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CHAIN-FINITE OPERATORS AND LOCALLY CHAIN-FINITE OPERATORS

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Abstract. We give algebraic conditions characterizing chain-finite operators and locally chain-finite operators on Banach spaces. For instance, it is shown that T is a chain-finite operator if and only if some power of T is relatively regular and commutes with some generalized inverse; that is there are a bounded linear operator B and a positive integer k such that

$$T^k B T^k = T^k \text{ and } T^k B = B T^k. \tag{1}$$

Moreover, we obtain an algebraic characterization of locally chain-finite operators similar to (1).

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1. Introduction. The problem we are concerned with in this paper is the algebraic characterization of chain-finite operators (global case) and of locally chain-finite operators (local case).

In the global case, recall that a bounded linear operator T on a Banach space X ($T \in L(X)$) is a *chain-finite operator*, denoted by $T \in CF(X)$, if there exists a non negative integer k such that $N(T^k) = N(T^{k+1})$ and $R(T^k) = R(T^{k+1})$, where $N(T)$ and $R(T)$ denote the kernel and the range of T, respectively. The smallest non negative integer k for which this occurs will be denoted by $l(T)$. The following characterizations of chain-finite operators are well known. Given $T \in L(X)$, T is a chain-finite operator with $l(T) = k$ if and only if 0 is a pole of the resolvent operator $(\lambda - T)^{-1}$ of T of order k [**17**, Theorems V.10.1 & V.10.2]. For convenience we shall say that 0 is a pole of the resolvent operator of T of order 0 if $0 \in \rho(T)$. Moreover, T is a chain-finite operator if and only if

$$X = N(T^k) \oplus R(T^k) \tag{2}$$

for some $k \in \mathbb{N}$ ($k \geq l(T)$) [**13**, Proposition 38.4]. See [**13**,**17**] for more details.

In [**9**], González and Onieva prove the following algebraic property: if $T \in CF(X)$, then there exists a positive integer k and an operator $B \in L(X)$ such that

$$T^k B T^k = T^k \text{ and } TB = BT. \tag{3}$$

The following condition is similar and apparently weaker than (3):

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$$T^kBT^k = T^k \text{ and } T^kB = BT^k. \tag{4}$$

Also Laursen and Mbekhta [**14**] and Harte [**10**,**11**] prove that T is a chain-finite operator with $l(T) \leq 1$ if and only if T is relatively regular and commutes with some generalized inverse; namely there exists $S \in L(X)$ such that $T = TST$ and $ST = TS$; the operator STS is called the *Drazin inverse* of T [**10**, Definition 3.1].

In the local case, taking into account [**1**, Remark 1.5], we have

$$\sigma(Tx, T) \subset \sigma(x, T) \subset \sigma(Tx, T) \cup \{0\},$$

where $\sigma(x, T)$ denotes the local spectrum of T at x. We can easily derive the following chain of inclusions for the local spectra

$$\sigma(x, T) \supset \sigma(Tx, T) \supset \ldots \supset \sigma(T^kx, T) \supset \ldots, \tag{5}$$

where 0 is the only point which may make these subsets different. Hence there is at most one inclusion in (5) that is not an equality. Then it is said that T is a locally chain-finite operator at x if the chain given in (5) breaks. Namely, given $T \in L(X)$ and $x \in X$, we say that T is a *locally chain-finite operator at* x with $l(T, x) = k > 0$ if $\sigma(T^{k-1}x, T) \neq \sigma(T^kx, T)$ and with $l(T, x) = 0$ if $0 \notin \sigma(x, T)$ [**4**, Definition 4.1]. This notion is a localization of the concept of chain-finite operator: if T satisfies the *Single Valued Extension Property* (hereafter referred to as SVEP), then T is a chain-finite operator if and only if T is a locally chain-finite operator at x for every $x \in X$ [**4**, Theorem 4.2]. Moreover, locally chain-finite operators have the properties that 0 is a pole of the local resolvent function and that the vector has a unique decomposition similar to (2). Indeed, given $T \in L(X)$ and $x \in X$, if T has the SVEP and $0 \in \sigma(x, T)$ then, by [**3**, Theorem 1], 0 is a pole of order k of the local resolvent function if and only if

$$0 \in \sigma(T^{k-1}x, T) \setminus \sigma(T^kx, T); \tag{6}$$

equivalently, there exists a unique decomposition $x = x_1 + x_2$ such that $x_1 \in N(T^k) \setminus N(T^{k-1})$ and $\sigma(x_2, T) = \sigma(x, T) \setminus \{0\}$ by [**4**, Theorem 3.3]. For convenience, we shall say that 0 is a pole of the local resolvent function of T at x of order 0 if $0 \in \rho(x, T)$. Note that T is locally chain-finite at x with $l(T, x) = k$ if and only if 0 is a pole of the local resolvent function of T at x of order k.

In this paper, we give some algebraic characterizations of chain-finite operators and of locally chain-finite operators. In Theorem 1, we prove that (3) and (4) are algebraic characterizations of chain-finite operators. By using a local result we give one more characterization (Corollary 2). Under certain conditions, we prove that T is a locally chain-finite operator with $l(T, x) \leq k$ if and only if there exists $B \in L(X)$ such that, for every $n \geq 1$ we have

$$T^kB^nT^kx = B^{n-1}T^kx. \tag{7}$$

Moreover, Corollary 3 proves that condition (7) implies the existence of $S \in L(X)$ such that

$$T^kST^kx = T^kx \text{ and } TSx = STx, \tag{8}$$

and by Example 2 we have that (7) and (8) are not equivalent. Indeed, we prove that (8) is a necessary condition (Proposition 3), (7) is a sufficient condition (Theorem 2) and under certain additional conditions is a characterization (Proposition 2, Remark 3) of locally chain-finite operators.

2. Preliminaries. Given $T \in L(X)$, a complex number λ belongs to the resolvent set $\rho(T)$ of T if there exists $(\lambda - T)^{-1} =: R(\lambda, T) \in L(X)$. We denote $\sigma(T) = \mathbb{C} \setminus \rho(T)$ the spectrum of T. The resolvent map $R(., T) : \rho(T) \longrightarrow L(X)$ is analytic; hence the following equation has an analytic solution on $\rho(T)$

$$(\mu - T)w(\mu) = x, \tag{9}$$

given by $w(\mu) = R(\mu, T)x$ for every $\mu \in \rho(T)$ and $x \in X$. This function may admit an analytic extension for some $x \in X$. We say that a complex number λ belongs to the *local resolvent set* of T at x, denoted by $\rho(x, T)$, if there exists an analytic function $w : U \longrightarrow X$, defined on a neighborhood U of λ, that satisfies (9), for every $\mu \in U$. The *local spectrum* of T at x is the complement $\sigma(x, T) := \mathbb{C} \setminus \rho(x, T)$.

Since w is not necessarily unique, a complementary property is needed to prevent ambiguity. An operator $T \in L(X)$ satisfies the SVEP if $h \equiv 0$ is the unique analytic solution of $(\lambda - T)h(\lambda) = 0$ on any open subset of the plane with values in X.

If T satisfies the SVEP, then for every $x \in X$ there exists a unique analytic function $\widehat{x}_T$ defined on $\rho(x, T)$ satisfying (9) that is called the *local resolvent function of T at x*.

In general, the local spectrum $\sigma(x, T)$ may be empty even for $x \neq 0$, but Finch [**8**] proved that $T \in L(X)$ satisfies the SVEP if and only if $\sigma(x, T) \neq \emptyset$ for every $x \in X \setminus \{0\}$. See [**6**,**7**,**15**] for further details.

Next, we recall some results that will be useful henceforth.

LEMMA 1. *Let $T \in L(X)$ and let $x \in X \setminus \{0\}$.*

*1. [**7**, Theorem 2.2]. $0 \in \rho(x, T)$ if and only if there are numbers $M > 0$, $R > 0$ and a sequence $\{x_n\} \subset X$ with the following properties:*

(a) $Tx_0 = x$,

(b) $Tx_n = x_{n-1}$,

(c) $\|x_n\| \leq MR^n$.

*2. [**16**, Theorem 2.3] & [**4**, Corollary 2.2] (Local Riesz Decomposition.) If T has the SVEP and $\sigma(x, T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint closed sets, then there exists a unique decomposition $x = x_1 + x_2$, where $\sigma(x_j, T) = \sigma_j$ $(j = 1, 2)$.*

*3. [**4**, Theorem 3.3] If T has the SVEP, then 0 is a pole of $\widehat{x}_T$ of order $k > 0$ if and only if there exists a unique decomposition $x = x_1 + x_2$ such that $T^k x_1 = 0$, $T^{k-1}x_1 \neq 0$ and $\sigma(x_2, T) = \sigma(x, T) \setminus \{0\}$.*

*4. [**3**, Theorem 1] & [**4**, Proposition 3.1] If T has the SVEP, then 0 is a pole of $\widehat{x}_T$ of order less than or equal to $k \geq 0$ if and only if $\sigma(T^k x, T) \neq \sigma(x, T)$ or $0 \in \rho(x, T)$.*

*5. [**4**, Theorem 4.2, Corollary 4.3] If T has the SVEP, then T is a chain-finite operator with $l(T) = k$ if and only if T is locally chain-finite at x with $l(T, x) \leq k$, for every $x \in X$.*

The condition $0 \in \rho(x, T)$ given in part (1) of Lemma 1 is described as follows: x *is in the holomorphic range of T* [**12**].

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3. Chain-finite operators. The following result proves that conditions (3) and (4) are algebraic characterizations of a chain-finite operator.

THEOREM 1. *Let $T \in L(X)$ and let k be a positive integer. The following assertions are equivalent.*

(a) *There exists $B \in L(X)$ such that $T^k BT^k = T^k$ and $BT = TB$.*

(b) *There exists $B \in L(X)$ such that $T^k BT^k = T^k$ and $BT^k = T^k B$.*

(c) *$T \in CF(X)$ and $l(T) \leq k$.*

Proof. (a)⇒(b) is obvious.

(b)⇒(c). Note that $N(T^k) \subset N(T^h) \subset N(T^{2k})$ and $R(T^k) \supset R(T^h) \supset R(T^{2k})$ for every $h = k, \ldots, 2k$. Consequently, it is sufficient to prove that $N(T^k) \supset N(T^{2k})$ and $R(T^k) \subset R(T^{2k})$. We have that

$$x \in N(T^{2k}) \Rightarrow T^k x = T^k BT^k x = BT^{2k} x = 0 \Rightarrow x \in N(T^k).$$

Moreover, if $x \in R(T^k)$, then there exists $y \in X$ such that

$$x = T^k y = T^k BT^k y = T^{2k} By \in R(T^{2k}).$$

Thus $T \in CF(X)$ and $l(T) \leq k$.

(c)⇒(a). This implication was proved by González and Onieva [**9**]. For the sake of completeness we give the proof here.

For $x \in X$, taking into account equation (2), we write $x = T^k y + z$, where $y \in R(T^k)$ and $z \in N(T^k)$ are determined uniquely since T^k is an isomorphism of $R(T^k)$ onto $R(T^{2k})$. Note that $T^k x = T^{2k} y$. We define

$$Bx = B(T^k y + z) := y.$$

Then

$$T^k BT^k x = T^k BT^{2k} y = T^{2k} y = T^k x.$$

Moreover

$$TBx = Ty = BT^{k+1} y = BTx.$$

Hence $T^k BT^k = T$ and $TB = BT$. □

In a wide context, it is said that $T \in L(X)$ is *polar* if condition (b) of the above theorem holds for some k and *simply polar* if it is polar with $k = 1$. See [**10**,**11**].

REMARK 1. By Theorem 1, it is clear that $T \in CF(X)$ with $l(T) \leq k$ if and only if $T^k \in CF(X)$ with $l(T^k) \leq 1$. Since $T \in CF(X)$ with $l(T) = k$ if and only if 0 is a pole of the resolvent operator of T of order k, we have that 0 is a pole of the resolvent operator of T of order less than or equal to k if and only if 0 is a pole of the resolvent operator of T^k of order less than or equal to 1.

As an immediate consequence of Theorem 1 we get the following result of Laursen and Mbekhta [**14**, Theorem 3] and Harte [**10**, Theorem 3.3] & [**11**, Theorem 7.3.6].

Corollary 1. *Let $T \in L(X)$. The following assertions are equivalent.*
1. *There exists $B \in L(X)$ such that $TBT = T$ and $BT = TB$.*
2. $X = N(T) \oplus R(T)$.

Note that Corollary 1 establishes that T is simply polar if and only if T is a chain finite operator with $l(T) \leq 1$.

Recall that $T \in L(X)$ is *relatively regular* if there exists an operator $B \in L(X)$, called a *generalized inverse* of T, such that $TBT = T$. Thus the chain-finite operators are characterized by the following condition: some power of T is relatively regular and commutes with some generalized inverse.

Next, we consider some classes of operators in $L(X)$: $CF(X)$ (*chain-finite operators*), $RR(X)$ (*relatively regular operators*) and $PRR(X)$ (*power relatively regular operators*). The three classes are related in the following way:

$$CF(X) \subset PRR(X) \supset RR(X).$$

The following examples show that the inclusions are strict.

Examples. (1) We consider $T_1, B \in L(\ell^2)$ defined by

$$T_1(\xi_1, \xi_2, \ldots) := (\xi_2, \xi_3, \ldots),$$

$$B(\xi_1, \xi_2, \ldots) := (0, \xi_1, \xi_2, \ldots).$$

Then $T_1BT_1 = T_1$ and so $T_1 \in RR(\ell^2) \subset PRR(\ell^2)$. Moreover, for every $k = 1, 2, \ldots,$ we have that $R(T_1^k) = \ell^2$, but $N(T_1^k) \neq \{0\}$; hence $T_1 \notin CF(\ell^2)$.

(2) The operator $T_2 \in L(\ell^2)$, defined by

$$T_2(\xi_1, \xi_2, \ldots) := (2^{-2}\xi_2, 0, 2^{-4}\xi_4, 0, \ldots),$$

is a compact operator [**13**, Example 13.2] and $R(T_2)$ is infinite dimensional, hence $R(T_2)$ is not closed [**17**, Theorem V.7.4]. Thus $T_2 \notin RR(\ell^2)$. Furthermore, it is obvious that $T_2^2 = 0$, hence $T_2 \in CF(\ell^2) \subset PRR(\ell^2)$.

(3) Define $T \in L(\ell^2 \times \ell^2)$ by

$$T((x_n), (y_n)) := (T_1(x_n), T_2(y_n))$$

where T_1 and T_2 are as defined in the examples above. Therefore $T \notin CF(\ell^2 \times \ell^2)$, since $R(T^k) = \ell^2 \times R(T_2^k)$ and $N(T^k) \neq \{0\} \times N(T_2^k)$. Also $T \notin RR(\ell^2 \times \ell^2)$ because $R(T)$ is not closed and $T \in PRR(\ell^2 \times \ell^2)$, since

$$S((x_n), (y_n)) := (B^2(x_n), 0)$$

satisfies $T^2ST^2 = T^2$, where B is as defined in the first example. □

4. Locally chain-finite operators. In this section, we give analogues of Theorem 1 for locally chain-finite operators.

The following proposition will be useful in the rest of the paper.

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Proposition 1. *Assume that $T \in L(X)$ has the SVEP. Let k be a positive integer and let $x \in X \setminus \{0\}$. Then T is a locally chain-finite operator with $l(T, x) \leq k$ if and only if T^k is locally chain-finite at x with $l(T^k, x) \leq 1$.*

Proof. By definition, if T is a locally chain-finite operator at x with $l(T, x) \leq k$ we have that $0 \in \rho(x, T)$ or $0 \in \sigma(x, T) \setminus \sigma(T^k x, T)$. Similarly, T^k is a locally chain-finite operator at x with $l(T^k, x) \leq 1$ if and only if $0 \in \rho(x, T^k)$ or $0 \in \sigma(x, T^k) \setminus \sigma(T^k x, T^k)$. Taking into account the local spectral mapping theorem for the functional calculus [**2**, Theorem 1.2] and [**6**, Theorem 1.5] (i.e $\sigma(y, f(T)) = f(\sigma(y, T))$ for admissible functions f) with $f(z) = z^k$, we obtain that the above conditions are equivalent. □

Henceforth, by the result above we have that 0 is a pole of the local resolvent function of T at x of order less than or equal to k if and only if 0 is a pole of the local resolvent function of T^k at x of order less than or equal to 1.

Next, we prove a sufficient condition for an operator to be locally chain-finite.

Theorem 2. *Assume that $T \in L(X)$ has the SVEP, let k be a positive integer and let $x \in X \setminus \{0\}$. If there exists $B \in L(X)$ such that $T^k B^n T^k x = B^{n-1} T^k x$ for all $n \in \mathbb{N}$, then T is locally chain-finite at x with $l(T, x) \leq k$.*

Proof. First, let us prove the result for $k = 1$. Construct a sequence of vectors in the following way: $x_n := B^{n+1} T x$ for all $n \in \mathbb{N}$ and $x_0 := BTx$. Then

$$Tx_n = TB^{n+1} Tx = B^n Tx = x_{n-1}$$

and

$$\|x_n\| = \|B^{n+1} Tx\| \leq R^n M,$$

where $R := \|B\|$ and $M := \|x_0\|$. By part (1) of Lemma 1 we have that $0 \in \rho(Tx, T)$. Then T is a locally chain-finite operator at x with $l(T, x) \leq 1$.

Let $k > 1$. If there exists $B \in L(X)$ such that $T^k B^n T^k x = B^{n-1} T^k x$ for all $n \in \mathbb{N}$ then, by the first part of this proof, we have that T^k is a locally chain-finite operator at x with $l(T^k, x) \leq 1$. Hence it is enough to apply Proposition 1 to complete the proof. □

Using the local result above we give a new global characterization of chain-finiteness for operators with the SVEP.

Corollary 2. *Assume that $T \in L(X)$ has the SVEP and let k be a positive integer. Then T is a chain-finite operator with $l(T) \leq k$ if and only if there exists $B \in L(X)$ such that $T^k B^n T^k = B^{n-1} T^k$ for all $n \in \mathbb{N}$.*

Proof. If T is a chain-finite operator with $l(T) \leq k$ then, by Theorem 1, there exists $B \in L(X)$ such that $T^k B T^k = T^k$ and $BT = TB$. Hence $T^k B^n T^k = B^{n-1} T^k$ for all $n \in \mathbb{N}$. Suppose that there exists $B \in L(X)$ such that $T^k B^n T^k x = B^{n-1} T^k x$ for all $x \in X$ and $n \in \mathbb{N}$. By Theorem 2, we have that T is a locally chain-finite operator with $l(T, x) \leq k$ for all $x \in X$. Taking into account part (5) of Lemma 1, we have that T is a chain-finite operator with $l(T, x) \leq k$. □

Remark 2. In the proof of Corollary 2, we do not need the hypothesis of the SVEP to show the necessity of the condition that characterizes chain-finite operators. On the contrary, this hypothesis cannot be neglected to establish that the condition is sufficient.

Example 1. Let T be the left shift operator on $\ell_2(\mathbb{N})$, i.e. $T(x_1, x_2, \ldots) := (x_2, x_3, \ldots)$. Let B be the right shift operator; i.e. $B(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$. It is clear that T is surjective but not injective. Hence by [**7**, Corollary 2.4] T does not have the SVEP. Moreover, $\sigma(T) = \overline{D(0,1)}$; (thus 0 is not a pole of the resolvent operator and hence T is not a chain-finite operator). Also $TB^nT = B^{n-1}T$, for all $n \in \mathbb{N}$. Notice that $TB \neq BT$. □

With some additional hypotheses we have the converse of Theorem 2 as shown in the following result.

Proposition 2. *Let $T \in L(X)$ with the SVEP such that 0 is an isolated point of $\sigma(T)$, let k be a positive integer and let $x \in X \setminus \{0\}$. Then T is a locally chain-finite operator at x with $l(T, x) \leq k$ if and only if there exists $B \in L(X)$ such that $T^k B^n T^k x = B^{n-1} T^k x$, for all $n \in \mathbb{N}$.*

Proof. Since 0 is an isolated point of $\sigma(T)$, we have that $X = X_1 \oplus X_2$, where X_i are invariant under T for $i = 1, 2$, $\sigma(T|X_1) = \{0\}$ and $\sigma(T|X_2) = \sigma(T) \setminus \{0\}$ by [**17**, Theorem V.9.1]. Define $B|X_1 := 0$ and $B|X_2 := (T^k|X_2)^{-1}$. If T is a locally chain-finite operator at x with $l(T, x) \leq k$, then 0 is a pole of $\widehat{x}_T$ of order less than or equal to k. By part (3) of Lemma 1, $x = x_1 + x_2$, where $T^k x_1 = 0$ and $\sigma(x_2, T) = \sigma(x, T) \setminus \{0\}$. Let $x_2 = y_1 + y_2$, where $y_i \in X_i$ for $i = 1, 2$. By [**6**, Proposition 1.3], $\sigma(x_2, T) = \sigma(y_1, T|X_1) \cup \sigma(y_2, T|X_2)$. If $y_1 \neq 0$, then $\sigma(x_2, T) = \sigma(x, T) \setminus \{0\} = \{0\} \cup \sigma(y_2, T|X_2)$. Hence $y_1 = 0$ and so $x_2 \in X_2$. Thus

$$T^k B^n T^k x = T^k B^n T^k x_2 = T^k B^{n-1} B T^k x_2 = T^k B^{n-1} x_2 = B^{n-1} T^k x_2 = B^{n-1} T^k x.$$

□

Remark 3. Let $T \in L(X)$ have the SVEP and let $x \in X \setminus \{0\}$. Suppose that $0 \in \rho(x, T)$. In order to get a result as in Proposition 2, we need to define $B \in L(X)$ such that $TB^nTx = B^{n-1}Tx$, for all $n \in \mathbb{N}$. The idea is that Bx must have a definition as the local resolvent function of T at x. Define

$$M := \langle \{Tx, x, \widehat{x}_T(0), \frac{d\widehat{x}_T}{d\lambda}(0), \frac{d^2\widehat{x}_T}{d\lambda^2}(0), \ldots\} \rangle,$$

and $By := -\widehat{y}_T(0)$ for all $y \in M$. If B is defined on M as above and could be extended as a bounded and linear operator on the whole of X, then there exists $B \in L(X)$ such that $TB^nTx = B^{n-1}Tx$, for all $n \in \mathbb{N}$. Indeed, $TBy = BTy$ for all $y \in M$. Notice that $0 \in \rho(x, T) \cap \rho(Tx, T)$. Moreover, by [**7**, Proposition 2.6], $\sigma(x, T) = \sigma(\widehat{x}_T(\lambda), T)$, for all $\lambda \in \rho(x, T)$. Hence $0 \in \rho(\widehat{x}_T(0), T)$. By [**5**, Remark 3.3]

$$\frac{d\widehat{x}_T}{d\lambda}(0) = -\widehat{\widehat{x}_T(0)}_T(0),$$

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and hence we have

$$0 \in \rho(x, T) = \rho(\widehat{x}_T(0), T) = \rho(\widehat{\widehat{x}_T(0)}_T(0), T) = \rho\left(\frac{d\widehat{x}_T}{d\lambda}(0), T\right).$$

By an induction argument we obtain that

$$0 \in \rho(x, T) = \rho\left(\frac{d^n\widehat{x}_T}{d\lambda^n}(0), T\right).$$

Suppose that 0 is a pole of $\widehat{x}_T$ of order less than or equal to $k > 0$. By part (3) of Lemma 1, $x = x_1 + x_2$ such that $T^k x_1 = 0$ and $\sigma(x_2, T) = \sigma(x, T) \setminus \{0\}$. Define

$$M_2 =: \langle \{Tx_2, x_2, \widehat{x_2}_T(0), \frac{d\widehat{x_2}_T}{d\lambda}(0), \frac{d^2\widehat{x_2}_T}{d\lambda^2}(0), \ldots\} \rangle,$$

and $By = -\widehat{y}_T(0)$, for all $y \in M_2$. Now, the argument is the same as above.

Proposition 2 and Remark 3 prove that with some additional conditions Theorem 2 is a characterization of locally chain-finite operators. However, we do not know if Theorem 2 is a characterization, in general.

In the next proposition, we give a necessary condition for an operator to be locally chain-finite similar to the necessary condition for chain-finiteness of operators given in Theorem 1.

Proposition 3. *Let $T \in L(X)$ have the SVEP, let k be a positive integer and let $x \in X \setminus \{0\}$. If T is a locally chain-finite operator at x with $l(T, x) \leq k$, then there exists $B \in L(X)$ such that $T^k B T^k x = T^k x$ and $TBx = BTx$.*

Proof. First, let us prove the result for $k = 1$. Suppose that $0 \in \rho(x, T)$. Define $M := \langle \{x, Tx\} \rangle$. Hence $X = M \oplus N$ for some $N \subset X$. Define $B(\alpha x + \beta Tx) := -\alpha \widehat{x}_T(0) + \beta x$, for all $\alpha, \beta \in \mathbb{C}$ and $By := 0$, for all $y \in N$. Then $B \in L(X)$, $TBTx = Tx$, and $TBx = BTx$. Suppose that T is a locally chain-finite operator with $l(T, x) = 1$, namely $0 \in \sigma(x, T) \setminus \sigma(Tx, T)$. By part (3) of Lemma 1 there exists a unique decomposition $x = x_1 + x_2$ such that $Tx_1 = 0$ and $\sigma(x_2, T) = \sigma(x, T) \setminus \{0\}$. Define $M_2 := \langle \{x_2, Tx_2\} \rangle$ and repeat the process as above. Hence $TBTx = TBTx_2 = Tx_2 = Tx$ and $BTx = TBx$.

Let $k > 1$ and T locally chain-finite operator at x with $l(T, x) \leq k$. By the proof above there exists $B \in L(X)$ such that $T^k B T^k x = T^k x$ and $T^k Bx = BT^k x$. In fact, $TBx = TBx_2 = x_2 = BTx_2 = BTx$. Thus, by Proposition 1, the result is proven. □

Corollary 3. *Assume that $T \in L(X)$ has the SVEP, let k be a positive integer and let $x \in X \setminus \{0\}$. If there exists $B \in L(X)$ such that $T^k B^n T^k x = B^{n-1} T^k x$ for all $n \in \mathbb{N}$, then there exists $S \in L(X)$ with $T^k S T^k x = T^k x$ and $TSx = STx$.*

Proof. Apply Theorem 2 and Proposition 3. □

The necessary condition given in Proposition 3 is not a sufficient condition. In fact, it is not equivalent to the sufficient condition given in Theorem 2, as shown in the following example.

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EXAMPLE 2. Let T be the right shift operator on $\ell_2(\mathbb{N})$, B the left shift operator and $x := (0, 1, 0, \ldots)$. It is easy to prove that T has the SVEP, $\sigma(x, T) = \overline{D(0, 1)}$ (hence 0 is not a pole of the local resolvent function so that T is not a locally chain-finite operator at x), $TBTx = Tx$ and $TBx = BTx$. Notice that $TB^3Tx \neq B^2Tx$. Moreover, Example 1 proves that there may exist $B \in L(X)$ such that $TBTx = Tx$ and $TBx = BTx$, but this need not imply that there exists $S \in L(X)$ such that $TS^nTx = S^{n-1}Tx$, for all $n \in \mathbb{N}$.

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