Can. J. Math., Vol. XXXIV, No. 3, 1982, pp. 759–764

PERFECT SETS OF UNIQUENESS ON THE GROUP 2^ω

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1. Introduction. Let $w_0, w_1, \ldots$ denote the Walsh-Paley functions and let G denote the dyadic group introduced by Fine [**3**]. Recall that a subset E of G is said to be a *set of uniqueness* if the zero series is the only Walsh series $\sum a_k w_k$ which satisfies

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} a_k w_k(x) = 0, \quad x \in G \sim E.$$

A subset E of G which is not a set of uniqueness is called a *set of multiplicity*.

It is known that any subset of G of positive Haar measure is a set of multiplicity [**5**] and that any countable subset of G is a set of uniqueness [**2**]. As far as uncountable subsets of Haar measure zero are concerned, both possibilities present themselves. Indeed, among perfect subsets of G of Haar measure zero there are sets of multiplicity [**1**] and there are sets of uniqueness [**5**].

There is a natural identification between the group G with its Haar measure and the unit interval $[0, 1]$ with Lebesgue measure. Moreover, Cantor sets $C(\xi)$ with constant ratio of dissection [**7**, p. 196], which can be described by

$$(1) \quad C(\xi) = \left\{ x \in [0, 1] : x = (1 - \xi) \cdot \sum_{k=1}^{\infty} \epsilon_k \xi^{k-1} \text{ where } \epsilon_k = 0 \text{ or } 1,\ k = 1, 2, \ldots \right\},$$

form an important class of perfect subsets of $[0, 1]$. Consequently, it is tempting to look at preimages of $C(\xi)$ in G and try to determine which of these are sets of uniqueness and which are sets of multiplicity. Šneider [**5**] did this in the case when $\xi = 2^{-n}$ for $n = 1, 2, \ldots$, showing that such Cantor sets are sets of uniqueness for G. It is still not known whether any other set of the form (1) is a set of uniqueness for G (see [**6**]).

The purpose of this paper is to show that if perfect sets $S(\xi)$ are defined analogously to (1) but with group operations replacing the sum $\sum_{k=1}^{\infty}$ and the products $\epsilon_k \cdot \xi^{k-1}$, then $S(\xi)$ is a set of uniqueness for G for all $\xi = (0, \xi_1, \xi_2, \ldots) \in G$. This is in sharp contrast to the trigonometric case where $C(\xi)$ is a set of uniqueness if and only if $1/\xi$ is a Pisot number [**4**].

Received April 23, 1981 and in revised form September 22, 1981.

Our method of proof is to show that all Dirichlet sets are $\mathring{H}$-sets and that all closed subgroups of G of Haar measure are Dirichlet sets. It will follow that a certain class of perfect subsets with variable ratios of dissection which includes the sets $S(\xi)$, $\xi = (0, \xi_1, \xi_2, \ldots)$, contains only $\mathring{H}$-sets. Since Wade [**6**] has shown that all $\mathring{H}$-sets are sets of uniqueness for G, it will follow that $S(\xi)$ is a set of uniqueness for G when $\xi = (0, \xi_1, \xi_2, \ldots)$.

2. Dirichlet sets. Analogous to the trigonometric case, a subset E of G is called a *Dirichlet set* if

(2) $\liminf_{n\to\infty} \sup_{x\in E} |1 - w_n(x)| = 0.$

In this section we shall show that every Dirichlet set is an $\mathring{H}$-set.

Recall [**6**] that given a non-negative integer $m = \sum_{k=0}^{\infty} \alpha_k 2^k (\alpha_k = 0 \text{ or } 1)$ and a point x which either belongs to G or to the set $\{1, 2, \ldots\}$, the *product* of m with x is defined by

$$m \otimes x = (\alpha_0 \otimes x) \dot{+} (\alpha_1 2 \otimes x) \dot{+} (\alpha_2 4 \otimes x) \dot{+} \ldots$$

where the symbols $\alpha 2^l \otimes x$ are defined as follows. If $x = (x_0, x_1, \ldots)$ is a point in the group G then

$$\alpha 2^l \otimes x \equiv (y_0, y_1, \ldots) \in G$$

where $y_j = 0$ for $j < l$ and $y_j = \alpha x_j$ for $j \geqq l$. If $x = \sum_{j=0}^{\infty} \beta_j 2^j$ ($\beta_j = 0$ or 1) is a non-negative integer, then

$$\alpha 2^l \otimes x = \sum_{j=0}^{\infty} \alpha \beta_j 2^{j+l}.$$

The important fact to remember is that if $x \in G$ and if m and k are non-negative integers then

(3) $w_k(m \otimes x) = w_{m\otimes k}(x).$

Motivated by the association of the group G and the interval $[0, 1]$, for each non-negative integer n and each integer $0 \leqq p < 2^n$, we shall denote those elements $x = (x_0, x_1, \ldots)$ of G which satisfy

$$p/2^n = \sum_{j=0}^{n-1} x_j/2^{j+1}$$

by $[p/2^n, (p + 1)/2^n]$. We observe that these sets are both open and closed in G, and that for each integer n,

$$G = \bigcup_{p=0}^{2^n-1} [p/2^n, (p + 1)/2^n].$$

THEOREM 1. *A necessary and sufficient condition for a set $E \subset G$ to be a Dirichlet set is that there exist integers $m_1 < m_2 < \ldots$ such that $m_k \otimes x \in [0, 1/2]$, that is to say $w_{m_k}(x) = 1$, for $k = 1, 2, \ldots$ and for $x \in E$.*

To prove Theorem 1, we observe that (2) holds if and only if there exist integers $n_1 < n_2 < \ldots$ such that $1 - w_{n_k}(x) \to 0$ uniformly for $x \in E$, as $k \to \infty$. But the Walsh functions assume only the values ± 1. Hence (2) holds if and only if there exist integers $m_1 < m_2 < \ldots$ such that $w_{m_k}(x) = 1$ for $x \in E$ and for $k = 1, 2, \ldots$. By (3), this condition is equivalent to $w_1(m_k \otimes x) = 1$ for $x \in E$ and for $k = 1, 2, \ldots$. Since w_1 only takes the value $+1$ on $[0, 1/2]$, it follows that (2) holds if and only if there exist integers $m_1 < m_2 < \ldots$ such that $m_k \otimes x \in [0, 1/2]$ for $k = 1, 2, \ldots$ and for $x \in E$. The theorem is proved.

Recall [**6**] that a subset E of G is an $\mathring{H}$-set if there exist integers $m_1 < m_2 < \ldots$ and an open connected set Δ of real numbers such that if $(y_0, y_1, \ldots) \equiv m_k \otimes x$ for some $x \in G$ and some integer k, then

$$\sum_{j=0}^{\infty} y_j 2^{-j-1} \notin \Delta.$$

If we use $\Delta = \{t \in [0, 1]: 1/2 < t < 1\}$, it follows from Theorem 1 that every Dirichlet set is an $\mathring{H}$-set. Since $\mathring{H}$-sets are sets of uniqueness [**6**], we have also established the following result.

COROLLARY 1. *If E is a Dirichlet set then E is a set of uniqueness.*

3. Closed subgroups and symmetric sets. Throughout this section let $I_n(x) = G$ when $n = 0$ and $x \in G$, and if n is a positive integer and $x = (x_0, x_1, \ldots)$ is a point in G, let

$$I_n(x) = [p/2^n, (p + 1)/2^n] \quad \text{where}$$

$$\sum_{j=0}^{n-1} x_j/2^{j+1} = p/2^n.$$

Given a closed subgroup H of G, set

$$H_n = \bigcup_{x \in H} I_n(x), \quad n = 0, 1, \ldots. \tag{4}$$

It is clear that for each positive integer n, H_n is a subgroup of H_{n-1}, and $H \subset H_n$. Moreover, each H_n can be expressed as a finite union of $I_n(x)$'s. Specifically for an integer $n \geqq 0$ there exist integers $0 < p_1 < \ldots < p_s < 2^n$ (depending upon n) such that

$$H_n = [0, 1/2^n] \cup [p_1/2^n, (p_1 + 1)/2^n] \cup \ldots \cup [p_s/2^n, (p_s + 1)/2^n]. \tag{5}$$

We shall call the sequence $\{H_n\}_{n=0}^{\infty}$ the *tower of subgroups* associated with H.

The main task of this section is to indicate a proof of the following:

THEOREM 2. *If H is a closed subgroup of G of Haar measure zero, then H is a Dirichlet set.*

To prove this result let $\{H_n\}_{n=0}^{\infty}$ be the tower of subgroups associated with H. It is our aim to show that there exist integers $m_1, m_2, \ldots$ such that $\limsup_{n\to\infty} m_n = \infty$ and such that $w_{m_n}(x) \equiv 1$ for $x \in H_n$. Since $H \subset H_n$ it will follow that $w_{m_n}(x) \equiv 1$ for $x \in H$, for $n = 1, 2, \ldots$. Since $\{m_n\}_{n=0}^{\infty}$ is unbounded we will have verified that (2) holds with H in place of E. In particular, H is a Dirichlet set.

Before indicating how to choose the integers $\{m_n\}_{n=0}^{\infty}$ it is necessary to look at the structure of each H_n more closely. Fix $n > 0$ and let $p_1, p_2, \ldots, p_s$ be determined by (5). Consider the collection $\Gamma_n = \{0, p_1, p_2, \ldots, p_s\}$. Recall that the dual group of G is the set of non-negative integers Γ where addition is binary addition with no carrying: if

$$t = \sum_{k=0}^{\infty} \alpha_k 2^k \quad \text{and} \quad u = \sum_{k=0}^{\infty} \beta_k 2^k$$

then

$$t \dotplus u = \sum_{k=0}^{\infty} |\alpha_k - \beta_k| 2^k.$$

Since H_n is a subgroup of G it is clear that Γ_n is a subgroup of $\tilde{\Gamma}_n = \{0, 1, \ldots, 2^n - 1\}$ which in turn is a subgroup of Γ. Moreover, since the order of $\tilde{\Gamma}_n$ is 2^n, the order of its subgroup Γ_n must be 2^k for some $0 \leqq k \leqq n$. Therefore Γ_n can be written as a direct sum of k cyclic subgroups of order 2, say

(6) $\quad \Gamma_n \equiv \{0, q_1\} \dotplus \{0, q_2\} \dotplus \ldots \dotplus \{0, q_k\}.$

Without loss of generality we suppose that $q_1 < q_2 < \ldots < q_k$.

Let $N_1 < N_2 < \ldots < N_k$ be integers determined by

$$2^{N_j} \leqq q_j < 2^{N_j+1} \quad \text{for } 1 \leqq j \leqq k.$$

Set

$$I_j = \{x : w_{q_j}(x) = 1\} \quad \text{and}$$
$$J_j = [\tilde{p}_j/2^{N_j+1}, (\tilde{p}_j + 1)/2^{N_j+1}] \quad \text{for } 1 \leqq j \leqq k$$

where

$$\tilde{p}_1 = \max\{p : [p/2^{N_1+1}, (p+1)/2^{N_1+1}] \subset I_1\} \quad \text{and}$$
$$\tilde{p}_j = \max\{p : [p/2^{N_j+1}, (p+1)/2^{N_j+1}] \subset J_{j-1} \cap I_j\} \quad \text{for } 1 < j \leqq k.$$

Since $J_1 \supsetneqq J_2 \supsetneqq \ldots \supsetneqq J_k$ and $N_k + 1 \geqq n$, there exists an integer $m_n < 2^n$ such that

(7) $\quad [m_n/2^n, (m_n + 1)/2^n] \subset J_k.$

In particular, the following identity holds for all $1 \leqq j \leqq k$:

(8) $\quad w_{q_j}(m_n/2^n) = 1.$

We are now prepared to show that $w_{m_n} \equiv 1$ on H_n and that

$$\limsup_{n\to\infty} m_n = \infty.$$

To show that $w_{m_n} \equiv 1$ on H_n, return to (6) to see that given $p \in \Gamma_n$ there exist numbers $\alpha_j = 0$ or 1 such that

$$p = \alpha_1 \cdot q_1 \dotplus \ldots \dotplus \alpha_k \cdot q_k.$$

Hence given $x \in H_n \cap [p/2^n, (p+1)/2^n]$ (see (5)) it is the case that

$$x = \alpha_1(q_1/2^n) \dotplus \alpha_2(q_2/2^n) \dotplus \ldots \dotplus \alpha_k(q_k/2^n) \dotplus \theta/2^n$$

where $0 \leqq \theta < 1$. Since $m_n < 2^n$ we have that

$$w_{m_n}(x) = w_{m_n}{}^{\alpha_1}(q_1/2^n) \ldots w_{m_n}{}^{\alpha_k}(q_k/2^n) \cdot 1.$$

But $w_{m_n}(q_j/2^n) \equiv w_{q_j}(m_n/2^n)$ so from (7) we conclude that $w_{m_n}(x) \equiv 1$ for all $x \in H_n$.

To show that $\limsup_{n\to\infty} m_n = \infty$ we first observe that since H is closed, it is the case that $H = \bigcap_{n=1}^{\infty} H_n$. Indeed, if $x \in H_n$ for all integers $n > 0$ then by (5) x belongs to a shrinking family of sets of the form $[p/2^n, (p+1)/2^n]$, which contains points $x^{(n)} \in H$ for $n = 1, 2, \ldots$. Since such a family of sets shrinks to the point x, it follows that $x^{(n)} \to x$ as $n \to \infty$. Since H is closed we have verified that $x \in H$.

Let $k_n = k$, $n = 1, 2, \ldots$ where k and n are related by the identity (6), $N_j{}^{(n)} = N_j$ and $q_j{}^{(n)} = q_j$. Since $m(H) = 0$, we can choose integers $n_1 < n_2 < \ldots$ such that $m(H_{n_i-1}) = 2 \cdot m(H_{n_i})$, so that

$$N_j{}^{(n_i)} = N_j{}^{(n_i-1)} + 1.$$

There happen three cases: (i) $2 \leqq q_1{}^{(n_i)}$, (ii) $q_1{}^{(n_i-1)} + 1$, $q_2{}^{(n_i-1)} \geqq 2$ and $q_1{}^{(n_i)} = 1$ and (iii) $\Gamma_{n_i-1} = \{0, 1\}$ for all i. In the case (i), from (7) we have

$$1/2 \leqq m_{n_i}/2^{n_i}.$$

In the case (ii), since

$$\Gamma_{n_i} = \{0, 1\} \dotplus \{0, q_2{}^{(n_i)}\} \dotplus \ldots$$

and $q_2{}^{(n_i)} \geqq 4$, we have $1/4 \leqq m_{n_i}/2^{n_i}$. Therefore in the cases (i) and (ii), we have $2^{n_i}/4 \leqq m_{n_i}$. In the case (iii), then, $H = \{0\}$ which is obviously a Dirichlet set. This completes the proof of Theorem 2.

The symmetric set $S(\xi_1, \xi_2, \ldots)$ associated with a sequence $\xi_1, \xi_2, \ldots$ of points in G is defined to be those points $x \in G$ which have the form

$$(9) \quad x = \epsilon_1 \otimes \xi_1 \dotplus \epsilon_2 \otimes \xi_2 \dotplus \ldots$$

where $\epsilon_k = 0$ or 1 for $k = 1, 2, \ldots$. If $\xi \in G$ is fixed, then the symmetric set $S(\xi)$ of constant ratio of dissection ξ is the set $S(\xi_1, \xi_2, \ldots)$ where $\xi_1 = \xi$ and $\xi_k = \xi \otimes \xi_{k-1}$ for $k > 1$. Clearly, $S(\xi)$ is the group analogue of the Cantor set described by (1).

COROLLARY 2. *Let $k_1 < k_2 < \ldots$ be a sequence of integers for which there exists a sequence $n_1 < n_2 < \ldots$ such that*

$$k_{n_j+1} < k_{n_{j+1}} \quad j = 1, 2, \ldots.$$

Suppose further that for each integer $n > 0$, ξ_n is a point in G whose first $n - 1$ coordinates are zero and whose nth coordinate is 1. Then $S(\xi_1, \xi_2, \ldots)$ is a set of uniqueness.

Indeed, under the imposed conditions, $S(\xi_1, \xi_2, \ldots)$ is a closed subgroup of Haar measure zero. Thus we need only apply Theorem 2, Corollary 1 and the previously cited result in [**6**].

Observe that the conditions of Corollary 2 are surely met by $S(\xi)$, when $\xi \in [0, 1/2]$. Hence we have proved the following result.

COROLLARY 3. *If $\xi \in [0, 1/2]$ then $S(\xi)$ is a set of uniqueness.*

We close by noting that if $\xi \in [1/2, 1]$ (i.e., if the first component of ξ is 1), then $S(\xi) \equiv G$. In particular, $S(\xi)$ is not a set of uniqueness.

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