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MOORE G-SPACES WHICH ARE NOT CO-HOPF G-SPACES

BY

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ABSTRACT. Let G be a finite group. By a Moore G-space we mean a G-space X such that for each subgroup H of G the fixed point space X^H is a simply connected Moore space of type (M_H, n), where M_H is an abelian group depending on H, and n is a fixed integer. By a co-Hopf G-space we mean a G-space with a G-equivariant comultiplication. In this note it is shown that, in contrast to the non-equivariant case, there exist Moore G-spaces which are not co-Hopf G-spaces.

1. **Introduction.** Let G be a finite group. G-spaces, G-actions, G-maps, and G-homotopies considered in this paper will be pointed. We shall work in the category of G-spaces having the G-homotopy type of a G-CW-complex [1], and we make tacit use of the standard strategies for keeping our constructions within this category.

DEFINITION 1.1. A *co-Hopf G-space* is a co-H-space X on which G acts in such a way that the comultiplication $\sigma : X \longrightarrow X \vee X$ is an equivariant map, and the composition $X \xrightarrow{\sigma} X \vee X \subset X \times X$ is G-homotopic to the diagonal map $\Delta : X \longrightarrow X \times X$.

Let O_G be the category of canonical orbits of G. The objects of O_G are the quotient spaces G/H, where H is a subgroup of G, and the morphisms are the G-maps between them, where G acts on G/H by left multiplication. A *coefficient system for G* is a contravariant functor from O_G into the category of abelian groups. A coefficient system will be called rational if its range is the category of $\mathbf{Q}$-vector spaces. For a G-space X, coefficient systems $\underline{\pi}_n(X)$ and $\underline{\tilde{H}}_n(X)$ can be defined by $\underline{\pi}_n(X)(G/H) = \pi_n(X^H)$, $\underline{\tilde{H}}_n(X)(G/H) = \tilde{H}_n(X^H)$, where $\tilde{H}_n(\ )$ denotes the reduced singular homology group with $\mathbf{Z}$-coefficients.

Let M be a coefficient system for G and $n \geqq 2$ an integer.

DEFINITION 1.2. A *Moore G-space* of type (M, n) is a G-space X such that each fixed point space X^H, H a subgroup of G, is simply connected and

$$\underline{\tilde{H}}_q(X) = \begin{cases} M & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Coefficient systems for G and their natural transformations form an abelian category with sufficiently many projectives and injectives [1]. The same holds for rational coefficient systems. By a result of P. J. Kahn [4], if M is a rational coefficient system for

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G, $n \geqq 2$ an integer, and proj.dim $M < n$, then, up to G-equivalence (= G-homotopy equivalence), there exists exactly one Moore G-space of type (M, n). Uniqueness, however, does not hold in general for Moore G-spaces. In [4] there is given an example of a rational coefficient system M and two Moore G-spaces L_1, L_2 of type $(M, 2)$ which are not G-equivalent. In [2] we have shown, by methods completely different from that of [4], that for the system M there exist infinitely many non G-equivalent Moore G-spaces of type $(M, 2)$.

The aim of this note is to show that all but one of the Moore G-spaces of type $(M, 2)$ constructed in [2] are not co-Hopf G-spaces. Thus, we show that the well known result that every simply connected Moore space is a co-H-space does not hold in the G-equivariant context.

2. **Constructing Moore G-spaces**. Let $G = \mathbf{Z}_2 \times \mathbf{Z}_2$, where $\mathbf{Z}_2$ denotes the cyclic group of order 2. A typical coefficient system for G can be represented as follows

$$\begin{array}{ccccc} & & M(G/H_1) & & \\ & \nearrow & & \searrow & \\ M(G/G) & \longrightarrow & M(G/H_2) & \longrightarrow & M(G/e) \\ & \searrow & & \nearrow & \\ & & M(G/H_3) & & \end{array}$$

where H_1, H_2, H_3 are the proper subgroups of G.

Henceforth, we shall assume that $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ and M will denote a rational coefficient system for G given by the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \searrow & \\ \mathbf{Q} & \longrightarrow & 0 & \longrightarrow & \mathbf{Q} \\ & \searrow & & \nearrow & \\ & & 0 & & \end{array}$$

in which the action on $\mathbf{Q} = M(G/e)$ is trivial.

We shall use the following property of the system M.

PROPOSITION 2.1. [4, 5.3.2] *Let* Ext^i *denote the i-th right derived functor of* Hom *in the category of rational coefficient systems. Then*

$$\mathrm{Ext}^i(M, M) = \begin{cases} \mathbf{Q} \oplus \mathbf{Q} & \text{if } i = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

We have observed in [2] that each Moore G-space of type $(M, 2)$ has only two non-trivial systems of homotopy groups: $\underline{\pi}_2(X) = \underline{\pi}_3(X) = M$ and that this implies that it is determined, up to G-homotopy type, by its equivariant k-invariant $k(X)$ which lies in the Bredon cohomology group $\tilde{H}^4_G(K(M, 2), M)$ [5]. Here $K(M, 2)$ denotes an Eilenberg-MacLane G-space of type $(M, 2)$ [1].

Let $i_1, i_2 \in \tilde{H}^2_G(K(M, 2), M)$ be the classes corresponding to (1,0), (0,1), respectively, under the identification $\tilde{H}^2_G(K(M, 2), M) = [K(M, 2), K(M, 2)]_G =$

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$\text{Hom}(M,M) = \text{Hom}(M(G/G), M(G/G)) \oplus \text{Hom}(M(G/e), M(G/e)) = \mathbf{Q} \oplus \mathbf{Q}$. Let us denote $i_k^2 = i_k \cup i_k, k .= 1, 2$, where $\cup : \tilde{H}_G^2(K(M,2), M) \otimes \tilde{H}_G^2(K(M,2), M) \longrightarrow \tilde{H}_G^4(K(M,2), M)$ is the cup-product in Bredon cohomology.

We shall use the following facts proved in [2].

PROPOSITION 2.2. *There is a functorial short exact sequence of* $\mathbf{Q}$*-vector spaces*

$$0 \longrightarrow \text{Ext}^2(\underline{\tilde{H}}_2(K(M,2), M) \longrightarrow \tilde{H}_G^4(K(M,2), M) \longrightarrow \text{Hom}(\underline{\tilde{H}}_4(K(M,2), M) \longrightarrow 0.$$

PROPOSITION 2.3. *For each element* $u \in \text{Ext}^2(\underline{\tilde{H}}_2(K(M,2), M) \subset \tilde{H}_G^4(K(M,2), M)$ *the* G*-space determined by the equivariant* k*-invariant* $u + i_1^2 + i_2^2$ *is a Moore* G*-space of type* $(M,2)$.

3. **Non-existence of co-Hopf G-structures.**

PROPOSITION 3.1. *Let* X *be a Moore* G*-space of type* $(M,2)$ *and* $p : X \longrightarrow K(M,2)$ *the equivariant Postnikov deomposition of* X [5]. *Then* $\tilde{H}_G^4(X, M) = \text{Ext}^2(\underline{\tilde{H}}_2(X), M)$ *and the homomorphism* $p^* : \tilde{H}_G^4(K(M,2), M) \longrightarrow \tilde{H}_G^4(X, M)$ *induced by* p *restricts to an isomorphism* $\text{Ext}^2(\underline{\tilde{H}}_2(K(M,2), M) \longrightarrow \tilde{H}_G^4(X, M)$, *where* $\text{Ext}^2(\underline{\tilde{H}}_2(K(M,2), M) \subset \tilde{H}_G^4(K(M,2), M)$ (*see Proposition* 2.2).

PROOF. The map $p : X \longrightarrow K(M,2)$ is a 2-G-equivalence. Thus, $p_* : \underline{\tilde{H}}_2(X) \longrightarrow \underline{\tilde{H}}_2(K(M,2))$ is an isomorphism. Clearly, a universal coefficient spectral sequence [1], gives for X an isomorphism $\text{Ext}^2(\underline{\tilde{H}}_2(X), M) \xrightarrow{\cong} \tilde{H}_G^4(X, M)$. Hence, the result follows from Proposition 2.2 and naturality of the spectral sequence. □

THEOREM 3.2. *Let* $u \in \text{Ext}^2(\underline{\tilde{H}}_2(K(M,2), M) \subset \tilde{H}_G^4(K(M,2), 2)$ *be any non-zero element. Then the Moore* G*-space of type* $(M,2)$ *determined by the equivariant* k*-invariant* $u + i_1^2 + i_2^2$ *is not a co-Hopf* G*-space.*

PROOF. Let X be a Moore G-space of type $(M,2)$ determined by the equivariant K-invariant $u + i_1^2 + i_2^2$, u a non-zero element of $\text{Ext}^2(\underline{\tilde{H}}_2(K(M,2), M)$. In the same way as in the non-equivariant case [6, p. 423], we can show that $p^*(u + i_1^2 + i_2^2) = 0$ in $\tilde{H}_G^4(X, M)$. It follows from Proposition 3.1 that $p^*(u) \neq 0$. Thus, $(p^*(i_1))^2 + (p^*(i_2))^2 = p^*(i_1^2 + i_2^2) \neq 0$. Hence, there are non-trivial cup-products in $\tilde{H}_G(X, M)$. Thus it follows, by the same arguments as in the non-equivariant case [3, p. 188], that X can not be a co-Hopf G-space. □

REMARK 3.3. We have shown in [2] that, varying u all over the group $\text{Ext}^2(\underline{\tilde{H}}_2(K(M,2), M)$, we can obtain infinitely many non G-equivalent Moore G-spaces of type $(M,2)$. Thus, it follows that there exist infinitely many non G-equivalent Moore G-spaces of type $(M,2)$ which are not co-Hopf G-spaces.

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