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CONVOLUTION ESTIMATES AND GENERALIZED DE LEEUW THEOREMS FOR MULTIPLIERS OF WEAK TYPE (1, 1)

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ABSTRACT. In the context of a locally compact abelian group, we establish maximal theorem counterparts for weak type (1, 1) multipliers of the classical de Leeuw theorems for individual strong multipliers. Special methods are developed to handle the weak type (1, 1) estimates involved since standard linearization methods such as Lorentz space duality do not apply to this case. In particular, our central result is a maximal theorem for convolutions with weak type (1, 1) multipliers which opens avenues of approximation. These results complete a recent series of papers by the authors which extend the de Leeuw theorems to a full range of strong type and weak type maximal multiplier estimates in the abstract setting.

1. **Introduction.** Throughout all that follows, G will be a locally compact abelian group with dual group Γ. The group Γ endowed with the discrete topology will be denoted by Γ_d. For $\psi \in L^\infty(\Gamma)$, we symbolize by T_ψ the corresponding multiplier transform on $L^2(G) : T_\psi f = (\psi \hat{f})\check{}$. If $1 \leq p < \infty$, then ψ is said to be a *multiplier of weak type* (p,p) (in symbols, $\psi \in M_p^{(w)}(\Gamma)$) provided that T_ψ is of weak type (p,p) on $L^2(G) \cap L^p(G)$. In this case, T_ψ extends uniquely from $L^2(G) \cap L^p(G)$ to a linear mapping $T_\psi^{(p)}$ of $L^p(G)$ into the measurable complex-valued functions on G such that whenever $f_n \longrightarrow f$ in $L^p(G)$, we have $T_\psi^{(p)} f_n \longrightarrow T_\psi^{(p)} f$ in measure. In this notation, the usual space $M_p(\Gamma)$ of strong type Fourier multipliers consists of the functions $\psi \in M_p^{(w)}(\Gamma)$ such that $T_\psi^{(p)}$ is of strong type (p,p). For a sequence $\{\psi_j\}_{j\geq 1} \subseteq M_p^{(w)}(\Gamma)$, we write $N_p^{(w)}(\{\psi_j\}_{j\geq 1})$ to symbolize the weak type (p,p) norm of the maximal operator on $L^p(G)$ defined by the sequence $\{T_{\psi_j}^{(p)}\}_{j\geq 1}$. We shall also denote by $\mathcal{C}(\Gamma)$ the Banach algebra of all bounded, continuous, complex-valued functions on Γ (with the usual uniform norm $\|\cdot\|_u$). The symbol $\mathbb{N}$ will stand for the set of positive integers, $\mathbb{Z}$ will be the additive group of integers, and $\mathbb{R}$ will be the additive group of real numbers. A given Haar measure on G will be indicated by μ_G.

In his celebrated article [7], K. de Leeuw introduced fundamental relationships between the individual multipliers of strong type (p,p) which are defined on $\mathbb{R}^n$, and those which are defined on $\mathbb{R}^n_d$, or on a quotient group or (by restriction) on a closed subgroup of $\mathbb{R}^n$. In [12], these theorems of de Leeuw were completely extended to individual multipliers and their strong type norms in the context of the general locally compact abelian group Γ. De Leeuw's restriction theorem concerning subgroups, as well as another seminal result in the classical setting of his paper [7, Corollary (4.6)], foreshadow the abstract

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Multiplier Homomorphism Theorem [9, Appendix B], which asserts that composition with any continuous homomorphism into Γ decreases the strong type norms of individual multipliers belonging to $M_p(\Gamma)\cap \mathcal{C}(\Gamma)$. Moreover, throughout the range $1 \leq p < \infty$, all the aforementioned abstract generalizations of de Leeuw's single multiplier results have counterparts for the strong type (p,p) norms of maximal multiplier operators (see [2, Theorem (4.1)] and [5, Theorems (4.25) and (6.5)]).

Abstract analogues of de Leeuw's theorems have also been obtained when $1 < p < \infty$ for weak type multipliers and for the weak type norms of corresponding maximal operators ([4, Theorem (4.1)-(i)] and [5, Theorems (4.11) and (6.3)]). However, special difficulties arise when one attempts to carry over de Leeuw's theorems to weak type $(1,1)$ multipliers and their corresponding (individual and maximal) weak type norms, because standard linearization techniques such as Lorentz space duality cannot be applied in this case. Consequently when $p = 1$ only the de Leeuw theorem for multipliers defined on quotient groups has been generalized to weak type multipliers (see the statement of [5, Theorem (4.11)] reproduced in Theorem (3.1) below). The purpose of this article is to establish analogues of the other de Leeuw theorems in the refractory case of weak type $(1,1)$ multipliers, and thereby furnish locally compact abelian groups with the complete array of generalizations for weak type as well as for strong type maximal multiplier transforms over the range $1 \leq p < \infty$. This goal is accomplished by an appropriate theorem for $M_1^{(w)}(\Gamma_d)$ and a homomorphism theorem for weak type $(1,1)$ multipliers, which are stated in (1.3) and (1.5), respectively.

The key ingredient in these considerations is the following convolution theorem for $M_1^{(w)}(\Gamma)$ (established in Section 5). In essence, this result on convolutions extends to the weak type $(1,1)$ case a *genre* of approximation techniques useful in general multiplier theory.

THEOREM 1.1. *Let G be a locally compact abelian group with dual group* Γ. *Suppose that* $k \in L^1(\Gamma)$, *and* $\{\psi_j\}_{j=1}^{\infty} \subseteq \mathcal{C}(\Gamma)\cap M_1^{(w)}(\Gamma)$. *Then* $\{k * \psi_j\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\Gamma)$, *and*

$$N_1^{(w)}(\{k * \psi_j\}_{j=1}^{\infty}) \leq C\|k\|_{L^1(\Gamma)} N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}), \tag{1.2}$$

where $C > 0$ *is the absolute constant specified in Theorem 2.7 below.*

Our generalized de Leeuw theorem for the Bohr compactification $b(G)$ of G (and its dual Γ_d) is expressed in the following result (see Sections 3, 4 for the proof).

THEOREM 1.3. *Let G be a locally compact abelian group with dual group* Γ, *and let* $\iota: \Gamma_d \longrightarrow \Gamma$ *be the identity homomorphism. Suppose that* $\{\psi_j\}_{j=1}^{\infty} \subseteq \mathcal{C}(\Gamma)$. *Then* $\{\psi_j\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\Gamma)$ *if and only if the composite functions* $\psi_j \circ \iota$ *satisfy* $\{\psi_j \circ \iota\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\Gamma_d)$. *If this is the case, then*

$$N_1^{(w)}(\{\psi_j \circ \iota\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}) \leq C N_1^{(w)}(\{\psi_j \circ \iota\}_{j=1}^{\infty}), \tag{1.4}$$

where C is the absolute constant in (1.2).

As a consequence of Theorem 1.1, we obtain (in Section 6) a generalized de Leeuw theorem which asserts that composition with a homomorphism preserves maximal estimates for weak type $(1,1)$ multipliers. In particular, this automatically covers restrictions of multipliers to subgroups. The specific statement of the homomorphism theorem follows. This result is new even in the special case of a single multiplier of weak type (1, 1).

THEOREM 1.5. *Suppose that ρ is a continuous homomorphism of the locally compact abelian group Γ_1 into the locally compact abelian group Γ_2, and that $\{\psi_j\}_{j=1}^{\infty} \subseteq \mathcal{C}(\Gamma_2) \cap M_1^{(w)}(\Gamma_2)$. Then the composite functions $\psi_j \circ \rho$, $j \in \mathbb{N}$, belong to $M_1^{(w)}(\Gamma_1)$, and*

$$N_1^{(w)}(\{\psi_j \circ \rho\}_{j=1}^{\infty}) \leq C N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}),$$

where C is the absolute constant in (1.2).

Our procedure for deducing these main results can be outlined as follows. In Section 2 we establish the special case of Theorem 1.1 when G is compact. In Section 3 we deduce the left-hand inequality in (1.4) for groups having a special form. With the aid of the structure theory for locally compact abelian groups, these results are used in Section 4 to demonstrate Theorem 1.3, which in Section 5 enables us to deduce the Convolution Theorem 1.1. After carrying out the proof of Theorem 1.1, we apply it in Section 6 to infer a general theorem transferring weak type (1, 1) multipliers to measure spaces. We close Section 6 by showing how this general transference result leads directly to a proof of the Homomorphism Theorem 1.5.

We remark that a variant of the convolution Theorem 1.1 has been established in [1, Theorem (1.2)] for the special case $\Gamma = \mathbb{R}^n$. At the expense of replacing the absolute constant C in (1.2) by a constant depending on the dimension n, this variant of Theorem 1.1 drops the pointwise continuity requirement on the functions ψ_j. Although the methods in [1] do not generalize directly to the setting of locally compact abelian groups, the discussion in Section 2 below adapts the strategy of [1] to the case of compact G.

2. A preliminary convolution theorem.

As a forerunner to our main results, we establish in Theorem 2.7 below the special case of Theorem 1.1 for compact G.

For ease of reference we begin the considerations of this section with the statement of Khintchine's Theorem [15, Theorem V.8.4, p. 213], which will be used repeatedly.

THEOREM 2.1. *If $0 < r < \infty$, then there are positive real constants α_r, B_r, depending only on r, such that if $J \in \mathbb{N}$, and $c_1, \ldots, c_J$ are complex numbers, then*

$$\alpha_r \Big\{ \sum_{j=1}^{J} |c_j|^2 \Big\}^{1/2} \leq \Big[\int_{D^J} \Big| \sum_{j=1}^{J} \epsilon_j c_j \Big|^r \, d\epsilon \Big]^{1/r} \leq B_r \Big\{ \sum_{j=1}^{J} |c_j|^2 \Big\}^{1/2},$$

where D^J is the J-fold product of the multiplicative group $D = \{-1, 1\}$, with general element $\epsilon = (\epsilon_1, \ldots, \epsilon_J)$, and $d\epsilon$ is Haar measure on D^J normalized so that $d\epsilon$ has total mass 1.

REMARKS. Notice that for $0 < r \leq 2$, the constant B_r in Theorem 2.1 can be taken to be 1. This fact will affect the values of the constants obtained in our results.

To set the stage for Theorem 1.1 in the special case of compact G, we first take up three lemmas which will simplify and focus the discussion. The first of these is shown in [1, Lemma (2.1)] by a variation on the method used to prove [14, Lemma 1, pp. 146, 147].

LEMMA 2.2. *Let G be a compact abelian group, and let m be normalized Haar measure on G. Suppose that A is a measurable subset of G such that* $m(A) > 0$. *Let*

$$N = \max\{k \in \mathbb{N} : m(A) \leq k^{-1}\}.$$

Then there are elements $z_1, \ldots, z_N$ *of G such that the corresponding translates satisfy*

$$m\Big(\bigcup_{j=1}^{N} A + z_j\Big) > \Big(1 - \frac{2}{e}\Big), \tag{2.3}$$

where e is the base of natural logarithms.

The next lemma provides a variation on the theme of [13, Theorem 1]. While the latter provides only the conclusion that $\mathcal{M}$ is of weak type $(1, 1)$, the precise estimate we obtain in (2.5) is essential to all the reasoning used to establish the convolution Theorems 2.7 and 1.1.

LEMMA 2.4. *Suppose that G is a compact abelian group with normalized Haar measure m, and that* $0 < p < \infty$. *Suppose further that* $M \in \mathbb{N}$, *and that* T_k, $1 \leq k \leq M$, *is a bounded linear mapping of* $L^1(G)$ *into* $L^p(G)$ *which commutes with translations. Let* $\mathcal{M}$ *be the maximal operator defined on* $L^1(G)$ *by the operators* $\{T_k\}_{k=1}^{M}$, *and denote the strong type* $(1,p)$ *norm of* $\mathcal{M}$ *by* $\|\mathcal{M}\|_{1,p}$. *Then we have:*

$$m\{x \in G : (\mathcal{M}f)(x) > y\} \leq \Big(\frac{e}{e-2}\Big)^{1/p} \alpha_p^{-1} \|\mathcal{M}\|_{1,p} \|f\|_1 y^{-1}, \tag{2.5}$$

for all $f \in L^1(G)$, *and all* $y > 0$, *where* α_p *is the constant corresponding to p in Khintchine's Inequality (2.1).*

PROOF. Suppose that $f \in L^1(G)$ and $y > 0$. Put

$$A = \{x \in G : (\mathcal{M}f)(x) > y\}.$$

The non-trivial case occurs when $m(A) > 0$, and we let

$$N = \max\{k \in \mathbb{N} : m(A) \leq k^{-1}\}.$$

Choose elements $z_1, \ldots, z_N$ of G so that (2.3) is satisfied. For $u \in G$, we write u as a subscript to denote translation by u, and we denote the characteristic function of a subset S of G by χ_S. Let $A_0 = \bigcup_{j=1}^{N} A_{z_j}$, and notice that for $1 \leq j \leq N$, it follows by the translation invariance of the operators T_k that we can take

$$A_{z_j} = \Big\{x \in G : \max_{1 \leq k \leq M} \Big|\Big(T_k(f_{-z_j})\Big)(x)\Big| > y\Big\}.$$

It follows readily that we have pointwise on G:

$$\chi_{A_0} = \max_{1 \leq j \leq N} \chi_{A_{z_j}} \leq y^{-1} \max_{1 \leq k \leq M} \max_{1 \leq j \leq N} |T_k(f_{-z_j})| \leq y^{-1} \max_{1 \leq k \leq M} \Big\{\sum_{j=1}^{N} |T_k(f_{-z_j})|^2\Big\}^{1/2}.$$

By Khintchine's Inequality we infer from this that pointwise on G,

$$\begin{aligned}\chi_{A_0} &\leq \alpha_p^{-p} y^{-p} \max_{1\leq k\leq M} \int_{D^N} \Big|\sum_{j=1}^{N} \epsilon_j T_k(f_{-z_j})\Big|^p \, d\epsilon \\ &\leq \alpha_p^{-p} y^{-p} \int_{D^N} \Big[\mathcal{M}\Big(\sum_{j=1}^{N} \epsilon_j f_{-z_j}\Big)\Big]^p \, d\epsilon.\end{aligned}$$

Integrating this last inequality over G with respect to dm, we see with the aid of (2.3) that:

$$\begin{aligned}\frac{e-2}{e} &< \alpha_p^{-p} y^{-p} \int_{D^N} \Big\|\mathcal{M}\Big(\sum_{j=1}^{N} \epsilon_j f_{-z_j}\Big)\Big\|_p^p \, d\epsilon \\ &\leq \alpha_p^{-p} y^{-p} \|\mathcal{M}\|_{1,p}^p N^p \|f\|_1^p \\ &\leq \alpha_p^{-p} y^{-p} \|\mathcal{M}\|_{1,p}^p \|f\|_1^p \big(m(A)\big)^{-p}.\end{aligned}$$

This completes the proof of Lemma 2.4.

The next lemma ([1, Lemma 2.8]) involves a well-known circle of ideas, stated here in a form convenient for our purposes.

LEMMA 2.6. *Suppose that* (Ω, μ) *is a measure space,* $M \in \mathbb{N}$, *and* T_k, $1 \leq k \leq M$, *is a linear mapping of* $L^1(\Omega, \mu)$ *into the complex-valued measurable functions on* Ω *such that* T_k *is of weak type* $(1,1)$. *Let* $\mathcal{M}$ *be the maximal operator on* $L^1(\Omega, \mu)$ *defined by the operators* $\{T_k\}_{k=1}^M$, *and denote the weak type* $(1,1)$ *norm of* $\mathcal{M}$ *on* $L^1(\Omega, \mu)$ *by* $N_1^{(w)}(\mathcal{M})$. *If* $0 < p < 1$, *and* S *is a subset of* Ω *such that* $\mu(S) < \infty$, *then for each* $g \in L^1(\Omega, \mu)$:

$$\|\mathcal{M}g\|_{L^p(S,\mu)} \leq \mu(S)^{(1-p)/p} (1-p)^{-1/p} N_1^{(w)}(\mathcal{M}) \|g\|_{L^1(\Omega,\mu)}.$$

THEOREM 2.7. *Let* G *be a compact abelian group with dual group* Γ. *Suppose that* $\zeta \in L^1(\Gamma)$, *and* $\{\phi_k\}_{k=1}^\infty \subseteq M_1^{(w)}(\Gamma)$. *Then* $\{\zeta * \phi_k\}_{k=1}^\infty \subseteq M_1^{(w)}(\Gamma)$, *and*

$$N_1^{(w)}(\{\zeta * \phi_k\}_{k=1}^\infty) \leq C \|\zeta\|_{L^1(\Gamma)} N_1^{(w)}(\{\phi_k\}_{k=1}^\infty), \tag{2.8}$$

where $C > 0$ *is the absolute constant specified by*

$$C = \inf_{0<p<1} \Big(\frac{e}{e-2}\Big)^{1/p} \alpha_p^{-2} (1-p)^{-1/p},$$

α_p *being as described in Khintchine's Inequality.*

PROOF. Obviously it is enough to establish the inequality corresponding to (2.8) in the case of a finite sequence $\{\phi_k\}_{k=1}^M \subseteq M_1^{(w)}(\Gamma)$, and to suppose that $\{\gamma \in \Gamma : \zeta(\gamma) \neq 0\}$ can be arranged in a finite sequence of distinct terms $\{\gamma_n\}_{n=1}^N$. Let $g \in L^2(G)$. For $1 \leq k \leq M$, we have

$$T_{\zeta * \phi_k} g = \sum_{n=1}^{N} \zeta(\gamma_n) T_{(\phi_k)_{-\gamma_n}} g, \tag{2.9}$$

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where $(\phi_k)_{-\gamma_n}$ denotes the translate of ϕ_k by $-\gamma_n$.

Fix an index p such that $0 < p < 1$. We now proceed to show that

(2.10) $$\Big\| \max_{1\le k\le M} |T_{\zeta*\phi_k} g| \Big\|_p \le \alpha_p^{-1}(1-p)^{-1/p}\|\zeta\|_1 N_1^{(w)}(\{\phi_k\}_{k=1}^M)\|g\|_1.$$

The Cauchy-Schwarz Inequality implies that for $1 \le k \le M$, we have pointwise on G:

$$\Big|\sum_{n=1}^N \zeta(\gamma_n)T_{(\phi_k)_{-\gamma_n}}g\Big| = \Big|\sum_{n=1}^N \zeta(\gamma_n)\gamma_n T_{\phi_k}(\bar\gamma_n g)\Big| \le \|\zeta\|_1^{1/2}\Big\{\sum_{n=1}^N |\zeta(\gamma_n)|\,|T_{\phi_k}(\bar\gamma_n g)|^2\Big\}^{1/2}.$$

Application of Khintchine's Inequality to the majorant gives:

$$\Big|\sum_{n=1}^N \zeta(\gamma_n)T_{(\phi_k)_{-\gamma_n}}g\Big|^p \le \|\zeta\|_1^{p/2}\alpha_p^{-p}\int_{D^N}\Big|\sum_{n=1}^N \epsilon_n|\zeta(\gamma_n)|^{1/2}T_{\phi_n}(\bar\gamma_n g)\Big|^p\,d\epsilon.$$

Consequently we get pointwise on G:

$$\Big[\max_{1\le k\le M}\Big|\sum_{n=1}^N \zeta(\gamma_n)T_{(\phi_k)_{-\gamma_n}}g\Big|\Big]^p \le \|\zeta\|_1^{p/2}\alpha_p^{-p}\int_{D^N}\Big[\max_{1\le k\le M}\Big|\sum_{n=1}^N \epsilon_n|\zeta(\gamma_n)|^{1/2}T_{\phi_k}(\bar\gamma_n g)\Big|\Big]^p\,d\epsilon.$$

Integrating this inequality over G with respect to normalized Haar measure dm, we easily infer that:

(2.11)
$$\Big\|\max_{1\le k\le M}\Big|\sum_{n=1}^N \zeta(\gamma_n)T_{(\phi_k)_{-\gamma_n}}g\Big|\Big\|_p \le \|\zeta\|_1^{1/2}\alpha_p^{-1}\int_{D^N}\Big\|\max_{1\le k\le M}\Big|T_{\phi_k}\Big(\sum_{n=1}^N \epsilon_n|\zeta(\gamma_n)|^{1/2}\bar\gamma_n g\Big)\Big|\Big\|_p\,d\epsilon.$$

An application of Lemma 2.6 to the majorant in (2.11) shows that:

$$\Big\|\max_{1\le k\le M}\Big|\sum_{n=1}^N \zeta(\gamma_n)T_{(\phi_k)_{-\gamma_n}}g\Big|\Big\|_p$$
$$\le \|\zeta\|_1^{1/2}\alpha_p^{-1}(1-p)^{-1/p}N_1^{(w)}(\{\phi_k\}_{k=1}^M)\int_{D^N}\Big\|\sum_{n=1}^N \epsilon_n|\zeta(\gamma_n)|^{1/2}\bar\gamma_n g\Big\|_1\,d\epsilon.$$

Another application of Khintchine's Inequality gives:

$$\int_{D^N}\Big\|\sum_{n=1}^N \epsilon_n|\zeta(\gamma_n)|^{1/2}\bar\gamma_n g\Big\|_1\,d\epsilon = \int_G\int_{D^N}\Big|\sum_{n=1}^N \epsilon_n|\zeta(\gamma_n)|^{1/2}\bar\gamma_n g\Big|\,d\epsilon\,dm$$
$$\le \int_G\Big\{\sum_{n=1}^N |\zeta(\gamma_n)|\Big\}^{1/2}|g|\,dm = \|\zeta\|_1^{1/2}\|g\|_1.$$

Using this in the majorant of the inequality just preceding, we see from (2.9) that (2.10) is now established.

Since $L^2(G)$ is dense in $L^1(G)$, and $L^p(G)$ is a complete metric space, straightforward application of Lemma 2.4 to 2.10 enables us to deduce that $\{\zeta * \phi_k\}_{k=1}^M \subseteq M_1^{(w)}(\Gamma)$, and

$$N_1^{(w)}(\{\zeta*\phi_k\}_{k=1}^M) \le \Big(\frac{e}{e-2}\Big)^{1/p}\alpha_p^{-2}(1-p)^{-1/p}\|\zeta\|_1 N_1^{(w)}(\{\phi_k\}_{k=1}^M).$$

Since p has been chosen arbitrarily in the range $0 < p < 1$, the proof of Theorem 2.7 is now complete.

3. **Preliminaries for the Bohr Compactification** $b(G)$. Let K be a compact abelian group with dual group D. In this section we establish the left-hand inequality of (1.4) when G has the special form $G = \mathbb{R}^a \times \mathbb{Z}^b \times K$, where a and b are non-negative integers. We shall require the case $p = 1$ of the following maximal theorem from [5] regarding weak type multipliers defined on quotient groups.

THEOREM 3.1 ([5, THEOREM 4.11]). *Let G be an arbitrary locally compact abelian group with dual group Γ, and let Λ be a closed subgroup of Γ. Denote the canonical homomorphism of Γ onto Γ/Λ by π_Λ. Suppose that $\{\psi_j\}_{j=1}^\infty \subseteq C(\Gamma/\Lambda)$, and $1 \le p < \infty$. Then $\{\psi_j \circ \pi_\Lambda\}_{j=1}^\infty \subseteq M_p^{(w)}(\Gamma)$ if and only if $\{\psi_j\}_{j=1}^\infty \subseteq M_p^{(w)}(\Gamma/\Lambda)$. If this is the case, then*

$$N_p^{(w)}(\{\psi_j\}_{j=1}^\infty) = N_p^{(w)}(\{\psi_j \circ \pi_\Lambda\}_{j=1}^\infty).$$

The first step will be to obtain the desired inequality when $b = 0$ (Theorem 3.4). We start by considering a group W having the form $W = \mathbb{R}^n \times b(\mathbb{R}^n) \times K$, where $n \in \mathbb{N}$. The dual group $\mathbb{R}^n \times (\mathbb{R}_d)^n \times D$ of W will be symbolized by X. The result for the case $b = 0$ will be achieved by recourse to the properties of a certain "coordinate switching function" $\delta_0 : X \longrightarrow \mathbb{R}^n \times D$ introduced in Corollary 3.3 below. Let a be a positive real number, and define the closed subgroup Λ of X by writing

$$\Lambda = \{(u, -au, 0) : u \in \mathbb{R}^n\}.$$

Let π_Λ be the canonical homomorphism of X onto X/Λ, and notice that for each $(u, v, d) \in X$, the coset $\pi_\Lambda(u, v, d)$ contains a unique element belonging to $\mathbb{R}^n \times \{0\} \times D$. Moreover this unique element is expressed by $(u + a^{-1}v, 0, d)$. It is easy to see from this that there is a bicontinuous isomorphism θ of $\mathbb{R}^n \times D$ onto X/Λ given by

$$\theta(s, d) = \pi_\Lambda(a^{-1}s, 0, d).$$

Define the continuous homomorphism δ_a of X onto $\mathbb{R}^n \times D$ by writing

$$\delta_a(u, v, d) = (au + v, d).$$

It is easy to see that $\pi_\Lambda = \theta \circ \delta_a$. From this and Theorem 3.1 we immediately infer the following lemma.

LEMMA 3.2. *Suppose that $\{\psi_j\}_{j=1}^\infty \subseteq C(\mathbb{R}^n \times D) \cap M_1^{(w)}(\mathbb{R}^n \times D)$. Then $\{\psi_j \circ \delta_a\}_{j=1}^\infty \subseteq C(X) \cap M_1^{(w)}(X)$, and*

$$N_1^{(w)}(\{\psi_j\}_{j=1}^\infty) = N_1^{(w)}(\{\psi_j \circ \delta_a\}_{j=1}^\infty).$$

COROLLARY 3.3. *Let δ_0 be the continuous homomorphism of X onto $\mathbb{R}^n \times D$ specified by*

$$\delta_0(u, v, d) = (v, d).$$

If $\{\psi_j\}_{j=1}^{\infty} \subseteq C(\mathbb{R}^n \times D) \cap M_1^{(w)}(\mathbb{R}^n \times D)$, *then* $\{\psi_j \circ \delta_0\}_{j=1}^{\infty} \subseteq M_1^{(w)}(X)$, *and*

$$N_1^{(w)}(\{\psi_j \circ \delta_0\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}).$$

PROOF. For $k \in \mathbb{N}$, let $a_k = 1/k$. Applying Lemma 3.2 to a_k, and letting $k \longrightarrow \infty$, we readily obtain the desired result with the aid of Plancherel's Theorem.

THEOREM 3.4. *Let K be a compact abelian group with dual group D, and let $\mathcal{G} = \mathbb{R}^n \times K$, where $n \in \mathbb{N}$. Denote the dual group of $\mathcal{G}$ by $\hat{\mathcal{G}}$, and let $\iota: (\hat{\mathcal{G}})_d \longrightarrow \hat{\mathcal{G}}$ be the identity mapping. Suppose that $\{\psi_j\}_{j=1}^{\infty} \subseteq C(\hat{\mathcal{G}}) \cap M_1^{(w)}(\hat{\mathcal{G}})$. Then $\{\psi_j \circ \iota\}_{j=1}^{\infty} \subseteq M_1^{(w)}\big((\hat{\mathcal{G}})_d\big)$, and*

$$N_1^{(w)}(\{\psi_j \circ \iota\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}).$$

PROOF. Suppose that $f \in L^2\big(b(\mathbb{R}^n) \times K\big)$. Let χ denote the characteristic function, defined on $\mathbb{R}^n$, of $[0,1]^n$, and define $F \in L^2(W) \cap L^1(W)$ by writing

$$F(s,t,w) = \chi(s) f(t,w).$$

By Corollary 3.3 we see that for $N \in \mathbb{N}$, and $y > 0$:

$$\mu_W\{x \in W : \max_{1 \leq j \leq N} |(T_{\psi_j \circ \delta_0} F)(x)| > y\} \leq N_1^{(w)}(\{\psi_j\}_{j=1}^{N}) \|F\|_{L^1(W)} y^{-1}. \tag{3.5}$$

By taking Fourier transforms, it is easy to see from the definition of δ_0 that for each $j \geq 1$, we have for μ_W-almost all $(s,t,w) \in W$:

$$(T_{\psi_j \circ \delta_0} F)(s,t,w) = \chi(s)(T_{\psi_j \circ \iota} f)(t,w).$$

Using this in (3.5), we infer immediately that

$$\mu_{b(\mathbb{R}^n) \times K}\big\{z \in b(\mathbb{R}^n) \times K : \max_{1 \leq j \leq N} |(T_{\psi_j \circ \iota} f)(z)| > y\big\} \leq N_1^{(w)}(\{\psi_j\}_{j=1}^{N}) \|f\|_{L^1(b(\mathbb{R}^n) \times K)} y^{-1}.$$

The remainder of the proof for Theorem 3.4 is evident from this.

THEOREM 3.6. *Let $H = \mathbb{R}^a \times \mathbb{Z}^b \times K$, where a and b are non-negative integers, and K is a compact abelian group. Let $\hat{H}$ be the dual group of H, and let $\iota': (\hat{H})_d \longrightarrow \hat{H}$ be the identity mapping. Suppose that $\{\phi_j\}_{j=1}^{\infty} \subseteq C(\hat{H}) \cap M_1^{(w)}(\hat{H})$. Then $\{\phi_j \circ \iota'\}_{j=1}^{\infty} \subseteq M_1^{(w)}\big((\hat{H})_d\big)$, and*

$$N_1^{(w)}(\{\phi_j \circ \iota'\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\phi_j\}_{j=1}^{\infty}).$$

PROOF. If $b = 0$, then the desired conclusions are either trivial ($a = 0$), or follow from Theorem 3.4 ($a > 0$). So we suppose that $b \in \mathbb{N}$. Put $\mathcal{G} = \mathbb{R}^{a+b} \times K$. We regard H as a closed subgroup of $\mathcal{G}$, and we denote by Λ the annihilator of H in $\hat{\mathcal{G}}$. In this notation, we can identify $\hat{H}$ with $\hat{\mathcal{G}}/\Lambda$. Let π be the canonical homomorphism of $\hat{\mathcal{G}}$ onto $\hat{\mathcal{G}}/\Lambda$, and denote by π' the canonical homomorphism of $(\hat{\mathcal{G}})_d$ onto $(\hat{\mathcal{G}})_d/\Lambda = (\hat{\mathcal{G}}/\Lambda)_d$. Let ι be the identity mapping of $(\hat{\mathcal{G}})_d$ onto $\hat{\mathcal{G}}$, and notice that the mapping ι' in the hypotheses is the identity mapping of $(\hat{\mathcal{G}}/\Lambda)_d$ onto $\hat{\mathcal{G}}/\Lambda$.

By Theorem 3.1, $\{\phi_j \circ \pi\}_{j=1}^{\infty} \subseteq M_1^{(w)}(\hat{\mathcal{G}}) \cap \mathcal{C}(\hat{\mathcal{G}})$, and

$$N_1^{(w)}(\{\phi_j \circ \pi\}_{j=1}^{\infty}) = N_1^{(w)}(\{\phi_j\}_{j=1}^{\infty}). \tag{3.7}$$

Applying Theorem 3.4 to the sequence $\{\phi_j \circ \pi\}_{j=1}^{\infty}$, we see that $\{\phi_j \circ \pi \circ \iota\}_{j=1}^{\infty} \subseteq M_1^{(w)}\big((\hat{\mathcal{G}})_d\big)$, and

$$N_1^{(w)}(\{\phi_j \circ \pi \circ \iota\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\phi_j \circ \pi\}_{j=1}^{\infty}).$$

Clearly $\pi \circ \iota = \iota' \circ \pi'$. So we have just shown that $\{\phi_j \circ \iota' \circ \pi'\}_{j=1}^{\infty} \subseteq M_1^{(w)}\big((\hat{\mathcal{G}})_d\big)$, and

$$N_1^{(w)}(\{\phi_j \circ \iota' \circ \pi'\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\phi_j \circ \pi\}_{j=1}^{\infty}).$$

Using (3.7) in this, we get

$$N_1^{(w)}(\{\phi_j \circ \iota' \circ \pi'\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\phi_j\}_{j=1}^{\infty}). \tag{3.8}$$

Since $\{\phi_j \circ \iota' \circ \pi'\}_{j=1}^{\infty} \subseteq M_1^{(w)}\big((\hat{\mathcal{G}})_d\big)$, we see from an application of Theorem 3.1 to the functions $\{\phi_j \circ \iota'\}_{j=1}^{\infty} \subseteq \mathcal{C}\big((\hat{\mathcal{G}})_d/\Lambda\big)$ that $\{\phi_j \circ \iota'\}_{j=1}^{\infty} \subseteq M_1^{(w)}\big((\hat{\mathcal{G}})_d/\Lambda\big)$, and

$$N_1^{(w)}(\{\phi_j \circ \iota'\}_{j=1}^{\infty}) = N_1^{(w)}(\{\phi_j \circ \iota' \circ \pi'\}_{j=1}^{\infty}).$$

Using this in (3.8) completes the proof of Theorem 3.6.

4. **Proof of Theorem 1.3.** We now return to the setting described at the outset of Section 1. G will be an arbitrary locally compact abelian group, and Γ will denote the dual group of G. We shall denote by $\mathcal{C}_{oo}(\Gamma)$ the algebra consisting of all compactly supported functions belonging to $\mathcal{C}(\Gamma)$. Let H be a closed subgroup of G such that H is open in G, and H is topologically isomorphic to a group $\mathbb{R}^a \times \mathbb{Z}^b \times K$, where a and b are non-negative integers, and K is a compact abelian group. Since H is open we can take as the Haar measure of H the restriction to H of any Haar measure for G. In this section, Λ will denote the annihilator in Γ of H, and π_Λ will be the canonical homomorphism of Γ onto Γ/Λ. Since H is open, G/H is discrete, and hence $(G/H)^\wedge = \Lambda$ is compact. Consequently we can extend the normalized Haar measure μ_Λ of Λ from the Borel sets of Λ to a regular Borel measure λ_0 on Γ by putting $\lambda_0(A) = \mu_\Lambda(A \cap \Lambda)$ for every Borel set A of Γ.

THEOREM 4.1. *In terms of the foregoing notation, suppose that* $\{\psi_j\}_{j=1}^{\infty} \subseteq \mathcal{C}_{oo}(\Gamma) \cap M_1^{(w)}(\Gamma)$. *Then*

$$\{\psi_j * \lambda_0\}_{j=1}^{\infty} \subseteq \mathcal{C}_{oo}(\Gamma) \cap M_1^{(w)}(\Gamma),$$

and

$$N_1^{(w)}(\{\psi_j * \lambda_0\}_{j=1}^{\infty}) \leq N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}).$$

PROOF. It will be convenient at the outset to select compatible Haar measures on various groups needed in the demonstration. We choose Haar measures μ_G and μ_Γ which are normalized for Fourier inversion. Our choice of μ_Γ together with the prior choice of μ_Λ automatically fixes the choice of $\mu_{\Gamma/\Lambda}$ for the operation of the Weil Formula [11, Theorem 28.54]. We then take μ_H to be normalized for Fourier inversion with respect to this choice of $\mu_{\Gamma/\Lambda}$. It is not difficult to see that the Haar measures μ_G and μ_H standardized in this way coincide on H.

Temporarily fix $j \in \mathbb{N}$. Since $\psi_j \in \mathcal{C}_{oo}(\Gamma)$, the conclusion that $\psi_j * \lambda_0 \in \mathcal{C}_{oo}(\Gamma)$ is immediate from the relation

$$(\psi_j * \lambda_0)(y) = \int_\Lambda \psi_j(y-t)\, d\mu_\Lambda(t), \quad \text{for all } y \in \Gamma.$$

It is easy to see directly that $\hat{\lambda}_0$ coincides with χ_H, the characteristic function, defined on G, of H. Consequently, if $\check{}$ denotes the L^2-inverse Fourier transform, then we can take $\check{\psi}_j$ and $(\psi_j * \lambda_0)\check{}$ to be continuous functions on G satisfying:

$$(\psi_j * \lambda_0)\check{}\,(x) = \chi_H(x)\check{\psi}_j(x), \quad \text{for all } x \in G. \tag{4.2}$$

Next we observe that since $\psi_j * \lambda_0$ is constant on the cosets of Λ, and belongs to $\mathcal{C}_{oo}(\Gamma)$, there is a function $\psi_j' \in \mathcal{C}_{oo}(\Gamma/\Lambda)$ such that

$$\psi_j * \lambda_0 = \psi_j' \circ \pi_\Lambda \quad \text{on } \Gamma. \tag{4.3}$$

If $x \in H$, then, with the aid of the Weil Formula, we see from (4.3) that:

$$\begin{aligned}
(\psi_j * \lambda_0)\check{}\,(x) &= \int_\Gamma (\psi_j * \lambda_0)(y)\langle x, y\rangle\, d\mu_\Gamma(y) \\
&= \int_{\Gamma/\Lambda}\int_\Lambda (\psi_j' \circ \pi_\Lambda)(\gamma + t)\langle x, \gamma + t\rangle\, d\mu_\Lambda(t)\, d\mu_{\Gamma/\Lambda}(\gamma + \Lambda) \\
&= \int_{\Gamma/\Lambda} \psi_j'(\gamma + \Lambda)\langle x, \gamma + \Lambda\rangle\, d\mu_{\Gamma/\Lambda}(\gamma + \Lambda) \\
&= (\psi_j')\check{}\,(x).
\end{aligned}$$

This together with (4.2) gives us:

$$(\psi_j')\check{}\,(x) = (\psi_j * \lambda_0)\check{}\,(x) = \check{\psi}_j(x), \quad \text{for all } x \in H. \tag{4.4}$$

Next suppose that $f \in L^1(H) \cap L^2(H)$, and extend f to a function $F \in L^1(G) \cap L^2(G)$ by taking F to vanish on the complement of H in G. For $j \in \mathbb{N}$ and $x \in H$, we see with the aid of (4.4) that:

$$\begin{aligned}
(T_{\psi_j'} f)(x) &= \int_H f(x-t)(\psi_j')\check{}\,(t)\, d\mu_H(t) \\
&= \int_G F(x-t)\check{\psi}_j(t)\, d\mu_G(t) \\
&= (T_{\psi_j} F)(x).
\end{aligned}$$

Hence for $N \in \mathbb{N}$, and $y > 0$:

$$\begin{aligned}\mu_H\{x \in H : \max_{1\le j\le N} |(T_{\psi_j'} f)(x)| > y\} &= \mu_G\{x \in H : \max_{1\le j\le N} |(T_{\psi_j} F)(x)| > y\} \\ &\le \mu_G\{x \in G : \max_{1\le j\le N} |(T_{\psi_j} F)(x)| > y\} \\ &\le N_1^{(w)}(\{\psi_j\}_{j=1}^N) y^{-1} \|F\|_{L^1(G)} \\ &= N_1^{(w)}(\{\psi_j\}_{j=1}^N) y^{-1} \|f\|_{L^1(H)}.\end{aligned}$$

It follows that $\{\psi_j'\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma/\Lambda)$, with

$$N_1^{(w)}(\{\psi_j'\}_{j=1}^\infty) \le N_1^{(w)}(\{\psi_j\}_{j=1}^\infty). \tag{4.5}$$

The proof of Theorem 4.1 is now readily completed by applying Theorem 3.1 to $\{\psi_j'\}_{j=1}^\infty$ and taking (4.3) into account.

COROLLARY 4.6. *Under the hypotheses of Theorem 4.1 we have that* $\{(\psi_j * \lambda_0) \circ \iota\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma_d)$, *and*

$$N_1^{(w)}\big(\{(\psi_j * \lambda_0) \circ \iota\}_{j=1}^\infty\big) \le N_1^{(w)}(\{\psi_j\}_{j=1}^\infty),$$

where $\iota \colon \Gamma_d \longrightarrow \Gamma$ *is the identity mapping.*

PROOF. Let π' be the canonical homomorphism of Γ_d onto $\Gamma_d/\Lambda = (\Gamma/\Lambda)_d$, and let $\iota' \colon (\Gamma/\Lambda)_d \longrightarrow \Gamma/\Lambda$ be the identity mapping. By (4.5) and Theorem 3.6, we see that $\{\psi_j' \circ \iota'\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma_d/\Lambda)$, with

$$N_1^{(w)}(\{\psi_j' \circ \iota'\}_{j=1}^\infty) \le N_1^{(w)}(\{\psi_j'\}_{j=1}^\infty) \le N_1^{(w)}(\{\psi_j\}_{j=1}^\infty). \tag{4.7}$$

Applying Theorem 3.1 to $\{\psi_j' \circ \iota'\}_{j=1}^\infty$, we find that $\{\psi_j' \circ \iota' \circ \pi'\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma_d)$, and

$$N_1^{(w)}(\{\psi_j' \circ \iota' \circ \pi'\}_{j=1}^\infty) = N_1^{(w)}(\{\psi_j' \circ \iota'\}_{j=1}^\infty).$$

Observe, with the aid of (4.3), that for each $j \in \mathbb{N}$,

$$\psi_j' \circ \iota' \circ \pi' = \psi_j' \circ \pi_\Lambda \circ \iota = (\psi_j * \lambda_0) \circ \iota,$$

and so we have $\{(\psi_j * \lambda_0) \circ \iota\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma_d)$, with

$$N_1^{(w)}\big(\{(\psi_j * \lambda_0) \circ \iota\}_{j=1}^\infty\big) = N_1^{(w)}(\{\psi_j' \circ \iota'\}_{j=1}^\infty).$$

The proof of Corollary 4.6 is now completed by using (4.7) in this.

With the prior notation of this section remaining in effect, we now consider an arbitrary function $\phi \in \mathcal{C}_{oo}(\Gamma)$, and we let $\check{\phi}$ be the L^2-inverse Fourier transform of ϕ: $\check{\phi}(t) = \int_\Gamma \phi(y)\langle t, y\rangle \, d\mu_\Gamma(y)$, for all $t \in G$. From [10, Theorem 5.14] we obtain an increasing sequence $\{H_n\}_{n=1}^\infty$ of compactly generated open and closed subgroups of G such that $\check{\phi}$ vanishes on the complement in G of $\bigcup_{n=1}^\infty H_n$. By [10, Theorem 9.8], we see that for each n, H_n is topologically isomorphic with a group $\mathbb{R}^{a_n} \times \mathbb{Z}^{b_n} \times K_n$, where a_n and

b_n are non-negative integers, and K_n is a compact abelian group. Thus, for each n, H_n is a subgroup of G fulfilling the requirements listed at the outset of this section. Now let Λ_n be the annihilator in Γ of H_n. In particular, $\{\Lambda_n\}_{n=1}^{\infty}$ is a decreasing sequence of compact subgroups of Γ. We further define the compact subgroup Λ_0 of Γ by writing $\Lambda_0 = \bigcap_{n=1}^{\infty} \Lambda_n$. Suppose now that $\gamma \in \Lambda_0$, and denote by $\phi_\gamma \in \mathcal{C}_{oo}(\Gamma)$ the corresponding translate of ϕ: $\phi_\gamma(y) \equiv \phi(y+\gamma)$, $y \in \Gamma$. For each $t \in G$,

$$(\phi_\gamma)\check{}\,(t) = \int_\Gamma \phi(y+\gamma)\langle t, y\rangle \, d\mu_\Gamma(y) = \overline{\langle t, \gamma\rangle}\check{\phi}(t).$$

Since $\gamma = 1$ on $\bigcup_{n=1}^{\infty} H_n$, it follows that $(\phi_\gamma)\check{} = \check{\phi}$ on G, and consequently $\phi(y+\gamma) = \phi(y)$, for all $y \in \Gamma$, and all $\gamma \in \Lambda_0$. Hence there is $\phi' \in \mathcal{C}_{oo}(\Gamma/\Lambda_0)$ such that

$$\phi = \phi' \circ \pi_{\Lambda_0}, \tag{4.8}$$

where π_{Λ_0} is the canonical homomorphism of Γ onto Γ/Λ_0.

For each $n \in \mathbb{N}$, let μ_{Λ_n} be the normalized Haar measure of Λ_n, and, following the procedure described at the beginning of this section, let the regular Borel measure λ_n on Γ be the extension of μ_{Λ_n} to the Borel sets of Γ. Similarly, we let τ_n be the regular Borel measure on Γ/Λ_0 obtained by extending the normalized Haar measure of the compact subgroup $\pi_{\Lambda_0}(\Lambda_n)$. The uniqueness properties for Haar measure [10, (15.8)] readily show that

$$\tau_n(A) = \lambda_n\big(\pi_{\Lambda_0}^{-1}(A)\big), \quad \text{for all Borel subsets } A \text{ of } \Gamma/\Lambda_0. \tag{4.9}$$

We claim that $\phi * \lambda_n \longrightarrow \phi$ uniformly on Γ. This claim, which can be viewed as a simple variant of the martingale theorem in [11, Section 44], is established as follows. It follows directly from (4.9) that for each Borel measurable simple function f on Γ/Λ_0, we have

$$\int_{\Gamma/\Lambda_0} f \, d\tau_n = \int_\Gamma (f \circ \pi_{\Lambda_0}) \, d\lambda_n,$$

and consequently for each $y \in \Gamma$,

$$(\phi' * \tau_n)\big(\pi_{\Lambda_0}(y)\big) = (\phi * \lambda_n)(y). \tag{4.10}$$

Hence, by taking account of (4.8), we see that in order to have $\phi * \lambda_n \longrightarrow \phi$ uniformly on Γ, it is enough to show:

$$\phi' * \tau_n \longrightarrow \phi' \quad \text{uniformly on } \Gamma/\Lambda_0. \tag{4.11}$$

Since $\{\pi_{\Lambda_0}(\Lambda_n)\}_{n=1}^{\infty}$ is a decreasing sequence of compact subgroups of Γ/Λ_0 such that $\bigcap_{n=1}^{\infty} \pi_{\Lambda_0}(\Lambda_n) = \{0\}$, it is clear that for each open neighborhood U of the identity in Γ/Λ_0, we have $\pi_{\Lambda_0}(\Lambda_n) \subseteq U$ for all sufficiently large n. Applying this, and the uniform continuity of ϕ' on Γ/Λ_0, to the identity

$$(\phi' * \tau_n)(y) - \phi'(y) = \int_{\pi_{\Lambda_0}(\Lambda_n)} \big(\phi'(y-s) - \phi'(y)\big) \, d\tau_n(s), \quad \text{for all } y \in \Gamma/\Lambda_0,$$

we see that (4.11) holds. Hence the claim is established.

The foregoing observations can be summarized in the following lemma.

LEMMA 4.12. *Suppose that* $N \in \mathbb{N}$, *and* $\{\phi_j\}_{j=1}^N \subseteq C_{oo}(\Gamma)$. *Then there is an increasing sequence* $\{H_n\}_{n=1}^\infty$ *of compactly generated open and closed subgroups of* G *such that for* $1 \le j \le N$, *the continuous function* $\check{\phi}_j$ *vanishes on the complement in* G *of* $\bigcup_{n=1}^\infty H_n$. *For* $n \in \mathbb{N}$, *let* Λ_n *be the annihilator in* Γ *of* H_n, *and let* λ_n *be the regular Borel measure on* Γ *obtained by extending as described above the normalized Haar measure* μ_{Λ_n} *for the compact group* Λ_n. *For* $1 \le j \le N$, *we have:*

$$\phi_j * \lambda_n \longrightarrow \phi_j \text{ uniformly on } \Gamma, \quad \text{as } n \longrightarrow \infty. \tag{4.13}$$

Having set the stage with the foregoing considerations of this section, we now pass to the proof of the "only if" assertion in Theorem 1.3 together with the left-hand inequality in (1.4). For this purpose, we suppose that $\{\psi_j\}_{j=1}^\infty \subseteq C(\Gamma) \cap M_1^{(w)}(\Gamma)$. By [5, Lemma 4.1], we can assume without loss of generality that $\{\psi_j\}_{j=1}^\infty \subseteq C_{oo}(\Gamma)$. Let $N \in \mathbb{N}$, and apply Lemma 4.12 (with its notation) to $\{\psi_j\}_{j=1}^N$. For each $n \in \mathbb{N}$, we see from Corollary 4.6 that $\{(\psi_j * \lambda_n) \circ \iota\}_{j=1}^N \subseteq M_1^{(w)}(\Gamma_d)$, and

$$N_1^{(w)}\big(\{(\psi_j * \lambda_n) \circ \iota\}_{j=1}^N\big) \le N_1^{(w)}(\{\psi_j\}_{j=1}^N).$$

With the aid of (4.13), we can let $n \longrightarrow \infty$ in this, and then let $N \longrightarrow \infty$ to obtain the desired conclusions.

In order to complete the proof of Theorem 1.3, and to demonstrate Theorem 1.1 in the next section, we shall require the following technical lemma and corollary thereof. Henceforth we shall denote convolution on Γ_d by the symbol #. Given subsets A and B of Γ, we shall write $A - B$ to indicate the group-theoretic difference set $\{a - b : a \in A, b \in B\}$.

LEMMA 4.14. *Suppose that* G *is a locally compact abelian group with dual group* Γ, $k \in L^1(\Gamma)$, $N \in \mathbb{N}$, $\{\phi_j\}_{j=1}^N \subseteq C_{oo}(\Gamma)$, *and* $\epsilon > 0$. *Then there is a complex-valued function* k_ϵ *defined on* Γ_d *such that:* k_ϵ *vanishes outside a finite set:*

$$\|k_\epsilon\|_{L^1(\Gamma_d)} \le \|k\|_{L^1(\Gamma)}; \quad \text{and, for } 1 \le j \le N,\ \gamma \in \Gamma,$$
$$\Big|(k * \phi_j)(\gamma) - \big(k_\epsilon \,\#\, (\phi_j \circ \iota)\big)(\gamma)\Big| < \epsilon.$$

PROOF. Clearly we can choose $M \in \mathbb{N}$, complex numbers $\{\alpha_s\}_{s=1}^M$, and disjoint compact subsets $\{F_s\}_{s=1}^M$ of Γ (with corresponding characteristic functions denoted by χ_{F_s}) so that $\mu_\Gamma(F_s) > 0$ for $1 \le s \le M$, and so that the simple function f defined on Γ by

$$f = \sum_{s=1}^M \alpha_s \chi_{F_s} \tag{4.15}$$

satisfies $|f| \le |k|$ pointwise on Γ, and $\|f - k\|_{L^1(\Gamma)} < \epsilon$. Let U be a symmetric open neighborhood of the identity element in Γ such that whenever $u \in \Gamma$, $v \in \Gamma$, and $(u - v) \in$

U, then $|\phi_j(u) - \phi_j(v)| < \epsilon$ for $1 \leq j \leq N$. Choose a symmetric open neighborhood V of the identity element in Γ such that $V + V \subseteq U$. Since $F_s \subseteq \bigcup_{x \in F_s} x + V$, it follows by compactness that we can express F_s as the union of a finite family $\{W_{s,t}\}_{t \geq 1}$ of non-void disjoint Borel subsets of Γ with the property that for each $t \geq 1$, $W_{s,t} - W_{s,t} \subseteq V + V$. Using this in (4.15) for $1 \leq s \leq M$, we see that f can be represented on Γ in the form

$$f = \sum_{\nu=1}^{L} \beta_\nu \chi_{S_\nu},$$

where $L \in \mathbb{N}$, β_ν, $1 \leq \nu \leq L$, is a complex number, and $\{S_\nu\}_{\nu=1}^{L}$ is a family of non-void disjoint Borel sets such that $\mu_\Gamma(S_\nu) < \infty$, $S_\nu - S_\nu \subseteq V + V$, for $1 \leq \nu \leq L$.

For $1 \leq j \leq N$, and $\gamma \in \Gamma$, we have

$$|(k * \phi_j)(\gamma) - (f * \phi_j)(\gamma)| \leq \|k - f\|_{L^1(\Gamma)} \|\phi_j\|_u \leq \epsilon \|\phi_j\|_u. \tag{4.16}$$

For $1 \leq \nu \leq L$, pick $\eta_\nu \in S_\nu$, and define k_ϵ on Γ_d by:

$$k_\epsilon = \sum_{\nu=1}^{L} \beta_\nu \mu_\Gamma(S_\nu) \chi_{\{\eta_\nu\}}.$$

Clearly k_ϵ vanishes outside a finite set, and

$$\|k_\epsilon\|_{L^1(\Gamma_d)} = \|f\|_{L^1(\Gamma)} \leq \|k\|_{L^1(\Gamma)}.$$

For $1 \leq j \leq N$, $\gamma \in \Gamma$, we have:

$$\begin{aligned}
&\Big|(f * \phi_j)(\gamma) - \big(k_\epsilon \,\#\, (\phi_j \circ \iota)\big)(\gamma)\Big| \\
&\qquad = \Big|\sum_{\nu=1}^{L} \beta_\nu \int_\Gamma \chi_{S_\nu}(u)\, \phi_j(\gamma - u)\, d\mu_\Gamma(u) - \sum_{\nu=1}^{L} \beta_\nu \mu_\Gamma(S_\nu)\, \phi_j(\gamma - \eta_\nu)\Big| \\
&\qquad = \Big|\sum_{\nu=1}^{L} \beta_\nu \int_{S_\nu} [\phi_j(\gamma - u) - \phi_j(\gamma - \eta_\nu)]\, d\mu_\Gamma(u)\Big| \\
&\qquad \leq \sum_{\nu=1}^{L} |\beta_\nu| \mu_\Gamma(S_\nu) \epsilon \\
&\qquad = \epsilon \|f\|_{L^1(\Gamma)} \\
&\qquad \leq \epsilon \|k\|_{L^1(\Gamma)}.
\end{aligned}$$

Combining this with (4.16), we see that for $1 \leq j \leq N$, and all $\gamma \in \Gamma$,

$$\Big|(k * \phi_j)(\gamma) - \big(k_\epsilon \,\#\, (\phi_j \circ \iota)\big)(\gamma)\Big| \leq \epsilon\Big(\|k\|_{L^1(\Gamma)} + \max_{1 \leq j \leq N} \|\phi_j\|_u\Big).$$

This suffices to complete the proof of Lemma 4.14.

COROLLARY 4.17. *Suppose that G is a locally compact abelian group with dual group Γ, $k \in L^1(\Gamma)$, $N \in \mathbb{N}$, $\{\phi_j\}_{j=1}^{N} \subseteq \mathcal{C}_{oo}(\Gamma)$, and $\{\phi_j \circ \iota\}_{j=1}^{N} \subseteq M_1^{(w)}(\Gamma_d)$. Then $\{(k * \phi_j) \circ \iota\}_{j=1}^{N} \subseteq M_1^{(w)}(\Gamma_d)$, and*

$$N_1^{(w)}\big(\{(k * \phi_j) \circ \iota\}_{j=1}^{N}\big) \leq C \|k\|_{L^1(\Gamma)} N_1^{(w)}(\{\phi_j \circ \iota\}_{j=1}^{N}),$$

where C is the absolute constant in Theorem 2.7.

PROOF. For each $n \in \mathbb{N}$, let $\epsilon_n = n^{-1}$, and choose the function k_{ϵ_n} on Γ_d for k, $\{\phi_j\}_{j=1}^N$, and ϵ_n in accordance with Lemma 4.14. Then by Theorem 2.7 applied to k_{ϵ_n} and the functions $\phi_j \circ \iota$, we have for each $n \in \mathbb{N}$,

$$N_1^{(w)}\big(\{k_{\epsilon_n} \# (\phi_j \circ \iota)\}_{j=1}^N\big) \leq C\|k\|_{L^1(\Gamma)} N_1^{(w)}(\{\phi_j \circ \iota\}_{j=1}^N). \tag{4.18}$$

It is clear from the last conclusion of Lemma 4.14 that for $1 \leq j \leq N$, $k_{\epsilon_n} \# (\phi_j \circ \iota) \longrightarrow (k * \phi_j) \circ \iota$ uniformly on Γ_d, as $n \longrightarrow \infty$. Consequently the proof of Corollary 4.17 can be completed by letting $n \longrightarrow \infty$ in (4.18).

In order to avoid digressions later on, it will be convenient to collect in the following scholium some standard facts about approximate identities for the group algebra. The symbol $\setminus$ will denote set-theoretic complementation: $A \setminus B = \{a \in A : a \notin B\}$.

SCHOLIUM 4.19 ([11, THEOREMS 28.52 AND 33.12)]. *Let Y be a locally compact abelian group. The algebra $L^1(Y)$ has an approximate identity $\{u_\delta\}_{\delta \in \Delta}$ such that: (i) for each $\delta \in \Delta$, $u_\delta \geq 0$, $\int_Y u_\delta \, d\mu_Y = 1$, and $\hat{u}_\delta$ is compactly supported; (ii) for every open neighborhood W of the identity in Y, $\int_{Y \setminus W} u_\delta \, d\mu_Y \longrightarrow 0$ as δ runs through Δ; (iii) $\hat{u}_\delta \longrightarrow 1$ uniformly on compact subsets of the dual group of Y.*

From these properties it is easy to see that if $\theta \in \mathcal{C}(Y)$, then $u_\delta * \theta \longrightarrow \theta$ uniformly on compact subsets of Y. Moreover, if $\theta \in \mathcal{C}_{oo}(Y)$, then $u_\delta * \theta \longrightarrow \theta$ uniformly on Y.

Having attended to these preliminaries, we can now complete the proof of Theorem 1.3. For this purpose, suppose that $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{C}(\Gamma)$, and $\{\psi_j \circ \iota\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma_d)$. We wish to establish the right-hand member of (1.4). Consider first the special case in which $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{C}_{oo}(\Gamma)$, let $N \in \mathbb{N}$, and choose an approximate identity $\{k_\delta\}_{\delta \in \Delta}$ for $L^1(\Gamma)$ enjoying the properties listed in Scholium 4.19. For each $\delta \in \Delta$, it follows from Corollary 4.17 that $\{(k_\delta * \psi_j) \circ \iota\}_{j=1}^N \subseteq M_1^{(w)}(\Gamma_d)$, and

$$N_1^{(w)}\big(\{(k_\delta * \psi_j) \circ \iota\}_{j=1}^N\big) \leq C N_1^{(w)}(\{\psi_j \circ \iota\}_{j=1}^N). \tag{4.20}$$

Since $\hat{k}_\delta$ is compactly supported, it follows by Fourier inversion that $k_\delta * \psi_j$ is the Fourier transform of a function in $L^1(G)$, for $1 \leq j \leq N$, $\delta \in \Delta$. In particular, $(k_\delta * \psi_j)$ is a multiplier of strong type $(1,1)$, and it follows from this by [5, Theorem 5.1] that for $\delta \in \Delta$:

$$N_1^{(w)}(\{k_\delta * \psi_j\}_{j=1}^N) = N_1^{(w)}\big(\{(k_\delta * \psi_j) \circ \iota\}_{j=1}^N\big).$$

Using this in (4.20), we get for each $\delta \in \Delta$,

$$N_1^{(w)}(\{k_\delta * \psi_j\}_{j=1}^N) \leq C N_1^{(w)}(\{\psi_j \circ \iota\}_{j=1}^N). \tag{4.21}$$

For $1 \leq j \leq N$, $k_\delta * \psi_j \longrightarrow \psi_j$ uniformly on Γ. We can apply this fact to (4.21), and then let $N \longrightarrow \infty$, in order to obtain the desired conclusions of Theorem 1.3 in the special case where $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{C}_{oo}(\Gamma)$.

In the general case, we let $\{u_\alpha\}_{\alpha\in Q}$ be an approximate identity for $L^1(G)$ possessing the properties described in Scholium 4.19. For $N \in \mathbb{N}$, and each $\alpha \in Q$, we can apply the preceding special case to $\{\hat{u}_\alpha\psi_j\}_{j=1}^N$, and thereby infer that $\{\hat{u}_\alpha\psi_j\}_{j=1}^N \subseteq M_1^{(w)}(\Gamma)$, and

$$\begin{aligned} N_1^{(w)}(\{\hat{u}_\alpha\psi_j\}_{j=1}^N) &\le CN_1^{(w)}(\{\hat{u}_\alpha\psi_j)\circ\iota\}_{j=1}^N) \\ &\le CN_1^{(w)}(\{\psi_j\circ\iota\}_{j=1}^N). \end{aligned}$$

The proof of Theorem 1.3 is readily completed from this by taking into account the fact that $\hat{u}_\alpha \longrightarrow 1$ uniformly on compact subsets of Γ.

5. **Proof of the Convolution Theorem 1.1.** Our task in this section will be simplified by establishing the reduction described in the following lemma.

LEMMA 5.1. *Let G be a locally compact abelian group with dual group Γ, and suppose that A is a real constant such that whenever $k \in L^1(\Gamma)$ and $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{C}_{oo}(\gamma)\cap M_1^{(w)}(\Gamma)$, we have $\{k*\psi_j\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma)$, and*

$$N_1^{(w)}(\{k*\psi_j\}_{j=1}^\infty) \le A\|k\|_{L^1(\Gamma)}N_1^{(w)}(\{\psi_j\}_{j=1}^\infty). \tag{5.2}$$

Then whenever $k \in L^1(\Gamma)$, and $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{C}(\Gamma)\cap M_1^{(w)}(\Gamma)$, we have $\{k*\psi_j\}_{j=1}^\infty \subseteq M_1^{(w)}(\Gamma)$, and the inequality in (5.2) continues to hold with the same constant A.

PROOF. Let $f \in L^1(G)\cap L^2(G)$, and let $n \in \mathbb{N}$. Let K be a compact subset of Γ such that $\int_{\Gamma\setminus K}|\hat{f}|^2\,d\mu_\Gamma < n^{-2}$, and $\int_{\Gamma\setminus K}|k|\,d\mu_\Gamma < n^{-1}$. Pick $g_n \in L^1(G)$ such that: $\hat{g}_n$ is compactly supported, $\hat{g}_n = 1$ on $K-K$, and $\|g_n\|_{L^1(G)} < 1+n^{-1}$ [11, Theorem 31.37].

Suppose now that $\{\psi_j\}_{j=1}^\infty \subseteq \mathcal{C}(\Gamma)\cap M_1^{(w)}(\Gamma)$, and $N \in \mathbb{N}$. Clearly, for $1 \le j \le N$, we have:

$$\begin{aligned} \int_{\Gamma\setminus K}|k*(\hat{g}_n\psi_j)-k*\psi_j|^2\,|\hat{f}|^2\,d\mu_\Gamma &\le \|k\|_1^2(2+n^{-1})^2\|\psi_j\|_u^2\int_{\Gamma\setminus K}|\hat{f}|^2\,d\mu_\Gamma \\ &\le \|k\|_1^2(2+n^{-1})^2\|\psi_j\|_u^2 n^{-2}. \end{aligned}\tag{5.3}$$

For $\gamma \in K$, and $t \in K$, we have $(\gamma - t) \in K-K$, and so $\hat{g}_n(\gamma-t) = 1$. Hence for $\gamma \in K$, and $1 \le j \le N$,

$$[k*(\hat{g}_n\psi_j)-k*\psi_j](\gamma) = \int_{\Gamma\setminus K}k(t)[\hat{g}_n(\gamma-t)\,\psi_j(\gamma-t)-\psi_j(\gamma-t)]\,d\mu_\Gamma(t),$$

and consequently

$$\begin{aligned} |[k*(\hat{g}_n\psi_j)-k*\psi_j](\gamma)| &\le (2+n^{-1})\|\psi_j\|_u\int_{\Gamma\setminus K}|k|\,d\mu_\Gamma \\ &\le (2+n^{-1})\|\psi_j\|_u n^{-1}. \end{aligned}$$

It follows that for $1 \le j \le N$, we have:

$$\int_K |k*(\hat{g}_n\psi_j)-k*\psi_j|^2\,|\hat{f}|^2\,d\mu_\Gamma \le (2+n^{-1})^2\|\psi_j\|_u^2 n^{-2}\|f\|_{L^2(G)}^2.$$

Combining this with (5.3), we obtain for $1 \leq j \leq N$,

$$\|[k*(\hat{g}_n\psi_j) - k*\psi_j]\hat{f}\|^2_{L^2(\Gamma)} \leq [\|k\|_1(2+n^{-1})\|\psi_j\|_u n^{-1}]^2 + [(2+n^{-1})\|\psi_j\|_u n^{-1}\|f\|_{L^2(G)}]^2.$$

From this we infer with the aid of Plancherel's Theorem that, as $n \longrightarrow \infty$,

$$(5.4) \qquad \max_{1\leq j\leq N} |T_{k*(\hat{g}_n\psi_j)}f| \longrightarrow \max_{1\leq j\leq N} |T_{k*\psi_j}f|,$$

in the norm topology of $L^2(G)$. If we now apply the lemma's assumptions for the case of compact supports to

$$\{\hat{g}_n\psi_j\}_{j=1}^N \subseteq C_{oo}(\Gamma) \cap M_1^{(w)}(\Gamma),$$

and observe that $N_1^{(w)}(\{\hat{g}_n\psi_j\}_{j=1}^N) \leq (1+n^{-1})N_1^{(w)}(\{\psi_j\}_{j=1}^N)$, then we deduce from (5.4) that $\{k*\psi_j\}_{j=1}^N \subseteq M_1^{(w)}(\Gamma)$, and

$$N_1^{(w)}(\{k*\psi_j\}_{j=1}^N) \leq A\|k\|_{L^1(\Gamma)}N_1^{(w)}(\{\psi_j\}_{j=1}^N).$$

Since N is arbitrary, the conclusions of Lemma 5.1 are now evident.

PROOF OF THEOREM 1.1. In view of Lemma 5.1, it is enough to show that if $N \in \mathbb{N}$, $\{\psi_j\}_{j=1}^N \subseteq C_{oo}(\Gamma) \cap M_1^{(w)}(\Gamma)$, $k \in L^1(\Gamma)$, and $\hat{k}$ has compact support, then $\{k*\psi_j\}_{j=1}^N \subseteq M_1^{(w)}(\Gamma)$, and

$$(5.5) \qquad N_1^{(w)}(\{k*\psi_j\}_{j=1}^N) \leq C\|k\|_{L^1(\Gamma)}N_1^{(w)}(\{\psi_j\}_{j=1}^N).$$

By Fourier inversion, $k*\psi_j$, $1 \leq j \leq N$, is the Fourier transform of a function belonging to $L^1(G)$, and hence is a multiplier of strong type $(1,1)$. It follows from this and [5, Theorem 5.1] that

$$(5.6) \qquad N_1^{(w)}\big(\{(k*\psi_j)\circ\iota\}_{j=1}^N\big) = N_1^{(w)}(\{k*\psi_j\}_{j=1}^N),$$

where, as previously $\iota : \Gamma_d \longrightarrow \Gamma$ is the identity mapping.

By Theorem 1.3, $\{\psi_j \circ \iota\}_{j=1}^N \subseteq M_1^{(w)}(\Gamma_d)$ and

$$(5.7) \qquad N_1^{(w)}(\{\psi_j\circ\iota\}_{j=1}^N) \leq N_1^{(w)}(\{\psi_j\}_{j=1}^N).$$

It follows by Corollary 4.17 that $\{(k*\psi_j)\circ\iota\}_{j=1}^N \subseteq M_1^{(w)}(\Gamma_d)$, with

$$N_1^{(w)}\big(\{(k*\psi_j)\circ\iota\}_{j=1}^N\big) \leq C\|k\|_{L^1(\Gamma)}N_1^{(w)}(\{\psi_j\circ\iota\}_{j=1}^N).$$

Applying (5.6) and (5.7) to this, we obtain (5.5), and thereby complete the proof of Theorem 1.1.

6. **Maximal estimates of weak type** $(1,1)$ **on measure spaces.** In [4, Theorem 2.6] it was shown that for $1 < p < \infty$ "distributionally controlled" representations of the locally compact abelian group G transfer elements of $\mathcal{C}(\Gamma) \cap M_p^{(w)}(\Gamma)$ and their associated maximal estimates to $L^p(\Omega, \sigma)$, where (Ω, σ) is an arbitrary measure space. In this section we utilize Theorem 1.1 to extend these results to the case $p = 1$. A strongly continuous representation $u \longrightarrow R_u$ of G in $L^1(\sigma)$ is said to be σ-distributionally controlled ([3]) provided the following requirements are satisfied:

(6.1) R is *separation-preserving* on $L^1(\sigma)$ (that is, whenever $f \in L^1(\sigma)$, $g \in L^1(\sigma)$, and $fg = 0$ σ-a.e., then, for all $u \in G$, we have $(R_u f)(R_u g) = 0$ σ-a.e.);

(6.2) there is a positive real constant K_1 such that $\|R_u f\|_{L^1(\sigma)} \leq K_1 \|f\|_{L^1(\sigma)}$, for all $u \in G$, and all $f \in L^1(\sigma)$;

(6.3) there is a positive real constant K_∞ such that $\|R_u f\|_{L^\infty(\sigma)} \leq K_\infty \|f\|_{L^\infty(\sigma)}$, for all $u \in G$, and all $f \in L^1(\sigma) \cap L^\infty(\sigma)$.

The terminology "distributional control" stems from the fact that the foregoing definition implies:

$$\sigma\{x \in \Omega : |(R_u f)(x)| > y\} \leq (K_1 K_\infty)\sigma\{x \in \Omega : |f(x)| > y/K_\infty\},$$
$$\text{for all } u \in G,\ f \in L^1(\sigma), \text{ and } y \text{ satisfying } 0 < y < +\infty$$

(see [3, (2.3)-(ii), (2.13)-(i)]).

By [3, (2.2), (2.4), and (2.6)], there is a (necessarily unique) uniformly bounded strongly continuous representation $R^{(2)}$ of G in $L^2(\sigma)$ such that for all $f \in L^2(\sigma) \cap L^1(\sigma)$ and all $u \in G$, $R_u^{(2)} f = R_u f$. Since $R^{(2)}$ is similar to a unitary representation of G in $L^2(\sigma)$ [8, Theorem 8.1], we can apply Stone's Theorem to infer that there is a unique regular Borel spectral measure $\mathcal{E}(\cdot)$ on the dual group Γ of G which acts on $L^2(\sigma)$ and satisfies

$$R_u^{(2)} = \int_\Gamma \gamma(u)\, d\mathcal{E}(\gamma), \qquad \text{for all } u \in G.$$

For each complex-valued, bounded Borel measurable function ϕ on Γ, we define the bounded linear operator $\mathcal{T}_\phi : L^2(\sigma) \longrightarrow L^2(\sigma)$ by writing

$$\mathcal{T}_\phi = \int_\Gamma \phi(\gamma)\, d\mathcal{E}(\gamma).$$

If $\mathcal{T}_\phi$ is of weak type $(1,1)$ on $L^2(\sigma) \cap L^1(\sigma)$, then $\mathcal{T}_\phi$ has a unique extension from $L^2(\sigma) \cap L^1(\sigma)$ to a linear mapping $\mathcal{T}_\phi^{(1)}$ from $L^1(\sigma)$ to the complex-valued measurable functions on Ω (identified modulo equality σ-a.e.) such that whenever $\{f_n\}_{n=1}^\infty \subseteq L^1(\sigma)$, $f \in L^1(\sigma)$, and $\|f_n - f\|_1 \longrightarrow 0$, then, with respect to σ, the sequence $\{\mathcal{T}_\phi^{(1)} f_n\}_{n=1}^\infty$ converges to $\mathcal{T}_\phi^{(1)} f$ in measure. General transference methods for strong type multipliers (see [6, Theorem 2.1 and Remarks preceding (2.7)]) apply to the context at hand, where they show that if $\phi \in \mathcal{C}(\Gamma)$ is a multiplier of strong type $(1,1)$, then $\mathcal{T}_\phi$ is of strong type $(1,1)$ on $L^2(\sigma) \cap L^1(\sigma)$. The simplest case of this phenomenon arises when ϕ is the Fourier transform of a function $k \in L^1(G)$. In this instance, it is easy to see by Fubini's Theorem that $\mathcal{T}_k^{(1)}$ coincides with the *transferred convolution operator* H_k, which is defined on $L^1(\sigma)$ by using Bochner integration to write

$$H_k f = \int_G k(t) R_{-t} f \, d\mu_G(t), \quad \text{for all } f \in L^1(\sigma). \tag{6.4}$$

Having described the appropriate setting, we can now state the transference theorem for multipliers of weak type $(1,1)$.

THEOREM 6.5. *Let R be a strongly continuous distributionally controlled representation of the locally compact abelian group G in $L^1(\Omega,\sigma)$, where (Ω,σ) is an arbitrary measure space. Assume the preceding notation, and suppose that $\{\psi_j\}_{j=1}^{\infty} \subseteq C(\Gamma) \cap M_1^{(w)}(\Gamma)$. Then for each $j \in \mathbb{N}$, $\mathcal{T}_{\psi_j}$ is of weak type $(1,1)$ on $L^2(\sigma) \cap L^1(\sigma)$, and*

$$\sigma\Big\{\omega \in \Omega : \sup_{j \in \mathbb{N}} |(\mathcal{T}_{\psi_j}^{(1)} f)(\omega)| > y\Big\} \le (K_1 K_\infty)^2 C N_1^{(w)}(\{\psi_j\}_{j=1}^{\infty}) y^{-1} \|f\|_{L^1(\sigma)}$$
$$\textit{for all } f \in L^1(\sigma), \textit{ and all } y > 0,$$

where C is the absolute constant in Theorem 2.7.

PROOF. Suppose that $N \in \mathbb{N}$, and $\{\phi_j\}_{j=1}^{N} \subseteq \mathcal{C}_{oo}(\Gamma) \cap M_1^{(w)}(\Gamma)$. We show first that:

$$\sigma\Big\{\omega \in \Omega : \max_{1 \le j \le N} |(\mathcal{T}_{\phi_j} f)(\omega)| > y\Big\} \le (K_1 K_\infty)^2 C N_1^{(w)}(\{\phi_j\}_{j=1}^{N}) y^{-1} \|f\|_{L^1(\sigma)}, \tag{6.6}$$

for all $f \in L^2(\sigma) \cap L^1(\sigma)$ and all $y > 0$. Let $\{h_\delta\}_{\delta \in \Delta}$ be an approximate identity for $L^1(\Gamma)$ having the properties listed in Scholium 4.19. For $\delta \in \Delta$, and $1 \le j \le N$, there is, by Fourier inversion, a function $k_{j,\delta} \in L^1(G)$ such that $(k_{j,\delta})^\wedge = h_\delta * \phi_j$. The transference of weak type estimates for maximal convolution operators expressed in [3, Theorem 2.14] can be applied to $\{k_{j,\delta}\}_{j=1}^{N}$. In the notation of (6.4), this gives:

$$\sigma\Big\{\omega \in \Omega : \max_{1 \le j \le N} |(H_{k_{j,\delta}} f)(\omega)| > y\Big\} \le (K_1 K_\infty)^2 N_1^{(w)}(\{h_\delta * \phi_j\}_{j=1}^{N}) y^{-1} \|f\|_{L^1(\sigma)},$$

for all $f \in L^2(\sigma) \cap L^1(\sigma)$, and all $y > 0$. We can rewrite this in the form:

$$\sigma\Big\{\omega \in \Omega : \max_{1 \le j \le N} |(\mathcal{T}_{h_\delta * \phi_j} f)(\omega)| > y\Big\} \le (K_1 K_\infty)^2 N_1^{(w)}(\{h_\delta * \phi_j\}_{j=1}^{N}) y^{-1} \|f\|_{L^1(\sigma)},$$

for all $f \in L^2(\sigma) \cap L^1(\sigma)$, and all $y > 0$. Applying Theorem 1.1 to the right-hand side of the last inequality, we see that:

$$\sigma\Big\{\omega \in \Omega : \max_{1 \le j \le N} |(\mathcal{T}_{h_\delta * \phi_j} f)(\omega)| > y\Big\} \le (K_1 K_\infty)^2 C N_1^{(w)}(\{\phi_j\}_{j=1}^{N}) y^{-1} \|f\|_{L^1(\sigma)}, \tag{6.7}$$

for all $f \in L^2(\sigma) \cap L^1(\sigma)$, and all $y > 0$. For $1 \le j \le N$, $h_\delta * \phi_j \longrightarrow \phi_j$ uniformly on Γ, and consequently $\mathcal{T}_{h_\delta * \phi_j} \longrightarrow \mathcal{T}_{\phi_j}$ in the uniform operator topology of the bounded operators on $L^2(\sigma)$. Application of this fact to (6.7) readily establishes the claim made in (6.6).

We now consider the sequence $\{\psi_j\}_{j=1}^{\infty}$ in the hypotheses. Fix $N \in \mathbb{N}$, $f \in L^2(\sigma) \cap L^1(\sigma)$, and suppose that $\epsilon > 0$. There is a compact subset $\mathcal{K}$ of Γ such that:

$$\|\mathcal{E}(\mathcal{K}) f - f\|_{L^2(\sigma)} < \epsilon. \tag{6.8}$$

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Choose $k \in L^1(G)$ such that $\hat{k}$ is compactly supported, $\hat{k} = 1$ on $\mathcal{K}$, and $\|k\|_{L^1(G)} < 1 + \epsilon$. Clearly $\{\hat{k}\psi_j\}_{j=1}^N \subseteq C_{oo}(\Gamma) \cap M_1^{(w)}(\Gamma)$, and

$$N_1^{(w)}(\{\hat{k}\psi_j\}_{j=1}^N) \leq (1+\epsilon) N_1^{(w)}(\{\psi_j\}_{j=1}^N).$$

We have σ-a.e. on Ω,

$$\max_{1 \leq j \leq N} |\mathcal{T}_{\psi_j} f| \leq \max_{1 \leq j \leq N} |\mathcal{T}_{\hat{k}\psi_j} f| + \max_{1 \leq j \leq N} \left| \int_{\Gamma \setminus \mathcal{K}} \psi_j (1 - \hat{k})\, d\mathcal{E}f \right|.$$

It follows easily from this that for $y > 0$, $\eta > 0$:

(6.9)

$$\begin{aligned} \sigma\Big\{\omega \in \Omega : \max_{1 \leq j \leq N} |(\mathcal{T}_{\psi_j} f)(\omega)| > y\Big\} & \\ \leq \sigma\Big\{\omega \in \Omega : & \max_{1 \leq j \leq N} |(\mathcal{T}_{\hat{k}\psi_j} f)(\omega)| > y(1+\eta)^{-1}\Big\} \\ & + \sigma\Big\{\omega \in \Omega : \sum_{j=1}^N \Big| \Big(\int_{\Gamma \setminus \mathcal{K}} \psi_j (1 - \hat{k})\, d\mathcal{E}f \Big)(\omega) \Big| > \eta y (1+\eta)^{-1} \Big\}. \end{aligned}$$

We now specialize $\{\phi_j\}_{j=1}^N$ in (6.6) to be $\{\hat{k}\psi_j\}_{j=1}^N$. Applying this result to the first term on the right of (6.9), and Chebychev's Inequality for $L^2(\sigma)$, together with (6.8), to the second term on the right of (6.9), we find that:

$$\begin{aligned} \sigma\Big\{\omega \in \Omega : \max_{1 \leq j \leq N} |(\mathcal{T}_{\psi_j} f)(\omega)| > y\Big\} \leq (K_1 K_\infty)^2 C (1+\epsilon) N_1^{(w)}(\{\psi_j\}_{j=1}^N)(1+\eta) y^{-1} \|f\|_{L^1(\sigma)} & \\ + \Big[(1+\eta)(\eta y)^{-1} 4\epsilon (2+\epsilon) A \sum_{j=1}^N \|\psi_j\|_u \Big]^2 &, \end{aligned}$$

where A is the supremum, taken over all Borel sets α in Γ, of $\|\mathcal{E}(\alpha)\|_2$. Letting $\epsilon \longrightarrow 0$ and $\eta \longrightarrow 0$ successively in this inequality, we arrive at:

$$\sigma\Big\{\omega \in \Omega : \max_{1 \leq j \leq N} |(\mathcal{T}_{\psi_j} f)(\omega)| > y\Big\} \leq (K_1 K_\infty)^2 C N_1^{(w)}(\{\psi_j\}_{j=1}^N) y^{-1} \|f\|_{L^1(\sigma)},$$

for $N \in \mathbb{N}$, $f \in L^2(\sigma) \cap L^1(\sigma)$, and $y > 0$. The conclusions of Theorem 6.5 follow immediately from this.

We finish the discussion by showing that Theorem 6.5 implies the homomorphism theorem expressed in (1.5), which extends to the case $p = 1$ the results for $1 < p < \infty$ in [4, Theorem 4.1-(i)].

PROOF OF THEOREM 1.5. Let G_1 and G_2 be the dual groups of Γ_1 and Γ_2, respectively, and let $\hat{\rho}: G_2 \longrightarrow G_1$ be the dual homomorphism of ρ:

$$\big(\hat{\rho}(u)\big)(\gamma) = u\big(\rho(\gamma)\big), \quad \text{for } u \in G_2,\ \gamma \in \Gamma_1.$$

For $u \in G_2$, let R_u designate translation by $\hat{\rho}(u)$ on $L^1(G_1)$. Obviously, $u \longrightarrow R_u$ is a strongly continuous, distributionally controlled representation of G_2 in $L^1(G_1)$, with $K_1 = K_\infty = 1$. Moreover, the reasoning in [2, Proof of Theorem 4.1] shows that for each bounded Borel measurable function ϕ on Γ_2, the operator $\mathcal{T}_\phi$ on $L^2(G_1)$ is the $L^2(G_1)$-multiplier transform corresponding to the bounded Borel measurable function $\phi \circ \rho$ on Γ_1. Applying these facts to the functions $\{\psi_j\}_{j=1}^{\infty}$, we can invoke Theorem 6.5 to complete the proof.

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