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Congruence Class Sizes in Finite Sectionally Complemented Lattices

G. Grätzer and E. T. Schmidt

Abstract. The congruences of a finite sectionally complemented lattice L are not necessarily *uniform* (any two congruence classes of a congruence are of the same size). To measure how far a congruence Θ of L is from being uniform, we introduce Spec Θ, the *spectrum* of Θ, the family of cardinalities of the congruence classes of Θ. A typical result of this paper characterizes the spectrum $S = (m_j \mid j < n)$ of a nontrivial congruence Θ with the following two properties:

(S_1) $2 \leq n$ and $n \neq 3$.
(S_2) $2 \leq m_j$ and $m_j \neq 3$, for all $j < n$.

1 Introduction

1.1 Generalizing N_6

The classical result of R. P. Dilworth (see G. Grätzer and E. T. Schmidt [2] and G. Grätzer [1, Section II.3]) states that every finite distributive lattice D can be represented as the congruence lattice of a finite lattice A. In fact, the G. Grätzer and E. T. Schmidt [2] version claims that A can be constructed as a finite *sectionally complemented* lattice.

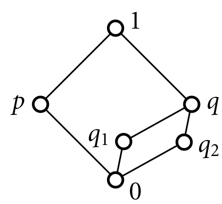

Figure 1: The lattices N_6.

The basic building stone of this lattice A is the lattice N_6 of Figure 1. This lattice has some crucial properties:

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(i) N_6 is sectionally complemented.
(ii) N_6 has exactly one nontrivial congruence Θ.
(iii) Θ has exactly two congruence classes: the prime ideal $\{0, q_1, q_2, q\}$ and the dual prime ideal $\{p, 1\}$.
(iv) $p \equiv 0\ (\Phi)$ implies that $q \equiv 0\ (\Phi)$, for every congruence Φ of N_6.

We can associate with Θ the pair $\langle 4, 2\rangle$ measuring the size of the two congruence classes. We started with the following question: Which pairs $\langle t_1, t_2\rangle$ can substitute for $\langle 4, 2\rangle$? In other words, for which pairs of integers $\langle t_1, t_2\rangle$ is there a finite lattice L such that

(1) L is sectionally complemented.
(2) L has exactly one nontrivial congruence Θ.
(3) Θ has exactly two congruence classes: the prime ideal P and the dual prime ideal Q satisfying that $|P| = t_1$ and $|Q| = t_2$.

(We did not add the fourth property from above since it follows from the three we have stated.)

This question is answered as follows:

Theorem 1 *Let $\langle t_1, t_2\rangle$ be a pair of natural numbers. Then there is a finite lattice L with properties* (1)–(3) *iff $\langle t_1, t_2\rangle$ satisfies the following three conditions:*

(P_1) $2 \leq t_1$ *and* $t_1 \neq 3$.
(P_2) $2 \leq t_2$ *and* $t_2 \neq 3$.
(P_3) $t_1 > t_2$.

Figure 2 illustrates the lattice we obtain for $\langle 5, 4\rangle$.

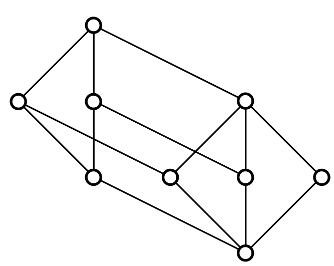

Figure 2: A lattice representing $\langle 5, 4\rangle$.

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1.2 Spectrum

The question answered by Theorem 1 is a very special case of a more general problem: What can we say about the cardinalities of the congruence classes of a nontrivial congruence in a finite sectionally complemented lattice?

Let L be a finite lattice, and let Θ be a congruence of L. We denote by $\operatorname{Spec}\Theta$ the *spectrum* of Θ, that is, the family of cardinalities of the congruence classes of Θ. So $\operatorname{Spec}\Theta$ has $|L/\Theta|$ elements, and each element is an integer ≥ 1.

It is clear that if S is a family of integers ≥ 1, then it is the spectrum of some congruence (take L as an appropriate chain). We are interested in the following problem: Characterize the spectra of nontrivial congruences of finite sectionally complemented lattices.

This problem is completely solved by the following result:

Theorem 2 *Let $S = (m_j \mid j < n)$ be a family of natural numbers, $n \geq 1$. Then there is a finite sectionally complemented lattice L with more than one element and a nontrivial congruence Θ of L such that S is the spectrum of Θ iff S satisfies the following conditions:*

(S_1) $2 \leq n$ *and* $n \neq 3$.
(S_2) $2 \leq m_j$ *and* $m_j \neq 3$, *for all* $j < n$.

Figure 3 illustrates the lattice we obtain for $S = (4, 4, 2, 2, 2)$.

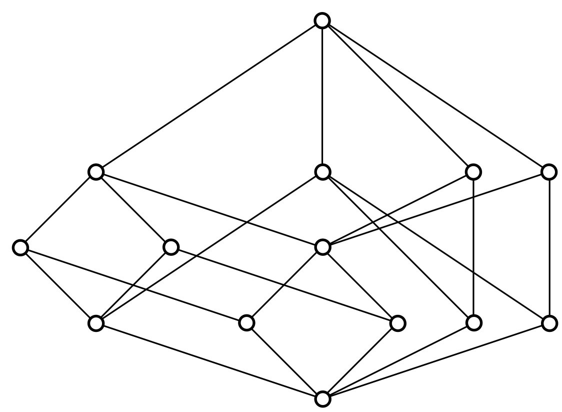

Figure 3: A lattice representing $S = (4, 4, 2, 2, 2)$.

This result is not a direct generalization of Theorem 1, since we did not assume that Θ is the only nontrivial congruence of K. This additional condition is easy to accommodate:

Corollary *Let $S = (m_j \mid j < n)$ be a family of natural numbers, $n > 1$. Then there is a finite sectionally complemented lattice L with more than one element with a* unique

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nontrivial congruence Θ *of* L *such that* S *is the spectrum of* Θ *iff* S *satisfies* (S_1) *and* (S_2)*, and additionally:*

(S_3) *S is not constant, that is, there are* $j,\ j' < n$ *satisfying that* $m_j \neq m_{j'}$.
(S_4) $n \neq 4$.

1.3 Valuation

There is a more sophisticated way of looking at spectra. Let L be a finite lattice, and let Θ be a congruence of L. Then there is a natural map $v\colon L/\Theta \to \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) defined as follows: Let $a \in L/\Theta$; then a is a congruence class of Θ, so we can define $v(a) = |a|$. We call v a *valuation* on L/Θ.

Now if L is a finite sectionally complemented lattice and Θ is a nontrivial congruence of L, then we obtain the finite sectionally complemented lattice $K = L/\Theta$ and the valuation v on K. The question is the following: Given a finite sectionally complemented lattice K and a map $v\colon K \to \mathbb{N}$, when is v a valuation?

Theorem 3 *Let K be a finite sectionally complemented lattice with more than one element, and let $v\colon K \to \mathbb{N}$. Then there exists a finite sectionally complemented lattice L and a nontrivial congruence Θ of L, such that there is an isomorphism $\varphi\colon K \to L/\Theta$ satisfying*

$$v(a) = |\varphi(a)|, \quad \text{for all } a \in K,$$

iff v satisfies the following conditions:

(V_1) *v is anti-isotone, that is, if $a \leq b$ in K, then $v(a) \geq v(b)$ in $\mathbb{N}$.*
(V_2) $2 \leq v(a)$ *and* $v(a) \neq 3$*, for all* $a \in K$.

As a very small example, let us start with $K = M_3$ with the valuation illustrated on Figure 4. The lattice L we construct from this valuation is the one shown in Figure 3.

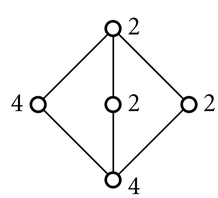

Figure 4: A valuation on M_3.

Again, we can ask about valuations induced by a finite sectionally complemented lattice L and the unique nontrivial congruence Θ of L.

Corollary *Let K be a finite sectionally complemented lattice with more than one element, and let $v\colon K \to \mathbb{N}$. Then there exists a finite sectionally complemented lattice L and a* unique *nontrivial congruence Θ of L, such that there is an isomorphism*

$\varphi\colon K \to L/\Theta$ satisfying

$$\nu(a) = |\varphi(a)|, \quad \textit{for all } a \in K,$$

iff ν satisfies the conditions (V_1) and (V_2), and additionally, ν satisfies the following two conditions:

(V_3) *ν is not a constant function.*
(V_4) *K is simple.*

1.4 Uniformity and Regularity

Let L be a lattice, and let Θ be a congruence on L. The congruence Θ is called *regular* if any congruence class determines Θ; it is called *uniform* if any two congruence classes are of the same size. A lattice L is called *regular* if any congruence of L is regular; a lattice L is called *uniform* if any congruence of L is uniform.

Sectionally complemented lattices are regular, however, in general, they are not uniform as witnessed by N_6. When are finite, sectionally complemented lattices uniform?

Theorem 4 *Let L be a finite sectionally complemented lattice. Then L is uniform iff L is a direct product of simple sectionally complemented lattices.*

Finite relatively complemented lattices are not very interesting from the point of view of congruence lattices; they all have Boolean congruence lattices. The following, however, holds:

Theorem 5 *A finite relatively complemented lattice is uniform. In fact, it is* isoform.

Isoform lattices were introduced in G. Grätzer and E. T. Schmidt [4]. Let L be a lattice, and let Θ be a congruence on L. The congruence Θ is called *isoform* if any two congruence classes of Θ are isomorphic as lattices. A lattice L is called *isoform* if any congruence of L is isoform.

Theorems 4 and 5 are quite easy to prove; they may even be folklore. Note that Theorems 5 implies that a finite, relatively complemented lattice is a direct product of simple relatively complemented lattices.

1.5 Outline

In Section 2, we prove a few useful lemmas on congruences of sectionally complemented finite lattices to lay the foundation for later proofs. We also prove Theorems 4 and 5. In Section 3, we present the basic lattice construction, and verify the relevant properties of the lattice constructed. In Section 4, we prove Theorem 3. This is easy, most of the work is done in Section 3. Most of this section is the proof of the Corollary of Theorem 3. Theorems 1 and 2 are proved in Section 5; they are easy consequences of Theorem 3. Finally, in Section 6, we list some open problems.

1.6 Acknowledgement

We would like to thank Bob Quackenbush and John Wedgewood for useful discussions on these topics.

2 Congruence Classes

We now prove a few lemmas that will be useful in proving the theorems of this paper.

In this section, let L be a finite sectionally complemented lattice with bounds 0_L and 1_L. Let Θ be a nontrivial congruence of L. We set $I = [0_L]\Theta$ (the congruence class containing 0_L). For any congruence class A of Θ, we set $A = [o_A, i_A]$.

Lemma 1 *The map $\varphi_A \colon x \mapsto x \vee o_A$ is a join-homomorphism of I onto A.*

Proof φ_A is obviously a join-homomorphism. Let $x \in A$. Let x' be a sectional complement of o_A in $[0_L, x]$. Then $x' \in I$ and $\varphi_A(x') = x$, so φ_A is onto. ∎

Corollary $|A| \leq |I|$.

Lemma 2 *Let A and B be congruence classes of Θ. If $A \leq B$ in L/Θ, then $|B| \leq |A|$.*

Proof Define a join-homomorphism $\varphi_{A,B}$ of A into B by $\varphi_{A,B} \colon x \mapsto x \vee o_B$. Obviously, $\varphi_A \circ \varphi_{A,B} = \varphi_B$. By Lemma 1 φ_B is onto; therefore, so is $\varphi_{A,B}$. It follows that $|B| \leq |A|$. ∎

Lemma 3 *Let us assume that Θ is uniform. Then $L \cong I \times L/\Theta$.*

Proof Let A be a congruence class of Θ. Then by Lemma 1, φ_A is a join-homomorphism of I onto A. However, by the uniformity of Θ, it follows that $|I| = |A|$. Therefore, $\varphi_A \colon I \to A$ is an isomorphism. For $x \in A$, let $x' \in I$ be the unique element with $\varphi_A(x') = x$. It is clear that

$$x \mapsto \langle x', A \rangle$$

establishes the required isomorphism $L \cong I \times L/\Theta$. ∎

Note that Lemma 3 utilizes only that Θ is a standard congruence (see [1, Section III.2]).

As a side result, we obtained:

Lemma 4 *A uniform congruence of a finite sectionally complemented lattice is also isoform.*

Now the results of Section 1.4 easily follow.

Proof of Theorem 4 Let L be a finite sectionally complemented lattice. We proceed by induction on $|L|$. If L is simple, then we are done. If L is not simple, then L has a nontrivial congruence Θ. By Lemma 3, we have the isomorphism $L \cong I \times L/\Theta$. Since $|I|$ and $|L/\Theta| < |L|$, the theorem follows by induction. ■

Proof of Theorem 5 Let L be a finite relatively complemented lattice. Let A be a congruence class of Θ. By Lemma 1, $|I| \geq |A|$. The dual of L is also sectionally complemented. Applying Lemma 1 again, we get that $|I| \leq |A|$. Hence Θ is uniform. We conclude from Lemma 4 that Θ is isoform. Therefore, so is L. ■

3 A Lattice Construction

3.1 The Construction

Let K be a finite sectionally complemented lattice with more than one element, with bounds 0 and 1. Let $\nu\colon K \to \mathbb{N}$ satisfy conditions (V_1) and (V_2) of Theorem 3.

Now we construct the lattice $L = L(K, \nu)$. Let $n = \nu(0) - 2$ and $M = M_n$ with bounds o and i, and atoms $p_1, \dots, p_n$. By (V_2), $n \geq 0$. If $\nu(0) = 2$, then $n = 0$; in this case, M_0 stands for the two-element lattice.

We form the direct product $K \times M$ and define $L = L(K, \nu)$ as a subset of $K \times M$. Let $\langle k, m\rangle \in K \times M$; then $\langle k, m\rangle \in L$ iff one of the following three conditions hold:

(i) $m = o$;
(ii) $m = i$;
(iii) $m = p_j$ and $j \leq \nu(k) - 2$.

3.2 The Closure Operator

We define a map $\varrho\colon K \times M \to K \times M$. Let $\langle k, m\rangle \in K \times M$; then

$$\varrho(\langle k, m\rangle) = \begin{cases} \langle k, m\rangle, & \text{if } \langle k, m\rangle \in L; \\ \langle k, i\rangle, & \text{if } \langle k, m\rangle \notin L. \end{cases}$$

Now recall that $\langle k, m\rangle \notin L$ means that $m = p_j$ and $j > \nu(k) - 2$.

We claim that ϱ is a closure operator on $K \times M$, that is

(a) $x \leq \varrho(x)$, for $x \in K \times M$.
(b) $\varrho\bigl(\varrho(x)\bigr) = \varrho(x)$, for $x \in K \times M$.
(c) $\varrho(x) \leq \varrho(y)$, if $x \leq y$ in $K \times M$.

(a) and (b) are clear by the definition of ϱ. Now let $x \leq y$ in $K \times M$. If $x \in L$, then $\varrho(x) = x$, so $\varrho(x) \leq \varrho(y)$ follows from (a). If $x \notin L$, then $x = \langle k, p_j\rangle$ and $j > \nu(k) - 2$. Let $y = \langle l, m\rangle$. Since $x \leq y$, it follows that $k \leq l$ in K and $p_j \leq m$ in M. By (V_1), $\nu(k) \geq \nu(l)$, so $\langle l, p_j\rangle \notin L$. It follows that $\varrho(x) = \langle k, i\rangle$ and $\varrho(y) = \langle l, i\rangle$, so $\varrho(x) \leq \varrho(y)$, verifying (c).

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3.3 L is a Lattice

A closure operator is always a meet-homomorphism, so L is a zero-preserving meet-homomorphic image of $K \times M$; hence, L is a meet-semilattice with unit, and therefore, a lattice.

Let $\wedge_\times$ and $\vee_\times$ denote the lattice operations in $K \times M$. Then the the lattice operations $\wedge$ and $\vee$ in L are described as follows. For $x, y \in L$, we have $x \wedge y = x \wedge_\times y$, while $x \vee y = \varrho(x \vee_\times y)$.

3.4 L is Sectionally Complemented

Take $\langle k, m\rangle, \langle k', m'\rangle \in L$ and let $\langle k, m\rangle < \langle k', m'\rangle$. We have to find $\langle k'', m''\rangle \in L$ satisfying

$$\langle k, m\rangle \wedge \langle k'', m''\rangle = \langle 0, o\rangle, \tag{1}$$

$$\langle k, m\rangle \vee \langle k'', m''\rangle = \langle k', m'\rangle. \tag{2}$$

Since $\langle k, m\rangle < \langle k', m'\rangle$, it follows that $k = k'$ or $k < k'$. We deal separately with these two cases.

First Case: $k = k'$**.** Let m'' be a sectional complement of m in $[o, m']$ in M. Since $v(k) \neq 1, 3$ by (V_2), we can choose m'' so that $\langle k, m''\rangle \in L$. It follows that $\langle 0, m''\rangle \in L$. It is now clear that we can choose $\langle 0, m''\rangle$ as the required sectional complement since (1) is obvious, and (2) holds in the stronger form

$$\langle k, m\rangle \vee_\times \langle k'', m''\rangle = \langle k', m'\rangle. \qquad \blacksquare$$

Second Case: $k < k'$**.** Let k'' be a sectional complement of k in $[0, k']$ in K.

If $v(k'') = 2$, then by (V_1) and (V_2), we also have that $v(k') = 2$, so we can choose $m'' = o$.

If $v(k'') \neq 2$, then by (V_2), we have that $v(k'') \geq 4$. Again, only the case $m = p_j$ is interesting. Now if $\langle k', p_j\rangle \in L$, then by (V_1), $\langle k'', p_j\rangle \in L$. So we can choose a sectional complement m'' of m in $[o, m']$ in M so that $\langle k'', m''\rangle \in L$. Obviously, $\langle k'', m''\rangle$ is the required sectional complement. If $\langle k', p_j\rangle \notin L$, then $\langle k'', o\rangle$ is the required sectional complement.

3.5 The Congruence Θ

Let Φ be the congruence kernel of the first projection map of $K \times M$, that is, let $\langle k, m\rangle \equiv \langle k', m'\rangle$ (Φ) iff $k = k'$. Let Θ be the restriction of Φ to L. Since Φ is a congruence of $K \times M$ with the property that $x \equiv \varrho(x)$ (Φ), for all $x \in K \times M$, it follows from the description of the operations in L in Section 3.3 that Θ is a congruence of L.

3.6 The Valuation $\overline{v}$ on L

The congruence Θ defines a valuation $\overline{v}$ on K. It is clear from the construction that $v = \overline{v}$.

4 Proving Theorem 3

Necessity Let K be a finite sectionally complemented lattice with more than one element, with bounds 0 and 1, and let $\nu\colon K \to \mathbb{N}$. Let us assume that there exists a finite sectionally complemented lattice L, with bounds 0_L and 1_L, and a nontrivial congruence Θ of L, such that there is an isomorphism $\varphi\colon K \to L/\Theta$ satisfying

$$\nu(a) = |\varphi(a)|, \quad \text{for all } a \in K.$$

We have to verify (V_1) and (V_2) for ν.

(V_1). This was proved in Lemma 2.

(V_2). We have to prove that $\nu(a) \neq 1, 3$. Indeed, if $\nu(a) = 1$, then the Θ congruence class $\varphi(a)$ of L is a singleton; this contradicts that Θ is a nontrivial regular congruence (because L is sectionally complemented).

Now let $\nu(a) = 3$. Let $\varphi(a) = [o_a, i_a]$, an interval of L. There is a unique $x \in L$ satisfying $o_a < x < i_a$. Since L is sectionally complemented, o_a has a sectional complement y in $[0_L, x]$. There is also a sectional complement z of y in $[0_L, i_a]$. Since $x \equiv o_a\ (\Theta)$, it follows that $y \equiv 0_L\ (\Theta)$, and so $z \equiv i_a\ (\Theta)$, that is, $z \in [o_a, i_a]$. Since $|[o_a, i_a]| = 3$, therefore, $z \leq x$ or $z = i_a$. The first would imply that $z \vee y \leq x$, contradicting that $z \vee y = i_a$, while the second would imply that $z \wedge y = y$, contradicting that $z \wedge y = 0_L$.

Sufficiency Let K be a finite sectionally complemented lattice with more than one element, and let $\nu\colon K \to \mathbb{N}$ satisfy (V_1) and (V_2). Let $L = L(K, \nu)$ be the lattice with the congruence Θ constructed in Section 3. All the required properties of L and Θ were proved in Section 3.

Proof of the Corollary of Theorem 3 Let K be a finite sectionally complemented lattice with more than one element, and let $\nu\colon K \to \mathbb{N}$ satisfy (V_1) and (V_2). By Theorem 3, there exists a finite sectionally complemented lattice L and nontrivial congruence Θ of L, such that there is an isomorphism $\varphi\colon K \to L/\Theta$ satisfying

$$\nu(a) = |\varphi(a)|, \quad \text{for all } a \in K.$$

We take for L and Θ the lattice and the congruence constructed in Section 3, respectively. We show that (V_3) and (V_4) are necessary and sufficient for Θ to be the unique nontrivial congruence Θ of L.

Necessity Let us assume that Θ is the unique nontrivial congruence of L.

To verify (V_3), assume to the contrary that ν is constant. Then Θ is uniform. By Lemma 3, then $L \cong I \times L/\Theta$, which contradicts that Θ is the unique nontrivial congruence of L.

To verify (V_4), assume to the contrary that K is not simple; let Ψ be a nontrivial congruence of K. Then the inverse image Ψ' of Ψ under the natural homomorphism $L \to L/\Theta$ is a nontrivial congruence of L different from Θ, a contradiction.

Sufficiency Let us now assume that conditions (V_3) and (V_4) hold. We shall prove that Θ is the unique nontrivial congruence of L.

We shall use the notation $\varphi(a) = [o_a, i_a]$, an interval of L, for $a \in K$. Equivalently,

$$[o_a, i_a] = \{\langle a, m\rangle \in L \mid m \in M\}.$$

Let $x/y \nearrow u/v$ be shorthand for $x \wedge v = y$ and $x \vee v = u$, see Section III.1 of [1].

The following statement is trivial:

Claim 1 *For any $a, b \in K$, with $a \leq b$, we have $i_a/o_a \nearrow i_b/o_b$.*

Let Ψ be a nontrivial congruence of L.

We continue with the following claim:

Claim 2 $\Theta \leq \Psi$.

Proof Let $x < x'$ in L and let $x \equiv x'(\Psi)$. Let $x = \langle k, m\rangle$ and $x' = \langle k', m'\rangle$. Since L is sectionally complemented, we can assume that $x = 0_L$ $(= o_0)$.

Now we distinguish two cases: $k' = 0$ and $k' > 0$.

First Case: $k' = 0$**.** In this case, $o_0 = \langle 0, o\rangle \equiv \langle 0, m'\rangle$ (Ψ), for some $m' > o$ in M.

We distinguish two subcases: $\nu(0) = 4$ and $\nu(0) \neq 4$.

First Subcase: $\nu(0) = 4$. By (V_1) and (V_2), it follows that $\nu(a) = 2$ or 4, for all $a \in K$. Since $\langle 0, o\rangle \equiv \langle 0, m'\rangle$ (Ψ), for some $m' > o$ in M, we can assume that $m' = p_1$ or $m' = p_2$.

Since $\nu(a) = 2$ or 4, for all $a \in K$, but by (V_3), ν is not constant, there is a $u \in K$ with $\nu(u) = 2$. Joining the congruence $\langle 0, o\rangle \equiv \langle 0, m'\rangle$ (Ψ) (where $m' = p_1$ or $m' = p_2$) with o_u, we obtain that $o_u \equiv i_u$ (Ψ). Meeting with i_0, by Claim 1 we get that $o_0 \equiv i_0$ (Ψ), and finally, for an arbitrary $a \in K$, joining with o_a, we conclude that

$$o_a \equiv i_a \quad (\Psi),$$

proving that $\Theta \leq \Phi$.

Second Subcase: $\nu(0) \neq 4$. Observe that in this case $\nu(0) \geq 5$. Indeed, if not, then by (V_2), we have that $\nu(0) = 2$. By (V_1) and (V_2), it follows that $\nu(a) = 2$, for all $a \in K$, contradicting (V_3). Since $\nu(0) \geq 5$, it follows that M is a simple lattice. Therefore, $o_0 \equiv i_0$ (Ψ). This implies that $o_a \equiv i_a$ (Ψ), for an arbitrary $a \in K$, proving that $\Theta \leq \Phi$.

Second case: $k' > 0$**.** We can assume that $m' = o$; otherwise, joining the congruence with $o_{k'}$ and taking a sectional complement, we are back to the first case. So we start with the congruence $o_0 \equiv o_{k'}$ (Ψ), that is, with

$$\langle 0, o\rangle \equiv \langle k', o\rangle \quad (\Psi),$$

and join it with $o_{a_1} = \langle a_1, o\rangle$, we get

$$\langle a_1, o\rangle \equiv \langle k' \vee a_1, o\rangle \quad (\Psi);$$

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meeting this with $o_{b_1} = \langle b_1, o\rangle$, we obtain that

$$\langle a_1 \wedge b_1, o\rangle \equiv \langle (k' \vee a_1) \wedge b_1, o\rangle \quad (\Psi).$$

In general, if $p(x)$ is a unary algebraic function on K (that is, a polynomial with constants from K), then

$$o_{p(0)} \equiv o_{p(k')} \quad (\Psi).$$

Since K is simple, for any $a \prec b$ in K, there is an algebraic function $p(x)$ such that $p(0) = a$ and $p(k') = b$ (see Section III.1 of [1]). This implies the congruence

$$o_a \equiv o_b \quad (\Psi).$$

By transitivity, this congruence holds for arbitrary a, b in K.

Since by (V_3) the function v is not constant, we can choose $a < b$ in K with $v(a) > v(b)$. This means that $\langle a, p_{v(a)-2}\rangle \in L$ but $\langle b, p_{v(a)-2}\rangle \notin L$. Join the congruence $o_a \equiv o_b$ (Ψ) with $\langle a, p_{v(a)-2}\rangle$ and observe that $o_b \vee \langle a, p_{v(a)-2}\rangle = i_b$, so we obtain the congruence

$$\langle a, p_{v(a)-2}\rangle \equiv i_b \quad (\Psi),$$

and meeting with i_a, we conclude that

$$\langle a, p_{v(a)-2}\rangle \equiv i_a \quad (\Psi).$$

By taking the sectional complement x of $\langle a, p_{v(a)-2}\rangle$ in $[o_0, i_a]$, we get $x \equiv o_0$ (Ψ), where $x \in [o_0, i_0]$ and $x > o_0$, reducing the second case to the first case.

Note that in this second case, we have proved that $\Psi = \iota$, the largest congruence. Indeed, we verified that

$$o_a \equiv o_b \quad (\Psi)$$

holds for arbitrary a, b in K, and also the conclusion of the first case holds, namely, that

$$o_a \equiv i_a \quad (\Psi),$$

for arbitrary a, b in K. The last two displayed congruences imply that $\Psi = \iota$. ∎

Continuing the proof of the sufficiency, let Ψ be a nontrivial congruence of L satisfying $\Psi \neq \Theta$. So there exist $x < x'$ in L such that $x \equiv x'$ (Ψ) but $x \not\equiv x'$ (Θ). Let $x = \langle k, m\rangle$ and $x' = \langle k', m'\rangle$. Since L is sectionally complemented, we can assume that $x = o_0$, the zero of L.

Now we cannot have $k' = 0$, because then $x = o_0 \equiv x' = \langle 0, m'\rangle$ (Θ). So $k' > 0$, the second case in Claim 2. However, in the second case we concluded that $\Psi = \iota$. So $\Psi = \iota$ holds, concluding the proof of the sufficiency. ∎

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5 Proving Theorems 1 and 2

We start by proving Theorem 2.

Let $S = (m_j \mid j < n)$ be a family of natural numbers, $n \geq 1$.

Necessity Let us assume that there is a finite sectionally complemented lattice L with more than one element and a nontrivial congruence Θ of L such that S is the spectrum of Θ. We have to verify that conditions (S_1) and (S_2) hold.

If (S_1) fails, then $n = 1$ or $n = 3$. But $n = 1$ contradicts that L has more than one element, and $n = 3$ is impossible because there is no sectionally complemented lattice with three elements. So (S_1) holds.

(S_2) follows from (V_2) of Theorem 3 applied to L/Θ. ■

Sufficiency Let us assume that conditions (S_1) and (S_2) hold. The valuation v on L/Θ satisfies (V_2), so the spectrum satisfies (S_2). Define $\max S = \max(m_j \mid j < n)$ and $\min S = \min(m_j \mid j < n)$. Let $K = M_{n-2}$, with bounds 0 and 1. We define a valuation v on K as follows:

$$v(0) = \max S,$$

$$v(1) = \min S,$$

and we arbitrarily assign the remaining $n-2$ elements of S as v-values to the atoms of K. Then v satisfies (V_1) and (V_2) because of (S_1) and (S_2) and the way we defined v. So Theorem 3 provides us with a finite sectionally complemented lattice L and a nontrivial congruence Θ of L realizing v. Obviously, S is the spectrum of Θ. ■

Moreover, it is clear that S is not constant iff the valuation v we have constructed from S is not constant. Observe that there is no four-element simple sectionally complemented lattice, so (S_4) is necessary. And in the presence of (S_4), the lattice K we constructed for S is always simple. So the Corollary of Theorem 2 follows from the Corollary of Theorem 3.

Finally, Theorem 1 is the special case of the Corollary of Theorem 2: $n = 2$ and $S = (t_1, t_2)$. Condition (S_1) then corresponds to (P_1), (S_2) to (P_2), and (S_3) to (P_3); (S_4) is trivially satisfied since $n = 2$.

6 Problems

6.1 Spectra

We only know that the congruence lattice $\operatorname{Con} L$ of the lattice L we construct in Theorem 2 has three or more elements. Can we prescribe its structure?

Problem 1 Let D be a finite distributive lattice. Can we construct the lattice L of Theorem 2 that represents the given $S = (m_j \mid j < n)$ as the spectrum of a nontrivial congruence Θ so that L also satisfies $D \cong \operatorname{Con} L$?

The following form is even harder:

Problem 2 Let D be a finite distributive lattice, and let $a \in D$, $a \neq 0, 1$. Can we construct the lattice L of Theorem 2 that represents the given $S = (m_j \mid j < n)$ as the spectrum of a nontrivial congruence Θ so that L also satisfies $D \cong \operatorname{Con} L$ and under this isomorphism the element a of D maps to the congruence Θ of L?

Note that the Corollary of Theorem 2 solves this problem for $D = C_3$, the three-element chain.

6.2 Valuations

We can state Problems 1 and 2 also for valuations.

Problem 3 Let D be a finite distributive lattice. Can we prove Theorem 3 with the additional condition: $\operatorname{Con} K \cong D$?

Problem 4 Let D be a finite distributive lattice, and let $a \in D$, $a \neq 0, 1$. Can we prove Theorem 3 with $\operatorname{Con} K \cong D$ so that under this isomorphism a maps to Θ?

Note that the Corollary of Theorem 3 solves this problem for $D = C_3$, the three-element chain.

6.3 Infinite Relatively Complemented Lattices

Does Theorem 5 extend to the infinite case?

Problem 5 Are infinite relatively complemented lattices uniform?

They are not isoform. For an example, take an infinite set X, and let L be the lattice of all finite and cofinite subsets of X. Let Θ be the congruence under which $a \equiv b\,(\Theta)$ iff $(a - b) \cup (b - a)$ is finite. Then Θ has two congruence classes, an ideal P and a dual ideal Q. Obviously, L is relatively complemented, but $P \ncong Q$.

6.4 Congruence Preserving Extensions

Let L be a lattice. A lattice L' is a *congruence-preserving extension* of L, if L' is an extension and every congruence Θ of L has *exactly one* extension Θ' to L'. Of course, then the congruence lattice of L is isomorphic to the congruence lattice of L'.

There is a large body of results on congruence-preserving extensions with special properties. See Appendix C of [1] for a survey of this field; the following example from G. Grätzer and E. T. Schmidt [3] is typical:

Theorem *Every finite lattice has a congruence-preserving extension to a finite* sectionally complemented *lattice.*

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Problem 6 Let L be a finite lattice and let Θ be a nontrivial congruence of L with spectrum v on $K = L/\Theta$. Let $v' \colon K \to \mathbb{N}$ satisfy (V_1) and (V_2). If $v \leq v'$ (*i.e.*, $v(a) \leq v'(a)$, for all $a \in K$), then does there exist a finite congruence-preserving extension L' of L such that the spectrum on L'/Θ' is v'?

6.5 Chopped Lattices

We need a few concepts.

Let M be a finite poset such that $\inf\{a, b\}$ exists in M, for all $a, b \in M$. We define in M:

$$a \wedge b = \inf\{a, b\}, \qquad \text{for all } a,\ b \in M;$$
$$a \vee b = \sup\{a, b\}, \qquad \text{whenever } \sup\{a, b\} \text{exists}.$$

This makes M into a finite *chopped lattice*.

An equivalence relation Θ on the chopped lattice M is a *congruence relation* iff for all $a_0, a_1, b_0, b_1 \in M$, $a_0 \equiv b_0$ (Θ) and $a_1 \equiv b_1$ (Θ) imply that $a_0 \wedge a_1 \equiv b_0 \wedge b_1$ (Θ) provided that $a_0 \wedge a_1$ and $b_0 \wedge b_1$ both exist, and $a_0 \vee a_1 \equiv b_0 \vee b_1$ (Θ), provided that $a_0 \vee a_1$ and $b_0 \vee b_1$ both exist.

The set $\operatorname{Con} M$ of all congruence relations of M is a lattice.

An *ideal* I of a finite chopped lattice M is a non-empty subset $I \subseteq M$ such that $i \wedge a \in I$, for $i \in I$ and $a \in M$; and $i \vee j \in I$, for $i, j \in I$, provided that $i \vee j$ exists in M. The ideals of the finite chopped lattice M form the finite lattice $\operatorname{Id} M$.

The following lemma was published in G. Grätzer [1].

***Lemma* (G. Grätzer and H. Lakser)** *Let M be a finite chopped lattice. Then for every congruence relation Θ of M, there exists exactly one congruence relation $\overline{\Theta}$ of* $\operatorname{Id} M$ *such that, for $a, b \in M$,*

$$(a] \equiv (b]\ (\overline{\Theta}) \quad \textit{iff} \quad a \equiv b\ (\Theta).$$

In particular, $\operatorname{Con} M \cong \operatorname{Con}(\operatorname{Id} M)$.

From the point of view of this paper, the significance of this lemma is that many finite sectionally complemented lattices with a given congruence lattice were constructed using this approach: We construct a finite sectionally complemented chopped lattice M, and then $\operatorname{Id} M$ is the desired lattice, see, for instance, [2] and [3]. Unfortunately, we do not know under what conditions $\operatorname{Id} M$ inherits from M the property of being sectionally complemented.

Problem 7 When is the ideal lattice of a finite sectionally complemented chopped lattice a sectionally complemented lattice?

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