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Automorphisms of Jacobian Kummer surfaces

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Abstract. We study automorphisms of a generic Jacobian Kummer surface. First we analyse the action of classically known automorphisms on the Picard lattice of the surface, then proceed to construct new automorphisms not generated by classical ones. We find 192 such automorphisms, all conjugate by the symmetry group of the (16,6)-configuration.

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0. Introduction

Let F denote a quartic surface with 16 nodes in $\mathbf{P}^3$ and $\widehat{F}$ its minimal resolution. We call F a *(singular) Jacobian Kummer surface*, as it is isomorphic to the Kummer surface of the Jacobian of a curve of genus 2.

The main purpose of this paper is to discuss automorphisms of $\widehat{F}$ (or, birational automorphisms of F). Some automorphisms of F are geometrically evident: namely, the sixteen translations induced by the translations of the Jacobian by a point of order 2, the sixteen projections of the surface upon itself from a node, and the sixteen correlations by means of tangent planes collinear with a trope. These automorphisms were introduced by F. Klein [Kl]. Another example is the composite of the dual map: $F \to F^*$ and a projective isomorphism: $F^* \to F$. This is called a *switch* because it switches the 16 nodes and the 16 tropes. A correlation is nothing but a conjugate of a projection by a switch. A question arose naturally as to whether any other automorphisms, not generated by translations, projections and a switch, exit. In 1897 Kantor constructed some projective automorphisms under the assumption that the surface F satisfies certain conditions [Kan]; it can be shown [Ke1] that such a surface has Picard number $\rho(\widehat{F}) \geqslant 18$. For F generic, the first explicit answer to the question was given in 1900 by Hutchinson ([Hut], [S-C]). He obtained, by referring the surface F to different Göpel tetrads, sixty involutions which are restrictions of cubic Cremona transformations of the space.

We assume $\rho(\widehat{F}) = 17$ (a generic case) and denote by G the group of automorphisms of $\widehat{F}$ generated by the 16 translations, the 16 projections, the 16 correlations,

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the 60 Cremona transformation, and a switch. Since Hutchinson, no other automorphisms have been provided and it has long been conjectured that $\mathrm{Aut}(\widehat{F}) = G$. In section 6 some counterexamples to this conjecture will be constructed.

The strategy used in the construction is as follows.

After studying the rich structure of the Picard lattice and the transcendental lattice of a generic Jacobian Kummer surface, we analyse the action of the classically known automorphisms on the Picard lattice and then proceed to construct new automorphisms by working in the reverse order: we find a (-4)-reflection of the Picard lattice and, using the Torelli theorem, show that it realizes as a new automorphism of the surface (Theorem (6.11)). The method used in the search of such a (-4)-reflection is explained in (6.10): noting that there are only finitely many elliptic pencils on $\widehat{F}$ up to the whole automorphism group (cf. [St]) and that an elliptic pencil is nothing but a primitive effective isotropic vector of the Picard lattice having a nonnegative intersection with any smooth rational curve, we focus on the set of primitive effective isotropic vectors having a nonnegative intersection with nodes and tropes and having minimum degree among vectors in its orbit under the group G and show that this set satisfies 109 inequalities and is still infinite. Adding more inequalities coming from rational curves other than nodes and tropes, we finally reach a (-4)-root.

Furthermore we prove that the new automorphism is a conjugate of a projection by the automorphism (which must be also new) induced by a certain linear system ((6.14) and (6.16)). There are 192 such linear systems corresponding to certain hexads of nodes. Such hexads are mutually conjugate by the symmetry group of the (16_6)-configuration.

1. (16_6)-Configuration on Jacobian Kummer surfaces

(1.1). Let A be the Jacobian of a curve C of genus 2, and let $\tau\colon A \to A, a \to -a$, be the involution automorphism. Riemann's theorem guarantees that C can be embedded in A as a theta-divisor Θ. Moreover we may assume $\tau(\Theta) = \Theta$: If $r \in C$ is a Weierstrass point, we can take the embedding: $C \to A \cong \mathrm{Pic}^\circ(C)$, $x \to [x - r]$, and set

$$\Theta = \{[x - r]\colon x \in C\} \subset A.$$

The following propositions are classical and well-known (cf. [Hud], [Beau], [G-H]). We state them without proof.

(1.2) PROPOSITION. (i) *The linear system $|2\Theta|$ on A defines a 2-to-1 morphism of A onto a surface in $\mathbf{P}^3$, which on passing to the quotient gives an isomorphism of A/τ with a quartic surface $F \subset \mathbf{P}^3$ having 16 nodes ($=$ ordinary double points).*

(ii) *The linear system $|4\varepsilon^*\Theta - \Sigma E_i|$ on A, where $\varepsilon\colon \widehat{A} \to A$ the blow-up of the 16 points of order 2, and $E_1, \dots, E_{16}$ the exceptional curves, defines a 2-to-1 morphism of $\widehat{A}$ into $\mathbf{P}^5$, which on passing to the quotient gives an isomorphism of*

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$\widehat{A}/\tau$ with a complete intersection of three quadrics $\sum \subset \mathbf{P}^5$. If $\sigma: \widehat{F} \to F$ is the resolution of the 16 double points, then the isomorphism: $\widehat{F} \to \Sigma$ is given by the linear system $|2\sigma^* H - \frac{1}{2}\Sigma N_i|$, where H is the class of a hyperplane section of F, and $N_1, \ldots, N_{16}$ the rational curves lying over the 16 double points.

(1.3) PROPOSITION. *Let F be a quartic surface in $\mathbf{P}^3$ with 16 nodes. Then there are 16 planes of $\mathbf{P}^3$ which touch F along a conic. Each of these conics, called a 'trope', passes through 6 of the 16 nodes of F, and each node lies on 6 conics. The double cover of any of the 16 conics branched along 6 nodes is a curve C of genus 2. F can be constructed from the Jacobian $A = J(C)$ of C by the same procedure as in $(1.2)(i)$, that is, F is isomorphic to the Kummer surface associated to the abelian surface $A = J(C)$.*

(1.4) DEFINITION. By a *singular Jacobian Kummer surface* (resp. *Jacobian Kummer surface*) we mean a quartic surface F in $\mathbf{P}^3$ having 16 nodes (resp. its minimal resolution $\widehat{F}$). The 16 nonsingular rational curves on $\widehat{F}$ lying over the 16 nodes of F and the proper transforms of the 16 tropes of F are also called nodes and tropes (of $\widehat{F}$) respectively. By H we denote the class of a hyperplane section of F or, interchangeably, its proper transform in $\widehat{F}$. The distinction will be clear from the context.

The configuration of 16 nodes and 16 tropes that each trope passes through 6 nodes and each node lies on 6 tropes is called the (16_6)- *configuration*.

(1.5). Let A be the Jacobian of a curve C of genus 2. Then there are 16 theta-divisors on A which form, with the 16 points of order 2, a (16_6)-configuration. This (16_6)-configuration on A induces the (16_6)-configuration on the singular Jacobian Kummer surface F associated to C and vice versa.

To explain the incidence relations of this (16_6)-configuration, think of C as the locus of

$$y^2 = \prod_{i=0}^{5}(x - \lambda_i),$$

with $p_i = (\lambda_i, 0)$ the Weierstrass points of C. Then, the points of A

$$\begin{aligned} \mu_i &= [p_i - p_0], \quad i = 0, \ldots, 5, \\ \mu_{ij} &= [p_i + p_j - 2p_0], \quad 1 \leqslant i < j \leqslant 5 \end{aligned}$$

are of order 2. Note that the standard theta-divisor

$$\Theta = \Theta_0 = \{[p - p_0]: p \in C\} \subset A$$

of course contains the six 2-torsion points $\{\mu_i\}$; likewise its translate

$$\Theta_i = \Theta + \mu_i$$

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contains the six points μ_0, μ_i and $\{\mu_{ij}\}_{j\neq i}$ and

$$\Theta_{ij} = \Theta + \mu_{ij}$$

contains the six points μ_i, μ_j, μ_{ij}, and $\{\mu_{\ell m}\}_{\ell, m \neq i,j}$.

Conversely, each of the points μ_i, μ_{ij} lies on exactly six of the divisors Θ_i, Θ_{ij};

$$\mu_i \in \Theta, \Theta_i \text{ and } \Theta_{ij} \quad \text{for } j \neq i$$

$$\mu_{ij} \in \Theta_i, \Theta_j, \Theta_{ij} \quad \text{and} \quad \Theta_{k\ell} \quad \text{for } k, \ell \neq i, j.$$

(1.6). Throughout this paper, we identify the group A_2 of 2-torsion points of A with the set of their indices, that is,

$$A_2 = \{i, jk \colon 0 \leqslant i \leqslant 5, 1 \leqslant j < k \leqslant 5\}.$$

The group law on this set is obvious:

$$i + j = ij, \qquad i + jk = mn, \qquad ij + km = n, \qquad ij + jk = ik$$

for i, j, k, m, n distinct.

We also regard $A_2 \cong (\mathbf{Z}/2)^4$ as a 4-dimensional affine space over $\mathbf{F}_2$, the field of two elements. There are 30 hyperplanes and 140 affine 2-planes in the affine space A_2.

(1.7). Note that the automorphism group of the (16_6)-configuration is isomorphic to $(\mathbf{Z}/2)^4 \rtimes \mathrm{Sp}(4, \mathbf{F}_2)$, where $(\mathbf{Z}/2)^4$ is the group of translations of the affine space A_2 and $\mathrm{Sp}(4, \mathbf{F}_2)$, the group preserving the symplectic form on A_2, can be identified with the permutation group of the set of 6 Weierstrass points of the curve C (see (1.5)) which induces the permutation group of the set of 6 theta-divisors containing a fixed 2-torsion point.

(1.8). One can write down the tropes T_α, in terms of the hyperplane section class H and the nodes N_α:

$$\begin{aligned}
T_0 &= \tfrac{1}{2}(H - N_0 - N_1 - N_2 - N_3 - N_4 - N_5),\\
T_1 &= \tfrac{1}{2}(H - N_0 - N_1 - N_{12} - N_{13} - N_{14} - N_{15}),\\
T_2 &= \tfrac{1}{2}(H - N_0 - N_2 - N_{12} - N_{23} - N_{24} - N_{25}),\\
T_{12} &= \tfrac{1}{2}(H - N_1 - N_2 - N_{12} - N_{45} - N_{35} - N_{34}),\\
T_3 &= \tfrac{1}{2}(H - N_0 - N_3 - N_{13} - N_{23} - N_{35} - N_{34}),\\
T_{13} &= \tfrac{1}{2}(H - N_1 - N_3 - N_{13} - N_{45} - N_{24} - N_{25}),\\
T_{23} &= \tfrac{1}{2}(H - N_2 - N_3 - N_{23} - N_{45} - N_{14} - N_{15}),\\
T_{45} &= \tfrac{1}{2}(H - N_{12} - N_{13} - N_{23} - N_{45} - N_4 - N_5),\\
T_4 &= \tfrac{1}{2}(H - N_0 - N_{45} - N_4 - N_{14} - N_{24} - N_{34}),\\
T_{14} &= \tfrac{1}{2}(H - N_1 - N_{23} - N_4 - N_{14} - N_{35} - N_{25}),
\end{aligned}$$

$$
\begin{aligned}
T_{24} &= \tfrac{1}{2}(H - N_2 - N_{13} - N_4 - N_{24} - N_{35} - N_{15}),\\
T_{35} &= \tfrac{1}{2}(H - N_{12} - N_3 - N_{14} - N_{24} - N_{35} - N_5),\\
T_{34} &= \tfrac{1}{2}(H - N_{12} - N_3 - N_4 - N_{34} - N_{25} - N_{15}),\\
T_{25} &= \tfrac{1}{2}(H - N_2 - N_{13} - N_{14} - N_{34} - N_{25} - N_5),\\
T_{15} &= \tfrac{1}{2}(H - N_1 - N_{23} - N_{24} - N_{34} - N_{15} - N_5),\\
T_5 &= \tfrac{1}{2}(H - N_0 - N_{45} - N_{35} - N_{25} - N_{15} - N_5).
\end{aligned}
$$

(1.9) DEFINITION. For a trope T_α, we denote by $I(T_\alpha)$ the sixtuple of indices of the nodes which T_α passes through. e.g. $I(T_1) = (0, 1, 12, 13, 14, 15)$.

(1.10) OBSERVATIONS.

(i) Every pair of nodes is contained in exactly two tropes.
(ii) Given any pair of tropes T_α, T_β, we have

$$|I(T_\alpha) \cap I(T_\beta)| = 2,$$

and the symmetric difference

$$I(T_\alpha)\Delta I(T_\beta) = I(T_\alpha) \cup I(T_\beta)\backslash I(T_\alpha) \cap I(T_\beta)$$

is a hyperplane of A_2.

(iii) Given a hyperplane J of A_2, there are exactly four pairs $\{T_\alpha, T_\beta\}$ of tropes such that $I(T_\alpha)\Delta I(T_\beta) = J$.

(iv) Given a trope T, and a hyperplane J of A_2, we have either $|I(T) \cap J| = 2$ and $I(T)\Delta J = A_2\backslash I(T')$ for some trope T', or $|I(T) \cap J| = 4$ and $I(T)\Delta J = I(T')$ for some trope T'.

2. Tetrads of nodes

(2.1). A Göpel tetrad of nodes is a tetrahedron whose vertices are nodes, but none of whose faces is a trope. In other words, it is a tetrad of nodes such that no trope contains three of the 4 nodes, e.g. $(0, 1, 23, 45)$. Each Göpel tetrad is an affine 2-plane in A_2 and there are 60 such tetrads, all forming a single orbit of the group $(\mathbf{Z}/2)^4 \rtimes \mathrm{Sp}(4, \mathbf{F}_2)$.

(2.2). A Rosenhain tetrad of nodes is a tetrahedron whose vertices are nodes and whose faces are tropes, e.g. $(0, 1, 15, 5)$. Each Rosenhain tetrad is an affine 2-plane in A_2 and there are 80 such tetrads, all forming a single orbit of the group $(\mathbf{Z}/2)^4 \rtimes \mathrm{Sp}(4, \mathbf{F}_2)$. Göpel and Rosenhain tetrads correspond to isotropic and non-isotropic planes in A_2, respectively.

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3. Picard lattice

Using standard theory of K3-lattices (see e.g. [B-P-V], [Nik 2]), one sees easily the following:

(3.1) LEMMA. *Let X be a Jacobian Kummer surface with $\rho(X) = 17$. Then*

(i) $\mathrm{Pic}(X) \cong D_8 \oplus D_8 \oplus \langle 4\rangle$, $T_X \cong \langle -4\rangle \oplus U(2) \oplus U(2)$, *where U is the even unimodular lattice of signature $(1,1)$ and D_8 the even negative definite lattice defined by the Cartan matrix of an irreducible root system of type D_8.*

(ii) $\mathrm{Pic}(X)$ *is generated, over* $\mathbf{Z}$, *by (the classes of) the 16 nodes N_α, the 16 tropes T_α, and the hyperplane section H.*

(iii) *The discriminant form D_S of S is generated by*

$$\begin{aligned} C_1 &= (N_2 + N_{12} + N_3 + N_{13})/2,\\ C_2 &= (N_1 + N_{12} + N_4 + N_{24})/2,\\ C_3 &= (N_2 + N_{12} + N_4 + N_{14})/2,\\ C_4 &= (N_1 + N_{12} + N_3 + N_{23})/2,\\ B &= H/4 + (N_0 + N_1 + N_2 + N_{12})/2. \end{aligned}$$

(3.2) PROPOSITION. *Let X be a Jacobian Kummer surface with $\rho(X) = 17$. Then the only roots in the lattice $S = \mathrm{Pic}(X)$ are (-2)- and (-4)-roots.*

Proof. Clearly, S contains (-2) and (-4)-roots, e.g. nodes and tropes;

$$H - 2N_\alpha, H - \sum_{\alpha\in G} N_\alpha, G \quad \text{a Göpel tetrad.}$$

Let e be a $(-2d)$-root of $S, d \geqslant 3$. Write $e = v_1 + v_2 + mf, v_1, v_2 \epsilon D_8, f$ a generator of $< 4 >, m \in \mathbf{Z}$. Then, since $\langle e, S\rangle \subset d\mathbf{Z}$, we see that

$$\langle v_i, D_8\rangle \subset d\mathbf{Z} \text{ and } d|\langle e, f\rangle = 4m.$$

We need the following:

(3.3) LEMMA. *If $v \in D_8$, and if $\langle v, D_8\rangle \subset d\mathbf{Z}$, then*

$$v = \begin{cases} d/2w & \text{for some } w \in D_8 \text{ if } d \text{ is even,}\\ dw & \text{for some } w \in D_8 \text{ if } d \text{ is odd.} \end{cases}$$

Proof. Let $e_1, \ldots, e_8$ be the canonical basis of D_8 and let $v = \Sigma a_i e_i$. Then the lemma follows from the system of linear equations (mod d) in $a_1, \ldots, a_8$:

$$\langle v, e_i\rangle = 0 \text{ (mod d)}, \quad i = 1, \ldots, 8.$$

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Now we split the proof of (3.2) into two cases:

(i) If a prime $p, p > 2$, divides d, then, by the above lemma, p divides $v_i, i = 1, 2$. Since d divides $4m$, p divides m and hence p divides e, which contradicts to the primitivity of e.

(ii) If $d = 2^k, k \geqslant 2$, then, by (3.3), $v_i = 2^{k-1}w_i$ for some $w_i \in D_8$. So,

$$-2^{k+1} = e^2 = v_1^2 + v_2^2 + m^2 f^2 = 2^{2(k-1)}(w_1^2 + w_2^2) + 4m^2.$$

Since $k \geqslant 2$ and since $2|(w_1^2 + w_2^2)$, $2^3|4m^2$. So, 2 divides m and hence divides e, which again contradicts to the primitivity of e.

4. The automorphism group

(4.1) THEOREM. *Let X be a Jacobian Kummer surface with $\rho(X) = 17$. Then*

$$\mathrm{Aut}(X) \cong \{\phi \in \mathrm{O}(S)^+ : \phi \text{ induces } \pm \text{ id on } D_S\},$$

where $\mathrm{O}(S)^+$ is the group of isometries of S which leaves the set of effective divisors invariant. In particular, every automorphism of X acts as either id *or* $-$id *on the transcendental lattice T.*

Proof. One can deduce this from Torelli theorem for K3 surfaces [P-S], Nikulin's result [Nik 3] on the character map $\chi\colon \mathrm{Aut}(X) \to \mathrm{GL}(\Omega^2(X)) \cong \mathbf{C}^*$, and the existence of a fixed point free involution on algebraic Kummer surfaces [Ke 2].

(4.2) *Remark.* It is a highly nontrivial arithmetic problem to calculate the generators of the group $\mathrm{O}(S)^+$. Recall that the only roots in the lattice $S = \mathrm{Pic}(X)$ are (-2)- and (-4)-roots. Denote by Γ the subgroup of the orthogonal group $\mathrm{O}(S)$ generated by all (-2)- and (-4)-reflections, and choose a fundamental polyhedron P of Γ in the Lobachevsky space $\Lambda^{16} :=$ (ample cone)/$\mathbf{R}_+$ in such a way that $P \subset \Delta$, the fundamental polyhedron of the group W generated by all (-2)-reflections. Then by a result of [Vin 2], we have $\mathrm{O}(S)^+ \cong \Gamma^+ \rtimes \mathrm{O}(S)^+_P$, where $\Gamma^+ = \Gamma \cap \mathrm{Sym}(\Delta) = \Gamma \cap \mathrm{O}(S)^+$, $\mathrm{O}(S)^+_P = \mathrm{O}(S)^+ \cap \mathrm{Sym}(P)$. Let $\mathcal{S}$ be the set of reflections with respect to the hyperplanes bounding P, i.e., the faces of P, and $\mathcal{S}_2$ (resp. $\mathcal{S}_4$) the set of (-2) (resp. (-4))-reflections belonging to $\mathcal{S}$. Obviously, $\mathcal{S} = \mathcal{S}_2 \cup \mathcal{S}_4$, and W is the minimal normal subgroup of Γ containing $\mathcal{S}_2$. Notice that the order of any product $s_\alpha \circ s_\beta, s_\alpha \in \mathcal{S}_2, s_\beta \in \mathcal{S}_4$, is even or infinite (cf. [Vin 2]).

Then it follows from a proposition in [Vin 3] that Γ^+ is the Coxeter group with a system of generators $\mathcal{S}_4$ and that the Coxeter diagram of Γ^+ can be obtained from the Coxeter diagram of Γ by removing the vertices belonging to $\mathcal{S}_2$.

One may employ Vinberg's algorithm [Vin 2] to compute the set $\mathcal{S}$ of the faces of P, or equivalently, the system of generators of Γ. The algorithm stops in a finite number of steps if and only if $\mathrm{O}(S)^+_P$ is finite. Unfortunately, it seems that in our case the algorithm goes forever. Indeed, Vinberg himself pointed out [Vin 1], on

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the basis of some experience, that it stops in finite steps only in exceptional cases where the discriminant of S is small. For example, in [Vin 3] Vinberg computed, using the algorithm, the set $\mathcal{S}$ and consequently the set of generators of $\mathrm{O}(S)^+$ for two examples of $K3$-surfaces with disc $(S) = 3, 4$ resp., both remarkably smaller than disc $(S) = 64$ in our case.

5. Classical automorphisms

Let F be a quartic surface with 16 nodes in $\mathbf{P}^3$ and $\widehat{F}$ its resolution of singularities. In this section, we discuss some birational automorphisms on F, known before the century, and their actions on the Picard lattice $S = \mathrm{Pic}(\widehat{F})$.

(5.1) (i) *16 Translations*

The translation of $A = J(C)$ by a 2-torsion point $\alpha \in A_2$ descends to give an automorphism t_α on F, called a translation. All translations are linear, i.e., they are induced by projective automorphisms of $\mathbf{P}^3$.

(ii) *16 Projections*

Projecting the surface $F \subset \mathbf{P}^3$ from one of its double points n_α, we obtain a ramified double cover: $F_\alpha \to \mathbf{P}^2$, where F_α is the blow-up of F at the double point n_α. We call the covering involution of the double cover a projection and denote it by p_α.

(iii) *Switches*

It is known (cf. [G-H], [Hud]) that the dual surface $F^* \subset (\mathbf{P}^3)^* \cong \mathbf{P}^3$ whose points are tangent planes of F is again a quartic surface with 16 nodes (= the image of the 16 tropes of F) and is projectively isomorphic to F. The composite of the dual map: $F \to F^*$ and the projective isomorphism: $F^* \to F$ is a birational involution of F, which switches the 16 nodes and the 16 tropes. We call such an involution *a switch*. If $\rho(\widehat{F}) = 17$, then there are exactly 16 switches on F as the composite of any pair of switches is projective (Remark (5.3) below) and hence, a translation. The 16 switches are of the form $\sigma \circ t_\beta, \beta \in A_2$, where σ is a switch sending N_α to $T_\alpha,\ \alpha \in A_2$.

(iv) *16 Correlations*

These are the correlative transformations by means of tangent planes collinear with a trope. In other word, they are the liftings onto F of the 16 projections of F^* and, hence, of the form $\sigma \circ p_\alpha \circ \sigma$.

(v) *60 Cremona transformations referred to Göpel tetrads*

The equation of F referred to a Göpel tetrad of nodes has the form

$$\begin{aligned} &A(x^2t^2 + y^2z^2) + B(y^2t^2 + z^2x^2) + C(z^2t^2 + x^2y^2) + Dxyzt \\ &\quad + F(yt + zx)(zt + xy) + G(zt + xy)(xt + yz) \\ &\quad + H(xt + yz)(yt + zx) = 0 \quad \text{[Hut]}, \end{aligned} \tag{*}$$

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and is unchanged by the standard Cremona transformation:

$$(x, y, z, t) \to (yzt, ztx, txy, xyz),$$

so by using different tetrads we obtain in this way 60 (birational) involutions of F. These involutions are denoted by $c_{\alpha,\beta,\gamma,\delta}$, where $(\alpha, \beta, \gamma, \delta)$ is a Göpel tetrad. Note that, for any translation t,

$$c_{t(\alpha,\beta,\gamma,\delta)} = t \circ c_{\alpha,\beta,\gamma,\delta} \circ t,$$

and that each 4 of the 60 Cremona transformations are mutually conjugate by translations.

(5.2). We compute the action of the classical automorphisms on the Picard lattice of $\widehat{F}$. The result is summarized in the following table. For simplicity, we assume $\rho(\widehat{F}) = 17$.

	t_α	p_α	σ	$c = c_{\alpha,\beta,\gamma,\delta}$
N_κ	$N_{\alpha+\kappa}$	N_κ, if $\kappa \neq \alpha$; $2H - 3N_\alpha$, if $\kappa = \alpha$	T_κ	$H - N_\alpha - N_\beta - N_\gamma - N_\delta + N_\kappa$, if $\kappa = \alpha, \beta, \gamma, \delta$; the remaining 12 nodes are permuted according to (*) below and (1.8)
T_κ	$T_{\alpha+\kappa}$	T_κ, if $\alpha \in I(T_\kappa)$; $T_\kappa + H - 2H_\alpha$, otherwise	N_κ	(*) c interchanges the two tropes containing any two of the four nodes $N_\alpha, N_\beta, N_\gamma, N_\delta$.
H	H	$3H - 4H_\alpha$	$3H - \Sigma_{\alpha \in A_2} N_\alpha$	$3H - 2(N_\alpha + N_\beta + N_\gamma + N_\delta)$

Note that p_α acts on the Picard lattice as the reflection with respect to a (-4)-root $H - 2N_\alpha$. Note also that $t_\alpha,\ p_\alpha,\ \sigma$, and $c_{\alpha,\beta,\gamma,\delta}$ induce $1, -1, -1$ and -1 on the transcendental lattice of $\widehat{F}$.

(5.3) *Remark.* It can be shown [Ke 1] that if $\rho(\widehat{F}) = 17$, then there are no other projective automorphisms on F than the 16 translations. If F is special in the sense that the six nodes lying on a trope satisfy certain configurations, then F admits other projective automorphisms than the 16 translations ([Kan] and [Hud; Sect. 120]). Such special surfaces all have Picard number $\rho(\widehat{F}) \geqslant 18$.

(5.4) *Remark.* The Cremona transformations referred to Göpel tetrads were provided in 1900 by Hutchinson [Hut] as an explicit answer to the question as to whether on a generic F any other automorphisms, not generated by translations,

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projections, and correlations, exist. He also pointed out, in the same paper that the Weddle surface

$$W\colon \begin{vmatrix} 1/x, & x, & a, & 1/a \\ 1/y, & y, & b, & 1/b \\ 1/z, & z, & c, & 1/c \\ 1/t, & t, & d, & 1/d \end{vmatrix} = 0,$$

into which the general Kummer surface F can be transformed birationally, is unchanged by the standard Cremona transformation. There are fifteen equations of this form, referred to different tetrahedra of nodes of W (W has exactly six nodes; $\sharp$ tetrahedra $= 15$), and he obtained in this way a group of automorphisms of F and claimed that this group is an infinite group. Later, Baker corrected this claim [B] and Sharpe and Craig proved that this group coincides with the group of 16 translations [S-C]. Another way of confirming the result of Sharpe and Craig is to note that the projection of W onto itself from a node corresponds to a switch of F.

6. Some new automorphisms

Throughout this section, we assume $\rho(\widehat{F}) = 17$ and denote by G the group of automorphisms of $\widehat{F}$ generated by the 16 translations t_α, the 16 projections p_α, the switch σ, the 16 correlations $\sigma \circ p_\alpha \circ \sigma$, and the 60 Cremona transformations $c_{\alpha,\beta,\gamma,\delta}$.

Since Hutchinson no other automorphisms have been provided and it has long been conjectured that $\mathrm{Aut}(\widehat{F}) = G$. In this section we give some counterexamples to this conjecture.

(6.1) PROPOSITION. *Let G_1 be the subgroup of G generated by $p_\alpha, \sigma \circ p_\alpha \circ \sigma$, $c_{\alpha,\beta,\gamma,\delta}$'s. Then G_1 is a normal subgroup of G and*

$$G \cong G_1 \rtimes < \sigma, t_\alpha : \alpha \in A_2 > \cong G_1 \rtimes (\mathbf{Z}/2\mathbf{Z})^5.$$

Proof. Translations commute with σ, i.e., $t_\alpha \circ \sigma = \sigma \circ t_\alpha$, so that the second isomorphism follows.

Since p_α is the (-4)-reflection w.r.t. $H - 2N_\alpha$, $t_\mu \circ p_\alpha \circ t_\mu$ is the (-4)-reflection w.r.t. $t_\mu(H - 2N_\alpha) = H - 2N_{\alpha+\mu}$, i.e.,

$$t_\mu \circ p_\alpha \circ t_\mu = p_{\alpha+\mu}$$

and consequently

$$t_\mu \circ (\sigma \circ p_\alpha \circ \sigma) \circ t_\mu = \sigma \circ p_{\alpha+\mu} \circ \sigma.$$

These two equalities together with

$$t_\mu \circ c_{\alpha,\beta,\gamma,\delta} \circ t_\mu = c_{\alpha+\mu,\beta+\mu,\gamma+\mu,\delta+\mu}$$

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imply that all translations t_μ normalize G_1.

It is straightforward to check that

$$\sigma \circ c_{\alpha,\beta,\gamma,\delta} \circ \sigma = c_{\kappa,\lambda,\mu,\nu},$$

where $\kappa, \lambda, \mu, \nu$ are the indices of the 4 tropes not passing through any of the nodes $n_\alpha, n_\beta, n_\gamma, n_\delta$. It follows that σ also normalizes G_1.

(6.2). By (4.1), we may regard $\mathrm{Aut}(\widehat{F})$ and G as subgroups of $\mathrm{O}(S)$. Let $(\alpha, \beta, \gamma, \delta)$ be a Göpel tetrad. We denote by $r_{\alpha,\beta,\gamma,\delta}$ the reflection of S w.r.t. a (-4)-root $H - N_\alpha - N_\beta - N_\gamma - N_\delta$, and by $t_{\alpha,\beta,\gamma,\delta}$ the composite $c_{\alpha,\beta,\gamma,\delta} \circ r_{\alpha,\beta,\gamma,\delta}$. Then

$$c_{\alpha,\beta,\gamma,\delta} = t_{\alpha,\beta,\gamma,\delta} \circ r_{\alpha,\beta,\gamma,\delta} = r_{\alpha,\beta,\gamma,\delta} \circ t_{\alpha,\beta,\gamma,\delta},$$

and $t_{\alpha,\beta,\gamma,\delta}$ fixes $H, N_\alpha, N_\beta, N_\gamma, N_\delta$, and permutes the remaining 12 nodes just in the same way that $c_{\alpha,\beta,\gamma,\delta}$ does.

Warning: Neither $r_{\alpha,\beta,\gamma,\delta}$ nor $t_{\alpha,\beta,\gamma,\delta}$ can realize as an automorphism of $\widehat{F}$.

(6.3) LEMMA. *Every element $f \epsilon G_1$ can be written in the form*

$$f = \phi_2 \circ \phi_1,$$

where ϕ_1 is a product of p_α's, $\sigma \circ p_\alpha \circ \sigma$'s, $r_{\alpha,\beta,\gamma,\delta}$'s, ϕ_2 is a product of $t_{\alpha,\beta,\gamma,\delta}$'s.

(6.4) LEMMA. *$H - N_\alpha - N_\beta - N_\gamma - N_\delta, \alpha, \beta, \gamma, \delta$ all distinct, is a (-4)-root of S if and only if $(\alpha, \beta, \gamma, \delta)$ is a Göpel tetrad.*

Proof. Straightforward.

Proof of (6.3). Since $t_{\alpha,\beta,\gamma,\delta}$ permutes nodes and tropes respectively (cf. (6.2)), and since p_μ and $r_{\kappa,\lambda,\mu,\nu}$ are reflections, we have

$$t_{\alpha,\beta,\gamma,\delta} \circ p_\mu \circ t_{\alpha,\beta,\gamma,\delta} = p_{\mu'},$$
$$t_{\alpha,\beta,\gamma,\delta} \circ (\sigma \circ p_\nu \circ \sigma) \circ t_{\alpha,\beta,\gamma,\delta} = \sigma \circ p_{\nu'} \circ \sigma,$$
$$t_{\alpha,\beta,\gamma,\delta} \circ r_{\kappa,\lambda,\mu,\nu} \circ t_{\alpha,\beta,\gamma,\delta} = r_{\kappa',\lambda',\mu',\nu'}$$

From these, by induction, we are done

$$\ldots p_\mu \circ t_{\alpha,\beta,\gamma,\delta} \ldots = \ldots t_{\alpha,\beta,\gamma,\delta} \circ p_{\mu'}, \ldots,$$
$$\ldots (\sigma \circ P_\nu \circ \sigma) \circ t_{\alpha,\beta,\gamma,\delta} \ldots = \ldots t_{\alpha,\beta,\gamma,\delta} \circ (\sigma \circ P_{\nu'} \circ \sigma) \ldots,$$
$$\ldots c_{\kappa,\lambda,\mu,\nu} \circ t_{\alpha,\beta,\gamma,\delta} \ldots$$
$$= t_{\kappa,\lambda,\mu,\nu} \circ t_{\alpha,\beta,\gamma,\delta} \circ r_{\kappa',\lambda',\mu',\nu'} \ldots.$$

(6.5) LEMMA. *Every element $g \in G$ can be written in the form*

$$g = t_\alpha \circ \phi_2 \circ \phi_1, \text{ or } t_\alpha \circ \phi_2 \circ \sigma \circ \phi_1,$$

where ϕ_1 and ϕ_2 the same as in (6.3).

Proof. Since $\sigma \circ t_{\alpha,\beta,\gamma,\delta} \circ \sigma = t_{\kappa,\lambda,\mu,\nu}$, this follows from (6.1) and (6.3).

(6.6). *Let E be a divisor of $\widehat{F}$.*

An isometry ϕ of S is said to *increase* (resp. *decrease*) E if $\langle \phi(E), H\rangle \geqslant \langle E, H >$ (resp. $< \phi(E), H\rangle \leqslant \langle E, H\rangle$).

E can be written in the form

$$E = aH - \sum b_\alpha N_\alpha, \quad a, b_\alpha \in \tfrac{1}{2}\mathbf{Z} \quad \text{(cf. (3.1))}$$

Then we see that

$$p_\alpha \text{ increases } E \Longleftrightarrow \langle E, H - 2N_\alpha\rangle \geqslant 0 \Leftrightarrow a \geqslant b_\alpha,$$

$$c_{\alpha,\beta,\gamma,\delta} \text{ increases } E \Longleftrightarrow r_{\alpha,\beta,\gamma,\delta} \text{ increases } E \Leftrightarrow 2a \geqslant b_\alpha + b_\beta + b_\gamma + b_\delta,$$

$$\sigma \text{ increases } E \Longleftrightarrow 4a \geqslant \sum b_\alpha.$$

Let us denote by $(**)$ the system of 77 inequalities

$$(**) \begin{cases} a \geqslant b_\alpha, \alpha \in A_2 & \text{(16 of them)} \\ 2a \geqslant b_\alpha + b_\beta + b_\gamma + b_\delta, (\alpha,\beta,\gamma,\delta) \text{a Göpel tetrad (60 of them)} \\ 4a \geqslant \sum_{\alpha \in A_2} b_\alpha \end{cases}$$

(6.7) PROPOSITION. *Let $E = aH - \Sigma b_\alpha N_\alpha, a, b_\alpha \in \frac{1}{2}\mathbf{Z}, a \geqslant 0, b_\alpha \geqslant 0$, be a divisor of $\widehat{F}$. Then E satisfies $(**)$ if and only if every element of G increases E.*

Proof. ($\Leftarrow$) Obvious. ($\Rightarrow$) Suppose E satisfies $(**)$. Then $p_\alpha, c_{\alpha,\beta,\gamma,\delta}$ and σ increase E.

Since $\langle t_\alpha(E), H\rangle = \langle E, H\rangle, t_\alpha$ increases E in a trivial sense. A correlation $\sigma \circ p_\alpha \circ \sigma$ is the (-4)-reflection w.r.t. $\sigma(H - 2N_\alpha)$, so that it increases E if and only if

$$\langle E, \sigma(H - 2N_\alpha)\rangle = 2\left(4a - \sum_{\beta \notin I(T_\alpha)} b_\beta\right) \geqslant 0.$$

But this inequality follows from the last inequality of $(**)$.

Now let g be an arbitrary element of G. Then, by (6.5), g can be written in the form

$$g = t_\alpha \circ \phi_2 \circ \phi_1, \quad \text{or} \quad t_\alpha \circ \phi_2 \circ \sigma \circ \phi_1,$$

where ϕ_1, ϕ_2 are the same as in (6.5).

Case 1. $g = t_\alpha \circ \phi_2 \circ \phi_1$.

Since ϕ_1 is a product of reflections, we see that $\phi_1(E)$, by induction, is of the form

$$\begin{aligned}\phi_1(E) &= E + \sum k_\alpha(H - 2N_\alpha) + \sum m_\alpha \sigma(H - 2N_\alpha)\\ &\quad + \sum n_{\alpha,\beta,\gamma,\delta}(H - N_\alpha - N_\beta - N_\gamma - N_\delta)\end{aligned}$$

with all $k_\alpha, m_\alpha, n_{\alpha,\beta,\gamma,\delta}$ non-negative. This proves that ϕ_1 increases E. But then, since $\langle g(E), H\rangle = \langle \phi_1(E), H\rangle$, g increases E.

Case 2. $g = t_\alpha \circ \phi_2 \circ \sigma \circ \phi_1$.

Using the formula for $\phi_1(E)$ as shown in Case 1, we see that

$$\begin{aligned}\langle g(E), H\rangle &= \langle \sigma \circ \phi_1(E), H\rangle\\ &= \langle \sigma(E), H\rangle + 4\left(2\sum k_\alpha + \sum m_\alpha + \sum n_{\alpha,\beta,\gamma,\delta}\right)\\ &\geqslant \langle E, H\rangle,\end{aligned}$$

and that g increases E.

The following lemma follows easily from the Riemann–Roch theorem.

(6.8) LEMMA. *Let* $E = aH - \Sigma b_\alpha N_\alpha$, $a, b_\alpha \in \frac{1}{2}\mathbf{Z}$, $a \neq 0$, *be a divisor of* $\widehat{F}$ *with* $E^2 \geqslant -2$. *Then* E *is effective if and only if* $a > 0$.

(6.9). Define

$$\mathcal{P} = \{E \in S\colon E \text{ primitive}, \langle E, H\rangle > 0\},$$

and

$$\mathcal{I} = \{E \in S\colon E \text{ primitive, effective}, E^2 = 0\}.$$

Then, by (6.8), $\mathcal{I} \subset \mathcal{P}$.

There is a one-to-one correspondence: $\mathcal{P} \to \mathbf{Q}^{16} \subset \mathbf{R}^{16}$,

$$E = aH - \sum_\alpha b_\alpha N_\alpha \in \mathcal{P} \to (b_\alpha/a)_{\alpha \in A_2} \in \mathbf{Q}^{16}.$$

The image of $\mathcal{I}$ under this correspondence is the set of rational points on the 15-dimensional sphere $\mathbf{S}^{15}_2$ of radius $= \sqrt{2}$. It is easy to see that the set of rational points on this sphere is dense in the sphere.

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(6.10). Let $\mathcal{E}$ be the set of elliptic pencils on $\widehat{F}$. Then

$$\begin{aligned}\mathcal{E} &= \{E \in \mathcal{I}: \langle E, R\rangle \geqslant 0 \text{ for all non-singular rational curve } R\}\\ &= \{E \in \mathcal{I}: \langle E, R\rangle \geqslant 0 \text{ for all effective divisor } R \text{ with } R^2 = -2\}.\end{aligned}$$

Consider a subset $\mathcal{E}_{\min}$ of $\mathcal{E}$: $\mathcal{E}_{\min} = \{E \in \mathcal{E}: E$ can not be decreased by any automorphism of $\widehat{F}$, i.e., E has minimum degree $(= \langle E, H\rangle)$ among elements in its orbit $\mathrm{Aut}(\widehat{F})E\}$. Then $\mathcal{E}_{\min}$ can be identified, by the correspondence $\mathcal{P} \to \mathbf{Q}^{16}$, with the set of rational points on $\mathbf{S}_2^{15} \cap P_\infty$, where $P_\infty \subset \mathbf{R}^{16}$ is the polyhedron defined by infinitely many hyperplanes (or linear inequalities)

$$\begin{cases}\langle E, H\rangle \leqslant \langle g(E), H\rangle, g \in \mathrm{Aut}(\widehat{F}),\\ \langle E, R\rangle \geqslant 0, R \quad \text{a non-singular rational curve.}\end{cases}$$

Since $\mathcal{E}/\mathrm{Aut}(\widehat{F})$ is finite (cf. [St]), we see that $\mathcal{E}_{\min}$ is finite. This fact together with the denseness of rational points of $\mathbf{S}_2^{15}$ implies that P_∞ is inscribed in $\mathbf{S}_2^{15}$, i.e., all points of P_∞ have length$^2 \leqslant 2$.

Let P be the polyhedron $\subset \mathbf{R}^{16}$ defined by the following 109 equalities:

$$\begin{cases}1 \geqslant x_\alpha, \alpha \in A_2 \quad \text{(16 of them)},\\ 2 \geqslant x_\alpha + x_\beta + x_\gamma + x_\delta, \ (\alpha, \beta, \gamma, \delta) \quad \text{a Göpel tetrad (60 of them)},\\ 4 \geqslant \sum_\alpha x_\alpha,\\ x_\alpha \geqslant 0, \quad \alpha \in A_2 \quad \text{(16 of them)}\\ 2 \geqslant x_\alpha + x_\beta + x_\gamma + x_\delta + x_\kappa + x_\lambda, \quad (\alpha, \beta, \gamma, \delta, \kappa, \lambda)\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a trope (16 of them).}\end{cases}$$

Then, by (6.7), rational points on $\mathbf{S}_2^{15} \cap P$ correspond to elements of the set $\{E \in \mathcal{I}: E$ can not be decreased by any element of G, and $\langle E, N_\alpha\rangle \geqslant 0,\ \langle E, T_\alpha\rangle \geqslant 0,$ for all $\alpha \in A_2\}$.

The above discussion suggests that, as long as a vertex of P has length$^2 > 2$, one should be able to find either an automorphism or a rational curve whose corresponding hyperplane cuts off the vertex.

Let $\{\alpha, \beta, \gamma, \delta, \kappa, \lambda\} \subset A_2$ be a sixtuple such that no four of $\alpha, \ldots, \lambda$ are contained in a Göpel tetrad or in a trope, e.g., $\{0, 14, 15, 23, 25, 34\}$. Then the point $(x_\nu)_{\nu \epsilon A_2} \in \mathbf{R}^{16}, x_\nu = \frac{2}{3}$, if $\nu = \alpha, \ldots, \lambda; x_\nu = 0$ otherwise, is a vertex of P. [Such a vertex was found by 'random walk' from a vertex $(1, 1, 0, \ldots, 0)$.] It has length$^2 = \frac{8}{3} > 2$ and is cut off by each of the six linear inequalities

$$\begin{aligned}&4 \geqslant 2x_\alpha + x_\beta + \cdots + x_\lambda,\\ &4 \geqslant x_\alpha + 2x_\beta + \cdots + x_\lambda,\\ &\quad\vdots\\ &4 \geqslant x_\alpha + x_\beta + \cdots + 2x_\lambda\end{aligned}$$

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corresponding to (-2)-curves

$$\begin{aligned}
&2H-2N_\alpha-N_\beta-\cdots-N_\lambda,\\
&2H-N_\alpha-2N_\beta-\cdots-N_\lambda,\\
&\ \vdots\\
&2H-N_\alpha-N_\beta-\cdots-2N_\lambda.
\end{aligned}$$

These new inequalities result in new vertices. Among them are the points $(x_\nu)_{\nu\in A_2}\in\mathbf{R}^{16}$, $x_\nu=\frac{4}{7}$, if $\nu=\alpha,\dots,\lambda$; $\frac{2}{7}$, if $\nu=\mu$; 0, otherwise, where μ is an element of A_2 different from $\alpha,\dots,\lambda$. These vertices have length$^2=\frac{100}{49}$ and their corresponding divisors.

$$7H-4(N_\alpha+N_\beta+N_\gamma+N_\delta+N_\kappa+N_\lambda)-2N_\mu$$

are (-4)-roots of $S=\mathrm{Pic}(\widehat{F})$.

(6.11) THEOREM. *Let $\alpha,\beta,\gamma,\delta,\kappa$ and λ be six elements of A_2 such that no four of them are contained in a Göpel tetrad or in a trope and let μ be another element of A_2. Then the reflection ϕ of $S=\mathrm{Pic}(\widehat{F})$ with respect to the (-4)-root*

$$V=7H-4(N_\alpha+N_\beta+N_\gamma+N_\delta+N_\kappa+N_\lambda)-2N_\mu,$$

realizes as an automorphism of $\widehat{F}$ not belonging to G.

Proof.

Step 1. The pair $(\phi,-\mathrm{id}_T)$ extends to an Hodge-isometry of $H_2(\widehat{F},\mathbf{Z})$. (See (4.1).) It is easy to check that ϕ induces -1 on the discriminant lattice of $\widehat{F}$.

Step 2. ϕ is effective. This part is postponed until (6.16). Now, by the Torelli theorem, ϕ realizes as an automorphism of $\widehat{F}$.

Step 3. $\phi\notin G$. This follows from the construction. Indeed, ϕ increases $E=aH-\Sigma b_\nu N_\nu$ if and only if

$$\langle E,V\rangle=4\{7a-2(b_\alpha+b_\beta+b_\gamma+b_\delta+b_\kappa+b_\lambda)-b_\mu\}\geqslant 0.$$

This is a new inequality because the divisor $3H-2(N_\alpha+N_\beta+N_\gamma+N_\delta+N_\kappa+N_\lambda)$ or the divisor V satisfies the 77 inequalities $(**)$ of (6.6), but not this one. So, by (6.7), $\phi\notin G$.

(6.12) DEFINITION. A hexad $(\alpha,\beta,\gamma,\delta,\kappa,\lambda)$ of elements of A_2 is called *good* if no four of the six elements are contained in a trope or in a Göpel tetrad. There are 192 such good hexads, each 16 of them mutually conjugate by translations. We give a list of 12 representatives:

$$\begin{aligned}&(0,1,2,13,24,34),(0,1,2,13,25,35),\\&(0,1,2,14,23,34),(0,1,2,14,25,45),\\&(0,1,2,15,24,45),(0,1,2,15,23,35),\\&(0,1,3,14,23,24),(0,1,3,14,35,45),\\&(0,1,3,15,23,25),(0,1,3,15,34,45),\\&(0,1,4,15,24,25),(0,1,4,15,34,35).\end{aligned}$$

All good hexads form a single orbit under the automorphism group of (16_6)-configuration.

(6.13) LEMMA. *Let* $M = aH - \Sigma b_\nu N_\nu, a, b_\nu \in \frac{1}{2}\mathbf{Z}, a > 0, b_\nu > 0$, *be a divisor of* $\widehat{F}$ *with self- intersection* $M^2 > 0$. *Then* M *is ample if and only if* $\langle M, D\rangle > 0$ *for any irreducible* (-2)*-curve* $D = yH - \Sigma x_\nu N_\nu$ *with* $y^2 \leqslant 2a^2/M^2 - \frac{1}{2}$.

Proof. This follows from Nakai-Moishezon criterion (cf [Har]) and Schwarz inequality.

(6.14) PROPOSITION. *The automorphism* ϕ *constructed in* (6.11) *is a conjugate of a projection, i.e.* $\phi = \psi^{-1}p_\varepsilon\psi$ *for some* $\varepsilon \in A_2$, *some* $\psi \in \mathrm{Aut}(\widehat{F})$.

Proof. The main part of the proof will be the following:

(6.15) LEMMA. *Let* V *be the divisor as in* (6.11). *Then the linear system* $|W|$ *of*

$$W = V + N_\mu = 7H - 4(N_\alpha + N_\beta + N_\gamma + N_\delta + N_\kappa + N_\lambda) - N_\mu,$$

defines a double covering: $\widehat{F} \to \mathbf{P}^2$, *which is branched along the sum of six lines. In particular, the covering involution is a conjugate of a projection.*

Proof. Note first that $\langle W, W\rangle = 2$.

We will prove that $\langle D, W\rangle \geqslant 0$ for each effective (-2)-curve D. This is obvious if $D = N_\alpha$. Write $D = yH - \Sigma x_\nu N_\nu$.

If $y \geqslant 7$, then $y^2 \geqslant 49 > 2\cdot 7^2/W^2 - \frac{1}{2}$, so, by the Schwarz inequality (cf. (6.13)), $\langle D, W\rangle > 0$.

If $0 < y < 7$, then for each integer or half-integer value of y, one can compute using the following Key facts the minimum value of $\langle D, V\rangle$ and show that it is nonnegative. Then $\langle D, W\rangle = \langle D, V\rangle + 2x_\mu \geqslant 0$.

KEY FACTS: (i) For each y fixed, one sees, inspired by Lagrange multiplier computation, that $< D, V > (= 4\{7y - 2(x_\alpha + \cdots + x_\lambda) - x_\mu\})$ takes on its minimum value when $\Sigma x_\nu N_\nu$ is 'almost proportional' to $4(N_\alpha + \cdots + N_\lambda) + 2N_\mu$, i.e., $x_\alpha \approx x_\beta \approx, \ldots, \approx x_\lambda \approx 2x_\mu, x_\nu \approx 0, \nu \neq \alpha, \ldots, \mu$.

(ii) If y is an integer, then $\{\nu \in A_2 \colon x_\nu \text{ is a half-integer}\} = A_2$ or a hyperplane J of A_2, and if y is a half-integer, then $\{\nu \in A_2 \colon x_\nu \text{ is a half-integer}\} = I(T_\zeta)$ or $A_2 \backslash I(T_\zeta)$ for some trope T_ζ (cf. (1.10) and (3.1)).

(iii) Let $\mathcal{H}$ be a good hexad, then

$$|\mathcal{H} \cap J| = 2 \text{ or } 4, \quad J \text{ a hyperplane,}$$

$$|\mathcal{H} \cap I(T)| = 1 \text{ or } 3, \quad T \text{ a trope.}$$

Now it can be shown that $\langle D, W\rangle = 0$ if and only if D is one of the following fifteen (-2)-curves:

$$\sharp \begin{cases} N_\nu, \nu \in A_2 \backslash \{\alpha, \beta, \gamma, \delta, \kappa, \lambda, \mu\}, \\ 2H - (N_\alpha + N_\beta + N_\gamma + N_\delta + N_\kappa + N_\lambda) - N_\nu, \nu \in \{\alpha, \beta, \gamma, \delta, \kappa, \lambda\}. \end{cases}$$

Note that these 15 curves are all irreducible (hence smooth) and mutually disjoint (The irreducibility of each of the last six curves follows from its indecomposability).

Now, by a result of Saint–Donat [SD], the linear system $|W|$ defines a double covering: $\widehat{F} \to \mathbf{P}^2$, which contracts exactly above 15 rational curves. Then it follows that the branch locus is a sextic with 15 nodes, and hence a sum of six lines in general position.

Let's complete the proof of (6.14).

Since $\langle V, W\rangle = 0, \phi(W) = W$, so ϕ acts on $\mathbf{P}^2$ Also the action of ϕ on $\mathbf{P}^2$ leaves invariant the sum of six lines (= branch locus), ϕ must be the identity of $\mathbf{P}^2$ and hence, the covering involution of the double cover. Now the result follows from (6.15).

(6.16) PROPOSITION. *The automorphism ψ as in* (6.14) *can be chosen to be the one induced by the linear system*

$$|7H - 4(N_\alpha + N_\beta + N_\gamma + N_\delta + N_\kappa + N_\lambda)|.$$

Proof. Let $f = \psi^{-1}$. Clearly f maps a collection of 15 nodes to the collection $\sharp$ of the 15 (-2)-curves (see the proof of (6.15)). Since $t_\nu p_\varepsilon t_\nu = p_{\varepsilon+\nu}$, we may assume that $\varepsilon = \mu$ and the collection of 15 nodes is $\{N_\nu \colon \nu \in A_2 \backslash \mu\}$. We need to determine $f(H)$ and $f(N_\mu)$.

Since $f p_\mu f^{-1}$ is the reflection w.r.t. $f(H - 2N_\mu), f(H - 2N_\mu) = V$ or $-V$. But this is not enough to determine $f(H)$ and $f(N_\mu)$, i.e. there are infinitely many choices for the pair. Among them we set

$$f(H) = 7H - 4(N_\alpha + N_\beta + N_\gamma + N_\delta + N_\kappa + N_\lambda) \quad \text{and} \quad f(N_\mu) = N_\mu.$$

This is a minimum possible choice in the sense that the leading coefficients of these two vectors are the smallest possible.

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We will prove that this choice actually works, i.e. by assigning in a suitable way each of the 15 remaining nodes to a curve in the collection $\sharp$ we can make such an f an isometry which realizes as an automorphism.
For such an f to be an isometry it is necessary and sufficient that

(i) $f(T_\nu) \in S$ for each trope T_ν (see (3.1)).
By (4.1), it is necessary for f to realize as an automorphism that
(ii) f must act as id or $-$id on the discriminant group of S.

These two conditions determine f uniquely:
Let $(\alpha, \beta, \gamma, \delta, \kappa, \lambda, \mu) = (0, 14, 15, 23, 25, 34, 12)$ for convenience.
From the equality $f(C_1) \equiv C_1(\mathrm{mod} S)$, where $C_1 = (N_2+N_{12}+N_3+N_{13})/2$, we get, using the key facts in the proof of (6.15),

$$f\{N_2, N_{12}, N_3, N_{13}\} = \{N_2, N_{12}, N_3, N_{13}\}.$$

Similar arguments for C_2, C_3, C_4 and B give us

$$\begin{aligned}
&f\{1, 12, 4, 24\} = \{1, 12, 4, 24\},\\
&f\{2, 4, 12, 14\} = \{2H - (N_0 + \cdots + N_{34}) - N_{25}, 3, 24, 12\},\\
&f\{1, 3, 12, 23\} = \{12, 4, 13, 2H - (N_0 + \ldots + N_{34}) - N_{15}\},
\end{aligned}$$

$$f\{0, 1, 2, 12\} = \begin{cases} \{12, 3, 4, 5\}(\text{ if } f = \text{ id on } D_S), \\ \{2H - (N_0 + \cdots + N_{34}) - N_{34}, 13, 24, 12\} \\ \quad (\text{if } f = -\text{id on } D_S). \end{cases}$$

If $f = -$id on D_S, then there is no compatibility, and if $f =$id on D_S, then

$$\begin{aligned}
f(H) &= 7H - 4(N_0 + N_{14} + N_{15} + N_{23} + N_{25} + N_{34}),\\
f(N_{12}) &= N_{12},\\
f(N_2) &= N_3,\\
f(N_4) &= N_{24},\\
f(N_{14}) &= 2H - (N_0 + \cdots + N_{34}) - N_{25},\\
f(N_3) &= N_{13},\\
f(N_{13}) &= N_2,\\
f(N_1) &= N_4,\\
f(N_{24}) &= N_1,\\
f(N_{23}) &= 2H - (N_0 + \cdots + N_{34}) - N_{15},\\
f(N_0) &= N_5
\end{aligned}$$

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and then the condition (i) implies

$$\begin{aligned} f(N_5) &= 2H - (N_0 + \cdots + N_{34}) - N_{34}, \\ f(N_{15}) &= N_{35}, \\ f(N_{25}) &= N_{45}, \\ f(N_{34}) &= 2H - (N_0 + \cdots + N_{34}) - N_0, \\ f(N_{35}) &= 2H - (N_0 + \cdots + N_{34}) - N_{23}, \\ f(N_{45}) &= 2H - (N_0 + \cdots + N_{34}) - N_{14}. \end{aligned}$$

Conversely, it is easy to check that the above assignment satisfies conditions (i) and (ii).

For the effectiveness of f, which proves step 2 of (6.11), we need to show that the image M of the ample divisor $2H - \Sigma N_\nu/2$ is ample, i.e.,

$$M = 8H - 9/2(N_\alpha + N_\beta + N_\gamma + N_\delta + N_\kappa + N_\lambda) - \frac{1}{2}\sum \text{(remaining 10 nodes)},$$

is ample. From (6.13), we see that M is ample if $\langle D, M\rangle > 0$ for any irreducible (-2)-curve $D = yH - \Sigma x_\nu N_\nu$ with $0 \leqslant y < 4$. This can be done by the same method used for the inequality $\langle D, V\rangle \geqslant 0$ as in (6.15). (In this case, Key fact (i) should read '$\cdots \langle D, M\rangle$ takes on its minimum value when $\Sigma x_\nu N_\nu$ is almost proportional to $9/2(N_\alpha + N_\beta + N_\gamma + N_\delta + N_\kappa + N_\lambda) + \frac{1}{2}\Sigma$ remaining 10 nodes.')

Now, by (4.1), f realizes as an automorphism.

It remains to show that $\psi = f^{-1}$ is induced by the linear system $|f(H)|$. Since H has the property that there is no primitive element E such that $\langle E, E\rangle = 0$ and $\langle E, H\rangle = 2$, so does $f(H)$. This implies, by the result of [SD], that $|f(H)|$ defines a birational map of $\widehat{F}$ onto a quartic surface in $\mathbf{P}^3$. Note that this map contracts sixteen mutually disjoint (-2)-curves, namely, N_μ and the collection $\sharp$. Finally, note that this quartic surface, by Torelli theorem, is projectively isomorphic to our original quartic surface F.

(6.17) REMARK. The following can be seen easily.

(i) The map ψ as in (6.16) has an infinite order,

(ii) The inverse of ψ is also such a map, but corresponds to a different hexad, e.g. if ψ corresponds to $(0, 14, 15, 23, 25, 34)$, then ψ^{-1} to $(5, 14, 23, 34, 35, 45)$.

(6.18) QUESTION. Is $\mathrm{Aut}(\widehat{F})$ generated by those classical automorphisms together with these 192 new ones corresponding to good hexads?

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