\right)$$

are

$$
\begin{aligned}
a_1 &= \frac{C}{13}(8z^6 - 3z^4 - 2z^2 - 9) - \frac{7}{55},\\
a_2 &= \frac{C}{715}(-56z^6 + 21z^4 + 14z^2 + 63) - \frac{161}{3025},\\
a_3 &= \frac{C}{39325}(-1288z^6 + 483z^4 + 322z^2 + 1449) - \frac{1379}{45375},\\
a_4 &= \frac{C}{589875}(-11032z^6 + 4137z^4 + 2758z^2 + 12411) - \frac{118027}{6655000},
\end{aligned}
$$

$$a_5 = \frac{C}{6655000}(-72632z^6 + 27237z^4 + 18158z^2 + 81711) - \frac{25165}{2108304},$$
$$a_6 = \frac{C}{27407952}(-201320z^6 + 75495z^4 + 50330z^2 + 226485) - \frac{219273755477}{26090262000000}.$$

Appendix C. Matrices for T_5 on $X_6^*(1)$ up to weight 48

Here we list the matrices for T_5 on $X_6^*(1)$ up to weight 48, computed using the recipe described in § 4.

Let $d_k = 1 - k + \lfloor k/4 \rfloor + \lfloor 3k/8 \rfloor + \lfloor 5k/12 \rfloor$ be the dimension of the space of automorphic forms of weight k on $X_6^*(1)$ and g_ℓ, $\ell = 0, \ldots, d_k - 1$ be the basis given in (16). The matrices

Table C.1.

k	M
8	-114
12	3630
16	77646
20	1953390
22	-23245050
24	$\begin{pmatrix} 10980750 & 3111696/5 \\ 55987200000 & 14267406 \end{pmatrix}$
28	1220703150
30	-21003872250
32	$\begin{pmatrix} 105068988750 & 376515216/5 \\ 12317184000000 & -39127734834 \end{pmatrix}$
34	-249151856250
36	$\begin{pmatrix} 33216768750 & 44743076784/5 \\ 169361280000000 & 2408347964910 \end{pmatrix}$
38	4477461318150
40	$\begin{pmatrix} -70619784011250 & 45558341136/5 \\ 30930128640000000 & 36422537206926 \end{pmatrix}$
42	9372398943750
44	$\begin{pmatrix} 896721261768750 & -10018108383696/5 \\ -321103388160000000 & -695085225669330 \end{pmatrix}$
46	$\begin{pmatrix} -1294661994656250 & -3322118218608 \\ -181517500800000000 & -5089104194777850 \end{pmatrix}$
48	$\begin{pmatrix} 100480725468750 & 225950546273760 & 2420662999104/5 \\ 512317872000000000 & 22159766272716750 & 5512559277456/5 \\ 2612138803200000000000 & -7950573190656000000 & -23013714467131314 \end{pmatrix}$

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listed in Table C.1 satisfy

$$T_5 \begin{pmatrix} g_0 \\ \vdots \\ g_{d_k-1} \end{pmatrix} = M \begin{pmatrix} g_0 \\ \vdots \\ g_{d_k-1} \end{pmatrix}.$$

Appendix D. Matrices for T_3 on $X_{10}^*(1)$ up to weight 32

Here we list the matrices for T_3 on $X_{10}^*(1)$ up to weight 32. Let $d_k = 1 - k + \lfloor k/3 \rfloor + 3\lfloor k/4 \rfloor$ be the dimension of the space of automorphic forms of weight k on $X_{10}^*(1)$ and g_ℓ, $\ell = 0, \dots, d_k - 1$ be the basis given in (35). The matrices listed in Table D.1 satisfy

$$T_3 \begin{pmatrix} g_0 \\ \vdots \\ g_{d_k-1} \end{pmatrix} = M \begin{pmatrix} g_0 \\ \vdots \\ g_{d_k-1} \end{pmatrix}.$$

TABLE D.1.

k	M
4	-8
8	28
12	$\begin{pmatrix} 468 & -98 \\ -1728 & 136 \end{pmatrix}$
16	$\begin{pmatrix} 1728 & 490 \\ 34560 & -3572 \end{pmatrix}$
18	-14976
20	$\begin{pmatrix} -2268 & -2450 \\ -328320 & 35992 \end{pmatrix}$
22	-21924
24	$\begin{pmatrix} 227772 & -272244 & 14406 \\ -388800 & -258192 & 12250 \\ 2985984 & 711936 & -199556 \end{pmatrix}$
26	162864
28	$\begin{pmatrix} 420552 & 949620 & -72030 \\ -933120 & 4479732 & -61250 \\ -104509440 & 31147200 & -196568 \end{pmatrix}$
30	$\begin{pmatrix} -6676344 & 4593750 \\ 14541120 & 3031596 \end{pmatrix}$
32	$\begin{pmatrix} 29821932 & -5456052 & 360150 \\ 95084928 & -48253536 & 306250 \\ 1803534336 & -618444288 & 19290988 \end{pmatrix}$

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