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ON AN IMPLICIT HIERARCHICAL FIXED POINT APPROACH TO VARIATIONAL INEQUALITIES

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Abstract

Moudafi and Maingé [Towards viscosity approximations of hierarchical fixed-point problems, *Fixed Point Theory Appl.* (2006), Art. ID 95453, 10pp] and Xu [Viscosity method for hierarchical fixed point approach to variational inequalities, *Taiwanese J. Math.* **13**(6) (2009)] studied an implicit viscosity method for approximating solutions of variational inequalities by solving hierarchical fixed point problems. The approximate solutions are a net $(x_{s,t})$ of two parameters $s, t \in (0, 1)$, and under certain conditions, the iterated $\lim_{t\to 0} \lim_{s\to 0} x_{s,t}$ exists in the norm topology. Moudafi, Maingé and Xu stated the problem of convergence of $(x_{s,t})$ as $(s, t) \to (0, 0)$ jointly in the norm topology. In this paper we further study the behaviour of the net $(x_{s,t})$; in particular, we give a negative answer to this problem.

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1. Introduction and preliminaries

A useful method for solving ill-posed nonlinear problems is to substitute the originally ill-posed problem by a family of regularized (well-posed) problems. A particular (viscosity) solution of the original problem is then obtained as limit of the solutions of the regularized problems. In [4, 7, 10] the authors used this idea to provide a viscosity method for solving variational inequality problems via a hierarchical fixed point approach.

Let T, V be two nonexpansive mappings from C to C, where C is a closed convex subset of a Hilbert space H. Consider the variational inequality (VI) of finding hierarchically a fixed point of T with respect to V, that is,

$$\text{Find } x^* \in \text{Fix}(T) \quad \text{such that } \langle x^* - Vx^*, y - x^*\rangle \geq 0,\ y \in \text{Fix}(T). \tag{1.1}$$

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Equivalently, $x^* = P_{\mathrm{Fix}(T)} V x^*$; that is, x^* is a fixed point of the nonexpansive mapping $P_{\mathrm{Fix}(T)} V$, where P_K denotes the metric projection from H on a closed convex subset K of H. The VI (1.1) covers several topics investigated in the literature (see [1, 3, 5, 6, 8, 11, 12] and the references cited therein).

Let S denote the solution set of (1.1) and assume throughout the rest of this paper that $S \neq \emptyset$. Note that $S = \mathrm{Fix}(P_{\mathrm{Fix}(T)} V)$. We also adopt the following notation: $x_n \to x$ means that (x_n) converges to x in the norm topology; $x_n \rightharpoonup x$ means that (x_n) converges to x in the weak topology.

Let $f : C \to C$ be a ρ-contraction and define, for $s, t \in (0, 1)$, two mappings W_t and $f_{s,t}$ by

$$W_t = tV + (1-t)T, \quad f_{s,t} = sf + (1-s)W_t.$$

It is easy to verify that W_t is nonexpansive and $f_{s,t}$ is a $[1-(1-\rho)s]$-contraction.

Let $x_{s,t}$ be the unique fixed point of $f_{s,t}$, that is, the unique solution of the fixed point equation

$$x_{s,t} = sf(x_{s,t}) + (1-s)W_t x_{s,t}. \tag{1.2}$$

Moudafi and Maingé [7] initiated the investigation of the iterated behaviour of the net $(x_{s,t})$ as $s \to 0$ firstly and $t \to 0$ secondly. They make the following assumptions:

(A1) for each $t \in (0, 1)$, the fixed point set $\mathrm{Fix}(W_t)$ of W_t is nonempty and the set $\{\mathrm{Fix}(W_t) : 0 < t < 1\} = \bigcup_{t\in(0,1)} \mathrm{Fix}(W_t)$ is bounded; and

(A2) $\emptyset \neq S \subset \|\cdot\| - \liminf_{t\to 0} \mathrm{Fix}(W_t) := \{z : \exists z_t \in \mathrm{Fix}(W_t) \text{ such that } z_t \to z\}$.

Moudafi and Maingé [7] (see also [9]) proved that, for each fixed $t \in (0, 1)$, as $s \to 0$, $x_{s,t} \to x_t$; moreover, as $t \to 0$, $x_t \rightharpoonup x_\infty$ which is the unique solution to the VI

$$x_\infty \in S, \quad \langle x_\infty - f(x_\infty), x - x_\infty \rangle \geq 0, \ x \in S. \tag{1.3}$$

The following theorem, due to Xu [10], improves the Moudafi–Maingé result since he proves that (x_t) actually strongly converges to x_∞. Moreover, Xu does not need the boundedness assumption of the set $\bigcup_{t\in(0,1)} \mathrm{Fix}(W_t)$.

THEOREM 1.1. [10] *Let the above assumption (A2) hold. Assume also that, for each $t \in (0, 1)$, $\mathrm{Fix}(W_t)$ is nonempty (but not necessarily bounded). Then the strong $\lim_{s\to 0} x_{s,t} =: x_t$ exists for each $t \in (0, 1)$. Moreover, the strong $\lim_{t\to 0} x_t =: x_\infty$ exists and solves the VI (1.3). Hence, for each null sequence (s_n) in $(0, 1)$, there is another null sequence (t_n) in $(0, 1)$ such that $x_{s_n,t_n} \to x_\infty$, as $n \to \infty$.*

In [7, 10], the authors stated the problem of the convergence of $(x_{s,t})$ when $(s, t) \to (0, 0)$ jointly. In this paper, we further investigate the behaviour of the net $(x_{s,t})$ along the curve $t = t(s)$ and our results point to a negative answer to this problem. Specifically, we prove that:

(i) if $t(s) = O(s)$, as $s \to 0$, then $x_{s,t(s)} \to z_\infty \in \mathrm{Fix}(T)$; and
(ii) if $t(s)/s \to \infty$, as $s \to 0$, then $x_{s,t(s)} \to x_\infty \in S$.

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We next include two lemmas which are pertinent to the proof of many convergence results of iterative methods. Let H be a real Hilbert space and C be a nonempty closed convex of H. Recall that the metric projection, P_C, from H onto C, assigns to each $x \in H$ a unique point $P_C x$ in C with the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|.$$

LEMMA 1.2. *Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if*

$$\langle x - z, y - z \rangle \le 0 \quad \forall y \in C. \tag{1.4}$$

LEMMA 1.3 ([2] Demiclosedness principle). *If $T : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, then the mapping $(I - T)$ is demiclosed; that is, if a sequence (x_n) in C is weakly convergent to x and if the sequence $((I - T)x_n)$ is strongly convergent to y, then $(I - T)x = y$.*

2. On convergence of $(x_{s,t})_{s,t\in(0,1)}$

In this section we study the convergence of the net $(x_{s,t})$ along the curve $t = t(s) =: t_s$, where $t_s = O(s)$, as $s \to 0$.

THEOREM 2.1. *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $V, T : C \to C$ be nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction with $\rho \in [0, 1)$. Assume that $t_s = O(s)$, as $s \to 0$, and let $l = \limsup_{s\to 0}(t_s/s)$. Then the net $(x_{s,t_s})_{s\in(0,1)}$ defined by*

$$x_{s,t_s} = sf(x_{s,t_s}) + (1 - s)W_{t_s} x_{s,t_s} \tag{2.1}$$

strongly converges to $z_\infty \in \mathrm{Fix}(T)$ which is the unique solution of the VI

$$z_\infty \in \mathrm{Fix}(T), \quad \langle [(I - f) + l(I - V)]z_\infty, x - z_\infty \rangle \ge 0, x \in \mathrm{Fix}(T). \tag{2.2}$$

PROOF. We first note that the VI (2.2) has a unique solution, due to the fact that the operator $(I - f) + l(I - V)$ is strongly monotone. The proof is divided into two steps.

The first step is to prove that the net $(x_{s,t_s})_{s\in(0,1)}$ is bounded. Let $z \in \mathrm{Fix}(T)$; then, from (2.1),

$$\begin{aligned}
\|x_{s,t_s} - z\|^2 &= \langle x_{s,t_s} - z, x_{s,t_s} - z \rangle \\
&= s\langle f(x_{s,t_s}) - z, x_{s,t_s} - z \rangle + (1 - s)\langle W_{t_s} x_{s,t_s} - z, x_{s,t_s} - z \rangle \\
&= s[\langle f(x_{s,t_s}) - f(z), x_{s,t_s} - z \rangle + \langle f(z) - z, x_{s,t_s} - z \rangle] \\
&\qquad + (1 - s)[\langle W_{t_s} x_{s,t_s} - W_{t_s} z, x_{s,t_s} - z \rangle + \langle W_{t_s} z - z, x_{s,t_s} - z \rangle] \\
&\le s\rho\|x_{s,t_s} - z\|^2 + s\langle f(z) - z, x_{s,t_s} - z \rangle \\
&\qquad + (1 - s)\|x_{s,t_s} - z\|^2 + t_s(1 - s)\langle Vz - z, x_{s,t_s} - z \rangle.
\end{aligned}$$

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Simplifying, we obtain

$$\|x_{s,t_s} - z\|^2 \le \frac{1}{1-\rho}\left[\langle f(z) - z, x_{s,t_s} - z\rangle + \frac{t_s(1-s)}{s}\langle Vz - z, x_{s,t_s} - z\rangle\right]. \tag{2.3}$$

In particular,

$$\|x_{s,t_s} - z\| \le \frac{1}{1-\rho}\left[\|f(z) - z\| + \frac{t_s}{s}\|Vz - z\|\right]. \tag{2.4}$$

Since $t_s = O(s)$, as $s \to 0$, (2.4) implies the boundedness of (x_{s,t_s}) and the first step is proved.

The second step is to prove that the net $x_{s,t_s} \to z_\infty \in \mathrm{Fix}(T)$, as $s \to 0$, where z_∞ is the unique solution of the VI (2.2). We observe that

$$\|x_{s,t_s} - Tx_{s,t_s}\| \le s\|f(x_{s,t_s})\| + (1-s)t_s\|Vx_{s,t_s}\| + (s + t_s - st_s)|\|Tx_{s,t_s}\|.$$

Since (x_{s,t_s}) is bounded when $s \to 0$ (hence $t_s \to 0$), we find that

$$\|x_{s,t_s} - Tx_{s,t_s}\| \to 0. \tag{2.5}$$

We now claim that $(x_{s,t_s})_{s\in(0,1)}$ is relatively compact as $s \to 0$ in the norm topology. To see this, assume (s_n) is null sequence in $(0, 1)$. Without loss of generality, we may assume that $x_{s_n,t_{s_n}} \rightharpoonup \widehat{x}$ which implies from (2.5) and Lemma 1.3 that $\widehat{x} \in \mathrm{Fix}(T)$. We thus immediately get from (2.3) that $x_{s_n,t_{s_n}} \to \widehat{x}$.

We next further claim that $\widehat{x} = z_\infty$, the unique solution to the VI (2.2), which then completes the proof. Indeed, observing

$$(I - f)x_{s,t} = -\frac{1-s}{s}(x_{s,t} - W_t x_{s,t}) = -\frac{1-s}{s}[t(I - V)x_{s,t} + (1-t)Tx_{s,t}],$$

we deduce that, for $z \in \mathrm{Fix}(T)$,

$$\begin{aligned}\langle (I - f)x_{s,t}, x_{s,t} - z\rangle = -\frac{1-s}{s}[&t\langle (I - V)x_{s,t}, x_{s,t} - z\rangle \\ &+ (1-t)\langle (I - T)x_{s,t}, x_{s,t} - z\rangle].\end{aligned}$$

However, since

$$\langle (I - T)x_{s,t}, x_{s,t} - z\rangle = \langle (I - T)x_{s,t} - (I - T)z, x_{s,t} - z\rangle \ge 0,$$

we obtain

$$\langle (I - f)x_{s,t}, x_{s,t} - z\rangle \le -\frac{t(1-s)}{s}\langle (I - V)x_{s,t}, x_{s,t} - z\rangle. \tag{2.6}$$

Now since $x_{s_n,t_{s_n}} \to \widehat{x}$, setting $s = s_n$ and $t = t_{s_n}$ in (2.6) and letting $n \to \infty$, we immediately see that $\widehat{x}$ satisfies the VI (2.2) and therefore we must have $\widehat{x} = z_\infty$ since z_∞ is the unique solution of (2.2). □

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REMARK 2.2. (i) If $t_s = o(s)$ (that is, $l = 0$), then the above argument shows that the net (x_{s,t_s}) actually converges in norm to the unique solution of the VI

$$x_\infty \in \mathrm{Fix}(T), \quad \langle x_\infty - f(x_\infty), p - x_\infty \rangle \geq 0, \ p \in \mathrm{Fix}(T), \tag{2.7}$$

which is also the unique fixed point of the contraction $P_{\mathrm{Fix}(T)} f$, $x_\infty = (P_{\mathrm{Fix}(T)} f) x_\infty$. This is Theorem 3.3 in Xu [10].

(ii) The net $(x_{s,t})_{s,t \in (0,1)}$ does not converge, in general, as $(s, t) \to (0, 0)$ jointly, to the unique solution $x_\infty \in S$ of the VI (1.3). As a matter of fact, if $(x_{s,t})_{s,t \in (0,1)}$ converged to x_∞ jointly as $(s, t) \to (0, 0)$, then (by (2.7) we would have the relation and (1.3))

$$x_\infty = P_S f(x_\infty) = P_{\mathrm{Fix}(T)} f(x_\infty)$$

for all ρ-contractions f. This implies that $S = \mathrm{Fix}(T)$ which is not true, in general.

(iii) Consider the case of $l > 0$. If x_∞, the unique solution of (2.7), belongs to S, then, clearly, $x_\infty = z_\infty$. If $x_\infty \notin S$, the following example shows that there are, in general, no links among z_∞, S and x_∞. Take

$$C = [0, 1], \quad T = I, \quad f(x) = \frac{x}{2}, \quad V(x) = 1 - x, \quad l = 1.$$

The unique solution x_∞ of the VI

$$x_\infty \in [0, 1], \quad \langle x_\infty - f(x_\infty), z - x_\infty \rangle \geq 0, \ z \in [0, 1],$$

is $x_\infty = 0$; the unique solution z_∞ of the VI

$$x_\infty \in [0, 1], \quad \langle (z_\infty - f(z_\infty)) + (z_\infty - V z_\infty), z - z_\infty \rangle \geq 0, \ z \in [0, 1],$$

is $z_\infty = \frac{2}{5}$, and the set S of the solutions of the VI

$$x \in [0, 1], \quad \langle x - Vx, z - x \rangle \geq 0, \ z \in [0, 1],$$

is the singleton $\{1/2\}$.

3. The case $l = \infty$

In this section we examine the convergence of the net $(x_{s,t_s})_{s \in (0,1)}$ along the curve where $t_s/s \to \infty$, as $s \to 0$. We shall prove that the net converges strongly to a point $x_\infty \in S$ which is the unique solution of the VI (1.3).

THEOREM 3.1. *Let H be a real Hilbert space and let C be a closed convex subset of H. Assume that $V, T : C \to C$ are nonexpansive mappings with* $\mathrm{Fix}(T) \neq \emptyset$ *and $f : C \to C$ is a ρ-contraction with $\rho \in [0, 1)$. Assume the condition (A2) in Section 1. Let $t_s = t(s)$ satisfy $\lim_{s \to 0} t_s/s = \infty$. Then the net $(x_{s,t_s})_{s \in (0,1)}$ defined by*

$$x_{s,t_s} = s f(x_{s,t_s}) + (1 - s) W_{t_s} x_{s,t_s} \tag{3.1}$$

strongly converges to $x_\infty \in S$ which is the unique solution of the VI (1.3).

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PROOF. The proof is divided into three steps, the first of which is to prove the boundedness of $(x_{s,t_s})_{s\in(0,1)}$. Let $z \in S$. By condition (A2) there exists $p_s \in \mathrm{Fix}(W_s)$ such that $p_s \to z$ as $s \to 0$. We then derive that

$$\begin{aligned}\|x_{s,t_s} - p_s\|^2 &= \|s(f(x_{s,t_s}) - f(p_s)) + s(f(p_s) - p_s) + (1-s)(W_{t_s}x_{s,t_s} - p_s)\|^2 \\ &\le \|s(f(x_{s,t_s}) - f(p_s)) + (1-s)(W_{t_s}x_{s,t_s} - p_s)\|^2 \\ &\qquad + 2s\langle f(p_s) - p_s, x_{s,t_s} - p_s\rangle \\ &\le s\|f(x_{s,t_s}) - f(p_s)\|^2 + (1-s)\|W_{t_s}x_{s,t_s} - p_s\|^2 \\ &\qquad + 2s\langle f(p_s) - p_s, x_{s,t_s} - p_s\rangle \\ &\le (1-(1-\rho^2)s)\|x_{s,t_s} - p_s\|^2 + 2s\langle f(p_s) - p_s, x_{s,t_s} - p_s\rangle.\end{aligned}$$

It follows that

$$\|x_{s,t_s} - p_s\|^2 \le \frac{2}{1-\rho^2}\langle f(p_s) - p_s, x_{s,t_s} - p_s\rangle. \tag{3.2}$$

This implies immediately that

$$\|x_{s,t_s} - p_s\| \le \frac{2}{1-\rho^2}\|f(p_s) - p_s\|. \tag{3.3}$$

From (3.3) the boundedness of $(x_{s,t_s})_{s\in(0,1)}$ follows since $\{p_s\}$ is bounded.

The second step is to prove that the set of weak cluster points of $(x_{s,t_s})_{s\in(0,1)}$, $\omega_w(x_{s,t_s})$, is a subset of S; moreover, $\omega_w(x_{s,t_s}) = \omega_s(x_{s,t_s})$. First observe that the boundedness of (x_{s,t_s}), (2.5), and Lemma 1.3 imply that $\omega_w(x_{s,t_s}) \subset \mathrm{Fix}(T)$.

Now let $w \in \omega_w(x_{s,t_s})$ and assume that $x_n := x_{s_n,t_{s_n}} \rightharpoonup w$, where $s_n \to 0$. For convenience, we write $t_n = t_{s_n}$ for all n; thus, $t_n/s_n \to \infty$ as $n \to \infty$. Noticing that

$$x_n = s_n f(x_n) + (1-s_n)[t_n V x_n + (1-t_n)T x_n],$$

we derive that, for each fixed $\widehat{x} \in \mathrm{Fix}(T)$ and for a constant $M \ge \sup_n\{\|f(x_n) - \widehat{x}\|\|x_n - \widehat{x}\|\}$,

$$\begin{aligned}\|x_n - \widehat{x}\|^2 &= s_n\langle f(x_n) - \widehat{x}, x_n - \widehat{x}\rangle \\ &\qquad + (1-s_n)(t_n\langle V x_n - \widehat{x}, x_n - \widehat{x}\rangle + (1-t_n)\langle T x_n - \widehat{x}, x_n - \widehat{x}\rangle) \\ &= s_n\langle f(x_n) - \widehat{x}, x_n - \widehat{x}\rangle + (1-s_n)t_n\langle V\widehat{x} - \widehat{x}, x_n - \widehat{x}\rangle \\ &\qquad + (1-s_n)[t_n\langle V x_n - V\widehat{x}, x_n - \widehat{x}\rangle + (1-t_n)\langle T x_n - T\widehat{x}, x_n - \widehat{x}\rangle] \\ &\le \|x_n - \widehat{x}\|^2 + (1-s_n)t_n\langle V\widehat{x} - \widehat{x}, x_n - \widehat{x}\rangle + s_n M.\end{aligned}$$

It follows that

$$\langle (I-V)\widehat{x}, x_n - \widehat{x}\rangle \le \frac{s_n M}{(1-s_n)t_n} \to 0$$

as $s_n/t_n \to 0$. But $x_n \rightharpoonup w$, and we get

$$\langle (I-V)\widehat{x}, w - \widehat{x}\rangle \le 0, \quad \widehat{x} \in \mathrm{Fix}(T). \tag{3.4}$$

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Upon replacing the $\widehat{x}$ in (3.4) with $w + \gamma(\widetilde{x} - w) \in \mathrm{Fix}(T)$, where $\gamma \in (0, 1)$ and $\widetilde{x} \in \mathrm{Fix}(T)$, we get

$$\langle (I - V)(w + \gamma(\widetilde{x} - w)), w - \widetilde{x} \rangle \leq 0.$$

Letting $\gamma \to 0$, we obtain the VI

$$\langle (I - V)w, w - \widetilde{x} \rangle \leq 0, \quad \widetilde{x} \in \mathrm{Fix}(T).$$

Therefore, $w \in S$.

Next using condition (A2) again, we have a sequence $p_n \in \mathrm{Fix}(W_{t_n})$ such that $p_n \to w$. Then in relation (3.2) we replace z and p_s with w and p_n, respectively, to get

$$\|x_n - p_n\|^2 \leq \frac{2}{1 - \rho^2} \langle f(p_n) - p_n, x_n - p_n \rangle. \tag{3.5}$$

Now since $f(p_n) - p_n \to f(w) - w$ and $x_n - p_n \rightharpoonup 0$, taking the limit in (3.5), we immediately get $x_n \to w$. Hence $w \in \omega_s(x_{s,t_s})$.

The third and final step is to prove that the net (x_{s,t_s}) converges in norm to $x_\infty = (P_S f)x_\infty$. It suffices to prove that each norm limit point $w \in \omega_s(x_{s,t_s})$ solves the VI (1.3). We still use the same subsequence $\{x_n\}$ of the net (x_{s,t_s}) such that $x_n \to w$ as shown in the second step. On the other hand, for every $p \in S$, by condition (A2), we have, for each n, $p_{t_n} \in \mathrm{Fix}(W_{t_n})$ such that $p_{t_n} \to p$ as $n \to \infty$.

Now since $I - W_{t_n}$ is monotone and since

$$(I - f)x_n = -\frac{1 - s_n}{s_n}(x_n - W_{t_n} x_n),$$

we get

$$\begin{aligned} \langle (I - f)x_n, x_n - p_{t_n} \rangle &= -\frac{1 - s_n}{s_n} \langle (x_n - W_{t_n} x_n), x_n - p_{t_n} \rangle \\ &= -\frac{1 - s_n}{s_n} \langle (I - W_{t_n})x_n - (I - W_{t_n})p_{t_n}, x_n - p_{t_n} \rangle \\ &\leq 0. \end{aligned}$$

Passing to the limit as $n \to \infty$ in the last inequality, we conclude that

$$\langle (I - f)w, w - p \rangle \leq 0, \quad p \in S.$$

This is the VI (1.3). Hence $w = x_\infty$, as required. □

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